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A CLASS OF STOCHASTIC PARTIAL DIFFERENTIAL EQUATIONS FOR INTERACTING SUPERPROCESSES ON A BOUNDED DOMAIN

YAN-XIA REN, RENMING SONG and HAO WANG

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Abstract

A class of interacting superprocesses on $\mathbb{R}$, called superprocesses with dependent spatial motion (SDSMs), were introduced and studied in Wang [32] and Dawson et al. [9]. In the present paper, we extend this model to allow particles moving in a bounded domain in $\mathbb{R}^d$ with killing boundary. We show that under a proper re-scaling, a class of discrete SPDEs for the empirical measure-valued processes generated by branching particle systems subject to the same white noise converge in $L^2(\Omega, \mathcal{F}, \mathbb{P})$ to the SPDE for an SDSM on a bounded domain and the corresponding martingale problem for the SDSMs on a bounded domain is well-posed.

1. Introduction

In this section, we will introduce our model and describe the difficulties and challenges we encounter and how we overcome them.

1.1. Model and preliminaries. A class of interacting superprocesses on $\mathbb{R}$ known as SDSMs were constructed and studied in Wang [32]. This class of SDSMs includes the super-Brownian motion as a special case. However, SDSMs may exhibit completely different features and properties from that of super-Brownian motion. For example, when the underlying dimension is one and the generator is degenerate, the state space of the SDSM consists of purely-atomic measures (see Wang [31]). The study of degenerate SDSMs is closely related to the theory of stochastic flows, see, for example, Dawson et al. [10], [8], Ma et al. [25] and Harris [18]. For a general reference on stochastic flows, the reader is referred to Kunita [22]. In the present paper, we extend this model to allow particles moving in a bounded domain in $\mathbb{R}^d$ with killing boundary. After we establish the existence and uniqueness of the limiting superprocess, we will derive a class of SPDEs for the limiting superprocesses on a bounded domain. Essentially we will follow the basic ideas of Dawson et al. [11] to construct our branching particle systems and to derive the corresponding discrete SPDEs on a bounded domain D in $\mathbb{R}^d$. However, due to the restriction to a bounded domain D our new model raises a sequence of challenges. First of all, since particles are killed upon exiting D, the branching mechanism

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$M^n(D \times (0, t])$ defined by (3.2) is no longer a martingale. Secondly, we have to choose the appropriate form of the infinitesimal generator as (1.2) to take care of the fact that particles are killed upon exiting the domain. Thirdly, in order that (1.2) is an infinitesimal generator of a measure-valued diffusion process, we have to choose a proper domain for it. This forces us to choose the test functions $\phi \in \mathscr{D}(D) = \bigcup_{K \subset D} C_K^\infty(D)$, the vector space of infinitely differentiable functions with compact support in D, endowed with the inductive limit of the topologies on $C_K^\infty(D)$. We will revisit this point in Subsection 1.2. The fourth problem is that Mitoma's theorem, a basic tool in deriving the limiting SPDE in Dawson et al. [11], is not applicable in the present case, since $\mathscr{D}(D)$ is not a Fréchet space. The fifth problem occurs in the proof of the uniqueness of the martingale problem for the SDSM on D when we use Dawson-Kurtz's duality argument. The difficulty lies in the verification of the invariant property of the dual semigroup. In the following we will explain how to overcome these difficulties. For convenience, the limiting superprocess will be abbreviated as SDSMB. Let D be a bounded domain (i.e., a connected open subset) in $\mathbb{R}^d$. We assume that D is regular, that is, a Brownian motion starting from any boundary point of D will, with probability 1, hit D^c, the complement of D immediately. The dynamics of each $\mathbb{R}^d$-valued particle is described by the following equation: for each $k \in \mathbb{N}$,

$$z_k^T(t) - z_k^T(0) = \int_0^t c(z_k(s))\, dB_k(s) + \int_0^t \int_{\mathbb{R}^d} h(y - z_k(s))\, W(dy, ds), \tag{1.1}$$

where $z_k(t) = (z_{k1}(t), \ldots, z_{kd}(t))$ is an $\mathbb{R}^d$-valued process, $\{B_k = (B_{k1}, \ldots, B_{kd})^T : k \geq 1\}$ are independent d-dimensional, standard Brownian motions, W is a Brownian sheet on $\mathbb{R}^d$ (see below for definition). The processes W and $\{B_k : k \geq 1\}$ are assumed to be independent of each other. $h(\,\cdot\,) := (h_i(\,\cdot\,)) = (h_1(\,\cdot\,), \ldots, h_d(\,\cdot\,))^T$ (written as a column vector, where H^T is the transpose of the vector H) is assumed to be an $\mathbb{R}^d$-valued Lipschitz function which belongs to $L^1(\mathbb{R}^d) \cap L^2(\mathbb{R}^d)$ and $c(\,\cdot\,) := (c_{ij}(\,\cdot\,))$ (a $d \times d$ matrix) is assumed to be an $\mathbb{R}^{d \times d}$-valued Lipschitz function. Then, by the standard Picard's iteration method we can prove that (1.1) has a unique strong solution which is denoted by $z_k(t)$.

Let $\mathcal{B}(\mathbb{R}^d)$ be the Borel σ-field. By abusing the notation, the Lebesgue measures on $\mathbb{R}^d$ and on $\mathbb{R}^{d+1}$ will both be denoted by m. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space with a right continuous filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A random set function W on $\mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ is called a *Brownian sheet* or a *space-time white noise* on $\mathbb{R}^d$ if

(i) for any $A \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ with $m(A) < \infty$, $W(A)$ is a Gaussian random variable with mean zero and variance $m(A)$;

(ii) for any $A_i \in \mathcal{B}(\mathbb{R}^d \times \mathbb{R}_+)$ with $A_1 \cap A_2 = \emptyset$ and $m(A_i) < \infty$, $i = 1, 2$, $W(A_1)$ and $W(A_2)$ are independent and

$$W(A_1 \cup A_2) = W(A_1) + W(A_2), \quad \mathbb{P}\text{-a.s.};$$

(iii) for any $A \in \mathcal{B}(\mathbb{R}^d)$ with $m(A) < \infty$, $M(A)_t := W(A \times [0,t])$, as a process in $t \geq 0$, is a square-integrable $\{\mathcal{F}_t\}$-martingale.

For more information on Brownian sheets, the reader is referred to Walsh [30, Chapter 2] and Dawson [6, Section 7.1].

From (1.1), we can see that each particle is subject to a random medium force, which is described by the common Brownian sheet W, and each particle has its own diffusion dynamics, which is described by an individual Brownian motion.

In this paper we will use the standard function space notation. $L^2(\mathbb{R}^d)$ stands for the Hilbert space of square-integrable functions on $\mathbb{R}^d$, and $L^\infty(\mathbb{R}^d)$ the space of bounded measurable functions on $\mathbb{R}^d$. $C((\mathbb{R}^d)^m)$ and $C^k((\mathbb{R}^d)^m)$ stand for the space of continuous functions on $(\mathbb{R}^d)^m$ and the space of continuous functions on $(\mathbb{R}^d)^m$ with continuous derivatives up to and including order k, respectively. $C_b((\mathbb{R}^d)^m)$ and $C_b^k((\mathbb{R}^d)^m)$ stand for the space of bounded continuous functions on $(\mathbb{R}^d)^m$ and the space of bounded continuous functions that have bounded continuous derivatives up to and including order k, respectively. We use $\mathrm{Lip}(\mathbb{R}^d)$ to denote the space of Lipschitz functions on $\mathbb{R}^d$; that is, $f \in \mathrm{Lip}(\mathbb{R}^d)$ if there is a constant $k > 0$ such that $|f(x) - f(y)| \leq k|x - y|$ for every $x, y \in \mathbb{R}^d$. The class of bounded Lipschitz functions on $\mathbb{R}^d$ will be denoted by $\mathrm{Lip}_b(\mathbb{R}^d)$.

Using the strong solution of (1.1), we can construct a family of branching particle systems in a bounded domain D. For each natural number $n \geq 1$, suppose that initially there are $m_0^{(n)}$ number of particles located at $z_i(0)$, $1 \leq i \leq m_0^{(n)}$ and each has mass θ^{-n}, where $\theta > 1$ is a fixed constant. These particles evolve according to (1.1) and branch independently at rate $\gamma\theta^n$ with identical offspring distribution in the domain D. Once a particle reaches the boundary of the domain D, it is killed and it disappears from the system. When a particle dies in the domain D, it immediately produces new particles. After branching, the offspring of each particle evolves according to (1.1) and then branches again in D. The common n-th stage branching mechanism $q^{(n)} := \{q_k^{(n)} : k = 0, 1, \ldots\}$, where $q_k^{(n)}$ stands for the probability that at the n-th stage an individual dies and has k offspring, is assumed to be critical (that is, the average number of offspring is 1), and it can not produce 1 or more than n number of children. Under the assumption that the initial distribution $\theta^{-n} \sum_{k=1}^{m_0^{(n)}} \delta_{z_k(0)}$ (for any $z \in \mathbb{R}^d$, δ_z stands for the δ-measure at the point z) of the particles converges weakly to a measure μ_0 on D and that the branching function $q^{(n)}$ converges uniformly to a limiting branching function with finite second moment, we will show that, under some additional conditions, the empirical process $\theta^{-n} \sum_{k \geq 1} \delta_{z_k(t)} 1_{\{t < \tau_k\}}$ converges weakly to a measure-valued process, where $\tau_k = \inf\{t > 0 : z_k(t) \notin D\}$ is the first time the k-th particle exiting D.

Let $M_F(D)$ denote the Polish space of all finite measures on D with weak topology and $C(M_F(D))$ be the space of all continuous functions on $M_F(D)$. Based on the assumption that motions are independent of branching, by Itô's formula and a formal calculation we can find that the limiting measure-valued processes have the following

formal generators (usually called pregenerators. See Section 2 of Dawson [5]):

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{1.2}$$

$$\mathcal{B}F(\mu) := \frac{1}{2}\gamma\sigma^2 \int_D \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\,\mu(dx), \tag{1.3}$$

and

$$\begin{aligned}\mathcal{A}F(\mu) :=& \frac{1}{2}\sum_{p,q=1}^{d}\int_D (a_{pq}(x) + \rho_{pq}(x,x))\left(\frac{\partial^2}{\partial x_p \partial x_q}\right)\frac{\delta F(\mu)}{\delta\mu(x)}\,\mu(dx)\\ &+ \frac{1}{2}\sum_{p,q=1}^{d}\int_D\int_D \rho_{pq}(x,y)\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial y_q}\right)\frac{\delta^2 F(\mu)}{\delta\mu(x)\delta\mu(y)}\,\mu(dx)\,\mu(dy)\end{aligned} \tag{1.4}$$

for $F(\mu) \in \mathcal{D}(\mathcal{L}) \subset C(M_F(D))$, where for $x = (x_1, \ldots, x_d)$, $y = (y_1, \ldots, y_d) \in \mathbb{R}^d$,

$$a_{pq}(x) := \sum_{r=1}^{d} c_{pr}(x)c_{qr}(x), \tag{1.5}$$

$$\rho_{pq}(x,y) := \int_{\mathbb{R}^d} h_p(u-x)h_q(u-y)\,du,$$

where the constant $\gamma > 0$ above is related to the branching rate of the particle system and $\sigma^2 > 0$ is the variance of the limiting offspring distribution, the variational derivative is defined by

$$\frac{\delta F(\mu)}{\delta\mu(x)} := \lim_{h\downarrow 0}\frac{F(\mu + h\delta_x) - F(\mu)}{h}, \tag{1.6}$$

and $\mathcal{D}(\mathcal{L})$, the domain of the pregenerator $\mathcal{L}$, consists of functions of the form

$$F(\mu) = f(\langle\phi_1, \mu\rangle, \ldots, \langle\phi_k, \mu\rangle)$$

satisfying following conditions:

(1) $\phi_i \in C_b^2(D)$ for $1 \le i \le k$ and $f \in C_b^2(\mathbb{R}^k)$;

(2) for any $1 \le i \le k$, ϕ_i has compact support in D.

Now let us give the motivation that why we need to choose the domain of the generator (1.2) in this way.

1.2. Motivation for the choice of domain. To simplify the situation and directly show the essential point of the problem, we consider two examples in the finite dimensional case.

(I) First, we consider a one dimensional reflecting Brownian motion. Let $\{B_t\}$ be an $\mathcal{F}_t$-Brownian motion on $(\Omega, \mathcal{F}, \mathbb{P})$. Then $X_t = |B_t|$ is the reflecting Brownian

motion. The infinitesimal generator of the semigroup T_t of the reflecting Brownian motion is given by $(\mathfrak{G}, \mathcal{D}(\mathfrak{G}))$, where

$$\mathfrak{G} f = \frac{1}{2} f''$$

and

$$\mathcal{D}(\mathfrak{G}) = \{ f \in C_b^2([0, \infty)) \colon f'' \text{ is uniformly continuous, } f'(0) = 0 \}$$

with $C_b^2([0, \infty))$ being the space of all bounded continuous functions on $[0, \infty)$ with bounded continuous derivatives up to and including order 2. The boundary condition $f'(0) = 0$ is forced upon us by the nature of the reflecting Brownian motion. See Section 4.2 of [19] for more information on the reflecting Brownian motion.

(II) The second example is the coalescing Brownian motion which can be described as follows. $(x_1(t), \ldots, x_m(t))$ is called a coalescing Brownian motion if the components move as independent Brownian motions until any pair, say $x_i(t)$ and $x_j(t)$, $(i < j)$, meet. After that, $x_j(t)$ assumes the values of $x_i(t)$ and the system continues to evolve in the same fashion. The infinitesimal generator of the transition semigroup of $(x_1(t), \ldots, x_m(t))$ is given by $(\mathfrak{C}^m, \mathcal{D}(\mathfrak{C}^m))$ with

$$\mathfrak{C}^m = \frac{1}{2} \sum_{1 \le i, j \le m} 1_{\{x_i = x_j\}} \frac{\partial^2}{\partial x_i \partial x_j}$$

and

$$\mathcal{D}(\mathfrak{C}^m) = \left\{ f \in C_b^2(\mathbb{R}^m) \colon \frac{\partial^2 f}{\partial x_i \partial x_j} = 0 \text{ if } x_i = x_j, \text{ for some } i \neq j \right\}.$$

Here again the domain $\mathcal{D}(\mathfrak{C}^m)$ is forced upon us by the nature of the coalescing Brownian motion. For more details on coalescing Brownian motion, one can see [24] and the references therein.

A similar problem needs to be handled for our measure-valued process. The following observation may shed some light on the present situation. For any function $f \in C_b^2(\mathbb{R}^k)$, if we choose $\phi_i \in \mathcal{D}(D)$, $i = 1, \ldots, k$ and

$$F(\mu) = f(\langle \phi_1, \mu \rangle, \ldots, \langle \phi_k, \mu \rangle),$$

then, we have

$$\frac{\delta F(\mu)}{\delta \mu(x)} = 0$$

for $x \in \partial D$ and

$$\frac{\delta^2 F(\mu)}{\delta \mu(x) \delta \mu(y)} = 0$$

for either $x \in \partial D$ or $y \in \partial D$. So we may choose $\mathscr{D}(D) = \bigcup_{K \subset D} C_K^\infty(D)$ as the space of the test functions for our generator. Recall that the vector space $\mathscr{D}(D) = \bigcup_{K \subset D} C_K^\infty(D)$ of infinitely differentiable functions with compact support in D is endowed with the inductive limit of the topologies on $C_K^\infty(D)$.

1.3. Basic ideas and organization of the paper. In the usual models (for example, (α, d, β)-superprocesses, see Chapter 4 of Dawson [6] and Perkins [27]), the motions of particles are independent and the motions are independent of branching, thus the particle systems have the following multiplicative property: If two branching Markov processes evolve independently with initial distribution m_1 and m_2 respectively, then their sum has the same distribution as the branching process with initial distribution $m_1 + m_2$. It is well-known that the log-Laplace functional (or evolution equation) technique can be applied to these models in order to construct the limiting measure-valued process. However, in our model and pregenerator, it is obvious that the motions of particles are not independent, this destroys the multiplicative property. Thus, just as in Wang [32] and Dawson et al. [11], the usual log-Laplace functional method is not applicable to our new model. Although Dynkin [13] and Le Gall [23] and other authors have already considered superprocesses on a bounded domain in $\mathbb{R}^d$, in their models particles' motions are independent and the log-Laplace functional method is applicable. There exists an essential difference between our model and their models. In order to construct the branching particle system in our model, by the Picard's iteration method we can show that under the assumption that the functions c and h satisfy the Lipschitz condition, the SDE (1.1) has a unique strong solution, which means that (1.1) has a strong solution and the pathwise uniqueness holds. Using the unique strong solutions of (1.1) with different initial positions and that a particle is killed once it exits the domain D, we can construct our branching particle system on D. After proving the tightness of the empirical measure-valued processes constructed from the branching particle system, the existence of the martingale problem for $\mathcal{L}$ on D will follow.

To prove the uniqueness of the martingale problem for $\mathcal{L}$ for the measure-valued interacting process on D, we use a duality method initiated by Dawson and Kurtz [7]. Let $\{P_t^n : t \geq 0\}$ be the transition semigroup of the underlying motion of n-particles given by (1.1), killed once one of the n particles exits D. Note that the infinitesimal generator of the n-particles $(z_1, \ldots, z_n)$ is given by

$$
\begin{aligned}
G_n f(x_1, \ldots, x_n) &:= \frac{1}{2} \sum_{i=1}^{n} \sum_{p,q=1}^{d} (a_{pq}(x_i) + \rho_{pq}(x_i, x_i)) \frac{\partial^2}{\partial x_{ip} \partial x_{iq}} f(x_1, \ldots, x_n) \\
&\quad + \frac{1}{2} \sum_{i \neq j,\, i,j=1}^{n} \sum_{p,q=1}^{d} \rho_{pq}(x_i, x_j) \left(\frac{\partial}{\partial x_{ip}} \right) \left(\frac{\partial}{\partial x_{jq}} \right) f(x_1, \ldots, x_n) \qquad (1.7) \\
&= \frac{1}{2} \sum_{i,j=1}^{n} \sum_{p,q=1}^{d} \Gamma_{pq}^{ij}(x_1, \ldots, x_n) \frac{\partial^2}{\partial x_{ip} \partial x_{jq}} f(x_1, \ldots, x_n),
\end{aligned}
$$

where $x_i = (x_{i1}, \ldots, x_{id}) \in \mathbb{R}^d$ and for $1 \le i \le n$,

$$\Gamma^{ij}_{pq}(x_1, \ldots, x_n) := \begin{cases} (a_{pq}(x_i) + \rho_{pq}(x_i, x_i)) & \text{if} \quad i = j, \\ \rho_{pq}(x_i, x_j) & \text{if} \quad i \neq j, \end{cases} \tag{1.8}$$

and

$$f \in \mathfrak{G}(G_n) := \{ f \in C_b^2((\mathbb{R}^d)^n) \text{: the support of } f \text{ is a compact subset of } D^n \} \subset \mathcal{D}(G_n),$$

where $D^n = D \times D \times \cdots \times D$, the n-fold product, and $\mathcal{D}(G_n)$ is the domain of the generator G_n.

The remainder of this paper is organized as follows. Section 2 is devoted to the construction of the branching particle system and the derivation of a discrete SPDE for the empirical measure-valued processes. In Section 3, the tightness of the corresponding empirical measure-valued processes and the uniqueness of the SDSMB will be discussed. Then, we prove the L^2-convergence of each term in the discrete SPDE and derive a SPDE for the SDSMB. Finally we use Dawson-Kurtz's duality method to show that the martingale problem for the generator corresponding to the SPDE is well-posed.

2. Branching Particle Systems

In order to construct the branching particle system, we need to introduce an index set to identify each particle in the branching tree structure. Let $\Re$ be the set of all multi-indices, i.e., strings of the form $\xi = n_1 \oplus n_2 \oplus \cdots \oplus n_k$, where the n_i's are non-negative integers. Let $|\xi|$ denote the length of ξ. We provide $\Re$ with the arboreal ordering: $m_1 \oplus m_2 \oplus \cdots \oplus m_p \prec n_1 \oplus n_2 \oplus \cdots \oplus n_q$ if and only if $p \le q$ and $m_1 = n_1, \ldots, m_p = n_p$. If $|\xi| = p$, then ξ has exactly $p - 1$ predecessors, which we shall denote respectively by $\xi - 1$, $\xi - 2, \ldots, \xi - |\xi| + 1$. For example, with $\xi = 6 \oplus 18 \oplus 7 \oplus 9$, we get $\xi - 1 = 6 \oplus 18 \oplus 7$, $\xi - 2 = 6 \oplus 18$ and $\xi - 3 = 6$. We also define an $\oplus$ operation on $\Re$ as follows: if $\eta \in \Re$ and $|\eta| = m$, for any given non-negative integer k, $\eta \oplus k \in \Re$ and $\eta \oplus k$ is an index for a particle in the $(m + 1)$-th generation. For example, when $\eta = 3 \oplus 8 \oplus 17 \oplus 2$ and $k = 1$, we have $\eta \oplus k = 3 \oplus 8 \oplus 17 \oplus 2 \oplus 1$.

Let $\{B_\xi = (B_{\xi 1}, \ldots, B_{\xi d})^T : \xi \in \Re\}$ be an independent family of standard $\mathbb{R}^d$-valued Brownian motions, where $B_{\xi k}$ is the k-th component of the d-dimensional Brownian motion B_ξ, and W a Brownian sheet on $\mathbb{R}^d$. Assume that W and $\{B_\xi : \xi \in \Re\}$ are defined on a common filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \ge 0}, \mathbb{P})$, and independent of each other. For every index $\xi \in \Re$ and initial data $z_\xi(0)$, by Picard's iteration method (see Lemma 3.1 of Dawson et al. [9]), one can easily show that there is a unique strong solution $z_\xi(t)$ to the equation

$$z_\xi^T(t) = z_\xi^T(0) + \int_0^t c(z_\xi(s))\, dB_\xi(s) + \int_0^t \int_{\mathbb{R}^d} h(y - z_\xi(s))\, W(dy, ds). \tag{2.1}$$

Since the strong solution of (2.1) only depends on the initial state $z_\xi(0)$, the Brownian motion $B_\xi := \{B_\xi(t): t \geq 0\}$ and the common W, we can write the strong solution of (2.1) as $z_\xi(t) = \Phi(z_\xi(0), B_\xi, t)$ for some measurable $\mathbb{R}^d$-valued map Φ (we omit W from the notation as it is selected and fixed once and for all). Let $\partial_p := \partial/\partial x_p$. For each $\phi \in \mathfrak{G}(G_1)$, we have by Itô's formula that for every $t > 0$,

$$
\begin{aligned}
&\phi(z_\xi(t)) - \phi(z_\xi(0)) \\
&= \sum_{p=1}^{d} \Bigg[\int_0^t (\partial_p \phi(z_\xi(s))) \sum_{i=1}^{d} c_{pi}(z_\xi(s))\, dB_{\xi i}(s) \\
&\qquad + \int_0^t \int_{\mathbb{R}}^{d} \partial_p \phi(z_\xi(s)) h_p(y - z_\xi(s))\, W(dy, ds) \Bigg] \\
&\quad + \frac{1}{2} \sum_{p,q=1}^{d} \int_0^t \left(\partial_p \partial_q \phi(z_\xi(s)) \sum_{i=1}^{d} c_{pi}(z_\xi(s)) c_{qi}(z_\xi(s)) \right) ds \\
&\quad + \frac{1}{2} \sum_{p,q=1}^{d} \int_0^t (\partial_p \partial_q \phi(z_\xi(s))) \int_{\mathbb{R}^d} h_p(y - z_\xi(s)) h_q(y - z_\xi(s))\, dy\, ds.
\end{aligned} \tag{2.2}
$$

We now consider the branching particle systems in which each particle's spatial motion is modeled by the SDE (2.1). For every positive integer $n \geq 1$, there is an initial system of $m_0^{(n)}$ particles. Each particle has mass $1/\theta^n$ and branches independently at rate $\gamma\theta^n$. Let $q_k^{(n)}$ denote the probability of having k offspring when a particle dies in D. The sequence $\{q_k^{(n)}\}$ is assumed to satisfy the following conditions:

$$q_k^{(n)} = 0 \quad \text{if} \quad k = 1 \quad \text{or} \quad k \geq n + 1,$$

and

$$\sum_{k=0}^{n} k q_k^{(n)} = 1 \quad \text{and} \quad \lim_{n\to\infty} \sup_{k\geq 0} |q_k^{(n)} - p_k| = 0,$$

where $\{p_k: k = 0, 1, 2, \dots\}$ is the limiting offspring distribution which is assumed to satisfy following conditions:

$$p_1 = 0, \quad \sum_{k=0}^{\infty} k p_k = 1 \quad \text{and} \quad m_2 := \sum_{k=0}^{\infty} k^2 p_k < \infty.$$

Let $m_c^{(n)} := \sum_{k=0}^{n} (k-1)^4 q_k^{(n)}$. The sequence $\{m_c^{(n)}: n \geq 1\}$ may be unbounded, but we assume that

$$\lim_{n\to\infty} \frac{m_c^{(n)}}{\theta^{2n}} = 0 \quad \text{for any} \quad \theta > 1.$$

We will see that the limiting offspring distribution is the offspring distribution of the SDSM on a bounded domain, the limiting measure-valued process that we will construct. We assume that $m_0^{(n)} \le \hbar\theta^n$, where $\hbar > 0$ and $\theta > 1$ are fixed constants. Define $m_2^{(n)} := \sum_{k=0}^{n} k^2 q_k^{(n)}$, $\sigma_n^2 := m_2^{(n)} - 1$ and $\sigma^2 := m_2 - 1$. Note that σ_n^2 and σ^2 are the variance of the n-th stage and the limiting offspring distribution, respectively. We have $\sigma_n^2 < \infty$ and $\lim_{n\to\infty} \sigma_n^2 = \sigma^2$.

For a fixed stage $n \ge 1$, let $\xi \in \Re$ and x be the death location of the particle ξ, $\{O_\xi^{(n)}: \xi \in \Re\}$ be a family of i.i.d. random variables with $\mathbb{P}(O_\xi^{(n)} = k) = q_k^{(n)}$ for $x \in D$ and $k = 0, 1, 2, \ldots$, otherwise $\mathbb{P}(O_\xi^{(n)} = 0) = 1$ for $x \notin D$ and $\{C_\xi^{(n)}: \xi \in \Re\}$ be a family of i.i.d. real-valued exponential random variables with parameter $\gamma\theta^n$, which will serve as lifetimes of the particles. We assume W, $\{B_\xi: \xi \in \Re\}$, $\{C_\xi^{(n)}: \xi \in \Re\}$ and $\{O_\xi^{(n)}: \xi \in \Re\}$ are all independent. In our model, once the particle ξ exits D, it is killed immediately and disappears from the system.

In the remainder of this section we are only concerned with stage n. To simplify our notation, we will use the convention of dropping the superscript (n) from the random variables. In later sections we will continue this convention for some random variables such as locations, birth times and death times. This will not cause any confusion, since the stage should be clear from the context.

If x, the death location of the particle $\xi - 1$, belongs to D, then the birth time $\beta(\xi)$ of the particle ξ is given by

$$\beta(\xi) := \begin{cases} \displaystyle\sum_{j=1}^{|\xi|-1} C_{\xi-j}, & \text{if } O_{\xi-j} \ge 2 \text{ for every } j = 1, \ldots, |\xi| - 1; \\ \infty, & \text{otherwise.} \end{cases}$$

The death time of the particle ξ is given by $\zeta(\xi) = \beta(\xi) + C_\xi$ and the indicator function of the lifespan of ξ is denoted by $l_\xi(t) := 1_{[\beta(\xi),\zeta(\xi))}(t)$.

Recall that ∂ denotes the cemetery point. Define $x_\xi(t) = \partial$ if either $t < \beta(\xi)$ or $t \ge \zeta(\xi)$. We make the convention that any function f defined on D is automatically extended to $D \cup \{\partial\}$ by setting $f(\partial) = 0$—this allows us to keep track of only those particles that are alive at any given time.

To avoid the trivial case, we assume that $\mu_0 \in M_F(D)$. Let $\mu_0^{(n)} := (1/\theta^n) \sum_{\xi=1}^{m_0^{(n)}} \delta_{x_\xi(0)}$ be constructed such that $\mu_0^{(n)} \Rightarrow \mu_0$ as $n \to \infty$. We are thus provided with a collection of initial starting points $\{x_\xi(0)\}$ for each $n \ge 1$.

For a given starting point $a \in D$, let $\tau_\xi(a) := \inf\{t: \Phi(a, B_\xi, t) \notin D\}$ be the first exit time of the diffusion process $\Phi(a, B_\xi, t)$ from the domain D, where Φ is defined in the paragraph below (2.1). Let $\mathcal{N}_1^n := \{1, 2, \ldots, m_0^{(n)}\}$ be the set of indices for the

first generation of particles. For any $\xi \in \mathcal{N}_1^n \cap \Re$, if $x_\xi(0) \in D$, define

$$x_\xi(t) := \begin{cases} \Phi(x_\xi(0), B_\xi, t), & t \in [0, C_\xi \wedge \tau_\xi(x_\xi(0))), \\ \partial, & t \geq C_\xi \wedge \tau_\xi(x_\xi(0)), \end{cases} \tag{2.3}$$

and

$$x_\xi(t) \equiv \partial \quad \text{for any} \quad \xi \in (\mathbb{N} \setminus \mathcal{N}_1^n) \cap \Re \quad \text{and} \quad t \geq 0.$$

If $\xi_0 \in \mathcal{N}_1^n \cap \Re$ and $x_{\xi_0}(\zeta(\xi_0)-) \in \partial D$, then $x_\xi(t) \equiv \partial$ for any $\xi \succ \xi_0$ and any $t \geq \zeta(\xi_0)$. Otherwise, if $x_{\xi_0}(\zeta(\xi_0)-) \in D$ and $O_{\xi_0}(\omega) = k \geq 2$, define for every $\xi \in \{\xi_0 \oplus i : i = 1, 2, \ldots, k\}$,

$$x_\xi(t) := \begin{cases} \Phi(x_{\xi_0}(\zeta(\xi_0)-), B_\xi, t), & t \in [\beta(\xi), \zeta(\xi) \wedge \tau_\xi(x_{\xi_0}(\zeta(\xi_0)-))), \\ \partial, & t \geq \zeta(\xi) \wedge \tau_\xi(x_{\xi_0}(\zeta(\xi_0)-)). \end{cases} \tag{2.4}$$

If $O_{\xi_0}(\omega) = 0$, define $x_\xi(t) \equiv \partial$ for $0 \leq t < \infty$ and $\xi \in \{\xi_0 \oplus i : i \geq 1\}$.

More generally for any integer $m \geq 1$, let $\mathcal{N}_m^n \subset \Re$ be the set of all indices for the particles in the m-th generation. If $\xi_0 \in \mathcal{N}_m^n$ and if $x_{\xi_0}(\zeta(\xi_0)-) \in \partial D$, then $x_\xi(t) \equiv \partial$ for any $\xi \succ \xi_0$ and any $t \geq \zeta(\xi_0)$. Otherwise, if $x_{\xi_0}(\zeta(\xi_0)-) \in D$ and $O_{\xi_0}(\omega) = k \geq 2$, define for $\xi \in \{\xi_0 \oplus i : i = 1, 2, \ldots, k\}$

$$x_\xi(t) := \begin{cases} \Phi(x_{\xi_0}(\zeta(\xi_0)-), B_\xi, t), & t \in [\beta(\xi), \zeta(\xi) \wedge \tau_\xi(x_{\xi_0}(\zeta(\xi_0)-))), \\ \partial, & t \geq \zeta(\xi) \wedge \tau_\xi(x_{\xi_0}(\zeta(\xi_0)-)). \end{cases} \tag{2.5}$$

If $O_{\xi_0}(\omega) = 0$, define

$$x_\xi(t) \equiv \partial \quad \text{for} \quad 0 \leq t < \infty \quad \text{and for} \quad \xi \in \{\xi_0 \oplus i : i \geq 1\}.$$

Continuing in this way, we obtain a branching tree of particles for any given ω with random initial state taking values in $\{x_1(0), x_2(0), \ldots, x_{m_0^{(n)}}(0)\}$. This gives us our branching particle systems in $D \cup \partial$, where particles undergo a finite-variance branching at independent exponential times and have interacting spatial motions powered by diffusions and a common white noise.

3. Tightness, uniqueness, and SPDE for SDSMB

Recall that $\{x_\xi\}$ is the branching particle system constructed in the last section. Define its associated empirical process by

$$\mu_t^{(n)}(A) := \frac{1}{\theta^n} \sum_{\xi \in \Re} \delta_{x_\xi(t)}(A) \quad \text{for} \quad A \in \mathcal{B}(D), \tag{3.1}$$

where $\mathcal{B}(D)$ denotes the family of Borel subsets of D. In the following, we will show that $\{\mu_t^{(n)} : t \geq 0\}$ converges weakly as $n \to \infty$ and its weak limit is the SDSM on D.

For any $t > 0$ and $A \in \mathcal{B}(D)$, define

$$M^{(n)}(A \times (0, t]) := \sum_{\xi \in \Re} \frac{\left[O_\xi^{(n)} - 1\right]}{\theta^n} 1_{\{x_\xi(\zeta(\xi)-) \in A, \zeta(\xi) \le t\}}, \tag{3.2}$$

which describes the space-time related branching in the set A up to time t.

Since in the present model the branching particle system and the related superprocess are restricted to a bounded domain in $\mathbb{R}^d$, the framework based on the whole space $\mathbb{R}^d$ (for example, Mitoma [26]) is no longer suitable for our new situation. In order to discuss the weak convergence of our empirical measure-valued processes, we introduce some new notation.

Let Q be a nonempty open subset of $\mathbb{R}^d$ and let $C^\infty(Q)$ be the set of real-valued functions on Q with continuous derivatives of all orders. For any compact subset K of Q, let $C_K^\infty(Q)$ be the set of functions in $C^\infty(Q)$ with support in K. Equipped with the topology given by the seminorms

$$p_i(\phi) := \sup\{|\partial^\alpha \phi(x)| : x \in K,\ |\alpha| \le i\}, \quad i \ge 0,$$

$C_K^\infty(Q)$ is a nuclear Fréchet space (see Schaefer [28] and Al-Gwaiz [1]). Let $\mathscr{D}(Q) = \bigcup_{K \subset Q} C_K^\infty(Q)$ be the vector space of infinitely differentiable functions with compact support in Q, endowed with the inductive limit of the topologies on $C_K^\infty(Q)$. Then, $\mathscr{D}(Q)$ is usually called the Schwartz space of test functions on Q. Its topological dual, $\mathscr{D}'(Q)$, is the vector space of all distributions or continuous linear functionals on $\mathscr{D}(Q)$. $\mathscr{D}'(Q)$ is called the Schwartz space of distributions on Q. (For more details, the reader is referred to Schwartz [29], Barros-Neto [3] or Al-Gwaiz [1].)

Let $\mathcal{S}(\mathbb{R}^d)$ be the Schwartz space of rapidly decreasing test functions and $\mathcal{S}'(\mathbb{R}^d)$ be the dual space of $\mathcal{S}(\mathbb{R}^d)$, the space of Schwartz tempered distributions. Mitoma's theorem ([26]) provides a convenient tool for studying the weak convergence of measure-valued processes. It is applicable to càdlàg processes whose state space is the dual of a nuclear Fréchet space. A typical case is the $\mathcal{S}'(\mathbb{R}^d)$-valued processes. However, in our case, $\mathscr{D}(D)$ is not a Fréchet space (see Al-Gwaiz [1]). Therefore, Mitoma's theorem is not applicable to $\mathscr{D}'(D)$-valued processes. Fortunately Fouque [16] has proved a nice generalization of Mitoma's theorem to the case which is applicable to càdlàg processes whose state space is the dual of an inductive limit topological space of a sequence of nuclear Fréchet spaces. So this works for $\mathscr{D}'(D)$-valued càdlàg processes. Since every Radon measure on D defines a distribution on D, we have $M_F(D) \subset \mathscr{D}'(D)$.

Note that for a given bounded domain D in $\mathbb{R}^d$, (2.2) implies that for every $\phi \in \mathscr{D}(D)$,

$$\langle \phi, \mu_t^{(n)} \rangle - \langle \phi, \mu_0^{(n)} \rangle = \frac{1}{\sqrt{\theta^n}} U_t^{(n)}(\phi) + X_t^{(n)}(\phi) + Y_t^{(n)}(\phi) + M_t^{(n)}(\phi), \tag{3.3}$$

where, recall that $l_\xi(s) = 1_{[\beta(\xi),\zeta(\xi))}(s)$,

$$U_t^{(n)}(\phi) := \frac{1}{\sqrt{\theta^n}} \sum_{\xi\in\mathfrak{N}} \sum_{p,i=1}^{d} \int_0^t l_\xi(s)\, \partial_p\phi(x_\xi(s)) c_{pi}(x_\xi(s))\, dB_{\xi i}(s),$$

$$X_t^{(n)}(\phi) := \sum_{p=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle h_p(y-\cdot)\, \partial_p\phi(\cdot),\, \mu_s^{(n)}\rangle\, W(dy, ds),$$

$$Y_t^{(n)}(\phi) := \sum_{p,q=1}^{d} \int_0^t \left\langle \frac{1}{2}\, \partial_p\partial_q\phi(\cdot) \left[\sum_{i=1}^{d} c_{pi}(\cdot)c_{qi}(\cdot) + \int_{\mathbb{R}}^{d} h_p(y-\cdot)h_q(y-\cdot)\, dy \right], \mu_s^{(n)} \right\rangle ds,$$

$$M_t^{(n)}(\phi) := \int_0^t \int_{\mathbb{R}}^{d} \phi(x)\, M^{(n)}(dx, ds) = \sum_{\xi\in\mathfrak{N}} \frac{[O_\xi^{(n)} - 1]}{\theta^n} \phi(x_\xi(\zeta(\xi)-)) 1_{\{\zeta(\xi)\le t\}}.$$

The four terms in (3.3) represent the respective contributions to the overall motion of the finite particle system $\langle\phi, \mu_t^{(n)}\rangle$ in D by the individual Brownian motions $(U_t^{(n)}(\phi))$, the random medium $(X_t^{(n)}(\phi))$, the mean effect of interactive and diffusive dynamics $(Y_t^{(n)}(\phi))$, the branching mechanism $(M_t^{(n)}(\phi))$. Using a result of Dynkin ([12] p. 325, Theorem 10.25), we immediately get the following theorem.

Theorem 3.1. *For any $n \in \mathbb{N}$, $\mu_t^{(n)}$ defined by* (3.1) *is a right continuous strong Markov process which is the unique strong solution of* (3.3) *in the sense that it is a unique solution of* (3.3) *for a given probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and given W, $\{B_\xi\}$, $\{C_\xi^{(n)}\}$, $\{O_\xi^{(n)}\}$ defined on* $(\Omega, \mathcal{F}, \mathbb{P})$. *Furthermore, $\{\mu_t^{(n)}: t \ge 0\}$ are all defined on the common probability space* $(\Omega, \mathcal{F}, \mathbb{P})$.

For each $t \ge 0$, let $\mathcal{F}_t^{(n)}$ denote the σ-algebra generated by the collection of processes

$$\{\mu_t^{(n)}(\phi),\, U_t^{(n)}(\phi),\, X_t^{(n)}(\phi),\, Y_t^{(n)}(\phi),\, M_t^{(n)}(\phi): \phi \in \mathscr{D}(D),\ t \ge 0\}.$$

Note that according to our assumption, the fourth moment of $O_\xi^{(n)}$, $m_c^{(n)} := \mathbb{E}\{(O_\xi^{(n)} - 1)^4\}$, is finite and $\lim_{n\to\infty} m_c^{(n)}/\theta^{2n} = 0$ for any $\theta > 1$.

Lemma 3.2. *With the notation above, we have the following.*
(i) *For every $\phi \in \mathscr{D}(D)$, $M^{(n)}(\phi) := \{M_t^{(n)}(\phi): t \ge 0\}$ is a purely discontinuous square integrable martingale with*

$$\langle M^{(n)}(\phi)\rangle_t = \gamma\sigma_n^2 \int_0^t \langle\phi^2, \mu_u^{(n)}\rangle\, du \quad \textit{for every} \quad t \ge 0.$$

(ii) *For any* $t \geq 0$ *and* $n \geq 1$, *we have*

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\langle 1, \mu_s^{(n)}\rangle^2\right] \leq 2\langle 1, \mu_0^{(n)}\rangle^2 + 8\gamma\sigma_n^2 t\langle 1, \mu_0^{(n)}\rangle.$$

Furthermore, there is a constant $\kappa > 0$ *such that for every* $t \geq 0$,

$$\mathbb{E}\left[\sup_{0\leq s\leq t}\langle 1, \mu_s^{(n)}\rangle^4\right]$$
$$\leq \kappa\left(\gamma^3\sigma_n^6 t^3\langle 1, \mu_0^{(n)}\rangle + \gamma^2\sigma_n^4 t^2\langle 1, \mu_0^{(n)}\rangle^2 + \frac{\gamma m_c^{(n)}}{\theta^{2n}} t\langle 1, \mu_0^{(n)}\rangle + \langle 1, \mu_0^{(n)}\rangle^4\right).$$

(iii) $\{\mu_t^{(n)}: t \geq 0\}$ *defined by* (3.1) *is tight as a family of processes with sample paths in* $D([0, \infty), \mathscr{D}'(D))$.

Proof. (i) Recall that, for each $n \geq 1$, $\{C_\xi^{(n)}: \xi \in \Re\}$ is a family of i.i.d. exponential random variables with parameter $\gamma\theta^n$, $\{O_\xi^{(n)} - 1: \xi \in \Re\}$ is a family of i.i.d. random variables with zero mean, and these two families are independent. Thus $\mathbb{E}\{M_t^{(n)}(\phi)\} = 0$ for every $t > 0$ and $\phi \in \mathscr{D}(D)$. Since this is valid for any initial distribution $\mu_0^{(n)}$, by the Markov property of $\{\mu_t^{(n)}: t \geq 0\}$, we have for every $t, s > 0$,

$$\mathbb{E}\big[M_{t+s}^{(n)}(\phi) - M_t^{(n)}(\phi) \,\big|\, \mathcal{F}_t^{(n)}\big] = \mathbb{E}_{\mu_t^{(n)}}\big[M_s^{(n)}(\phi) - M_0^{(n)}(\phi)\big] = 0.$$

This shows that $M^{(n)}(\phi)$ is a martingale. Clearly it is purely discontinuous.

$$\begin{aligned}
&\mathbb{E}[\phi^2(x_\xi(\zeta(\xi)-)); \ \zeta(\xi) \leq t] \\
&= \mathbb{E}\big[1_{[0,t]}\big(\beta(\xi) + C_\xi^{(n)}\big)\phi^2\big(x_\xi\big(\big(\beta(\xi) + C_\xi^{(n)}\big)-\big)\big)\big] \\
&= \mathbb{E}\left[\int_0^\infty 1_{[0,t]}(\beta(\xi) + u)\phi^2(x_\xi((\beta(\xi) + u)-))\gamma\theta^n e^{-\gamma\theta^n u}\, du\right] \\
&= \mathbb{E}\left[\int_0^\infty 1_{[0,t]}(\beta(\xi) + u)\phi^2(x_\xi((\beta(\xi) + u)-))\gamma\theta^n 1_{\{C_\xi^{(n)} > u\}}\, du\right] \\
&\qquad\qquad \text{(by independence)} \\
&= \mathbb{E}\left[\int_0^\infty 1_{[0,t]}(\beta(\xi) + u)\phi^2(x_\xi((\beta(\xi) + u)))\gamma\theta^n\, du\right] \\
&\qquad\qquad \text{(by the definition of } x_\xi\text{)} \\
&= \mathbb{E}\left[\int_{\beta(\xi)}^\infty 1_{[0,t]}(v)\phi^2(x_\xi(v))\gamma\theta^n\, dv\right] \\
&= \gamma\theta^n\mathbb{E}\left[\int_0^t 1_{[\beta(\xi),\zeta(\xi))}(v)\phi^2(x_\xi(v))\, dv\right].
\end{aligned}$$

As $\{C_\xi^{(n)}\colon \xi \in \Re\}$ and $\{O_\xi^{(n)}\colon \xi \in \Re\}$ are all independent and $\mathbb{E}O_\xi^{(n)} = 1$, we conclude that

$$\begin{aligned}\mathbb{E}[M_t^{(n)}(\phi)^2] &= \sum_{\xi\in\Re} \theta^{-2n}\mathbb{E}\{(O_\xi^{(n)}-1)^2\}\mathbb{E}[\phi^2(x_\xi(\zeta(\xi)-))1_{(\zeta(\xi)\le t)}] \\ &= \theta^{-2n}\sigma_n^2 \sum_{\xi\in\Re} \gamma\theta^n \mathbb{E}\left[\int_0^t 1_{[\beta(\xi),\zeta(\xi))}(v)\phi^2(x_\xi(v))\,dv\right] \\ &= \gamma\sigma_n^2\mathbb{E}\left[\int_0^t \langle\phi^2, \mu_v^{(n)}\rangle\,dv\right].\end{aligned} \tag{3.4}$$

Note that the identity (3.4) holds for any initial distribution $\mu_0^{(n)}$. By the Markov property of $\{\mu_t^{(n)}\colon t \ge 0\}$ again, we have for every $t, s > 0$,

$$\begin{aligned}&\mathbb{E}\left[M_{t+s}^{(n)}(\phi)^2 - M_t^{(n)}(\phi)^2 - \gamma\sigma_n^2\int_t^{t+s}\langle\phi^2, \mu_v^{(n)}\rangle\,dv|\mathcal{F}_t^{(n)}\right] \\ &= \mathbb{E}_{\mu_t^{(n)}}\left[M_s^{(n)}(\phi)^2 - \gamma\sigma_n^2\int_0^s\langle\phi^2, \mu_v^{(n)}\rangle\,dv\right] = 0.\end{aligned}$$

This shows that $M_t^{(n)}(\phi)^2 - \gamma\sigma_n^2\int_0^t\langle\phi^2, \mu_v^{(n)}\rangle\,dv$ is a martingale. Hence we conclude that $M^{(n)}(\phi)$ is a purely discontinuous square integrable martingale with

$$\langle M^{(n)}(\phi)\rangle_t = \gamma\sigma_n^2\int_0^t\langle\phi^2, \mu_u^{(n)}\rangle\,du \quad \text{for every} \quad t \ge 0.$$

(ii) The proof of this part is related the total number of particles of the system. Since the total number of particles of the system with boundary is bounded by the total number of particles of the system without boundary, in the following we only need to prove the result for the system without boundary. As in Section 2, we can reconstruct the branching particle systems without boundary as follows. For each $\xi \in \Re$, let

$$\hat{x}_\xi(t) := \begin{cases}\Phi(\hat{x}_{\xi-1}(\zeta(\xi-1)-), B_\xi, t), & t \in [\beta(\xi), \zeta(\xi)),\\ \partial, & t \ge \zeta(\xi).\end{cases} \tag{3.5}$$

Define

$$\hat{\mu}_t^{(n)}(A) := \frac{1}{\theta^n}\sum_{\xi\in\Re}\delta_{\hat{x}_\xi(t)}(A) \quad \text{for} \quad A \in \mathcal{B}(\mathbb{R}^d). \tag{3.6}$$

For any $t > 0$ and $A \in \mathcal{B}(\mathbb{R}^d)$, define

$$\hat{M}^{(n)}(A\times(0,t]) := \sum_{\xi\in\Re}\frac{[O_\xi^{(n)}-1]}{\theta^n}1_{\{\hat{x}_\xi(\zeta(\xi)-)\in A, \zeta(\xi)\le t\}}, \tag{3.7}$$

Therefore, the particles $\{\hat{x}_\xi\}$ live in $\mathbb{R}^d$. Since $\langle 1, \hat{\mu}_t^{(n)} - \hat{\mu}_0^{(n)}\rangle = \hat{M}_t^{(n)}(1)$ is a zero-mean martingale (for the systems without boundary), by Doob's maximal inequality, we have

$$\begin{aligned}\mathbb{E}\Big[\sup_{0\le s\le t}\langle 1, \hat{\mu}_s^{(n)}\rangle^2\Big] &\le 2\mathbb{E}\Big[\sup_{0\le s\le t}\hat{M}_s^{(n)}(1)^2\Big] + 2\langle 1, \hat{\mu}_0^{(n)}\rangle^2\\ &\le 8\mathbb{E}[\hat{M}_t^{(n)}(1)^2] + 2\langle 1, \hat{\mu}_0^{(n)}\rangle^2\\ &\le 8\gamma\sigma_n^2 t\langle 1, \hat{\mu}_0^{(n)}\rangle + 2\langle 1, \hat{\mu}_0^{(n)}\rangle^2.\end{aligned}$$

Note that $\hat{M}_t^{(n)}(1) = \sum_{\xi\in\Re}\big(O_\xi^{(n)} - 1\big)/\theta^n 1_{\{\zeta(\xi)\le t\}}$ is a purely discontinuous martingale and $\big\{O_\xi^{(n)} - 1\colon \xi\in\Re\big\}$ are i.i.d. random variables with zero mean and are independent of $\big\{C_\xi^{(n)}\colon \xi\in\Re\big\}$. Thus

$$\begin{aligned}\mathbb{E}[(\hat{M}_t^{(n)}(1))^4] &= \mathbb{E}\Bigg[\sum_{\xi,\eta\in\Re,\,\xi\ne\eta}\Bigg(\frac{\big(O_\xi^{(n)}-1\big)^2}{\theta^{2n}}1_{\{\zeta(\xi)\le t\}}\frac{(O_\eta^{(n)}-1)^2}{\theta^{2n}}1_{\{\zeta(\eta)\le t\}}\Bigg)\Bigg]\\ &\quad + \mathbb{E}\Bigg[\sum_{\xi\in\Re}\frac{\big(O_\xi^{(n)}-1\big)^4}{\theta^{4n}}1_{\{\zeta(\xi)\le t\}}\Bigg]\\ &= \frac{\sigma_n^4}{\theta^{4n}}\mathbb{E}\Bigg[\sum_{\xi,\eta\in\Re,\,\xi\ne\eta}1_{\{\zeta(\xi)\le t\}}1_{\{\zeta(\eta)\le t\}}\Bigg] + \frac{m_c^{(n)}}{\theta^{4n}}\mathbb{E}\Bigg[\sum_{\xi\in\Re}1_{\{\zeta(\xi)\le t\}}\Bigg]\\ &= \frac{\sigma_n^4}{\theta^{4n}}\sum_{\xi,\eta\in\Re,\,\xi\ne\eta}\mathbb{E}[1_{\{\zeta(\xi)\le t\}}1_{\{\zeta(\eta)\le t\}}] + \frac{\gamma m_c^{(n)}}{\theta^{2n}}\mathbb{E}\Bigg[\int_0^t\langle 1, \hat{\mu}_v^{(n)}\rangle\,dv\Bigg].\end{aligned}$$

For $\xi, \eta\in\Re$ with $\xi\ne\eta$, $C_\xi^{(n)}$ and $C_\eta^{(n)}$ are independent and so

$$\begin{aligned}&\mathbb{E}[1_{\{\zeta(\xi)\le t\}}1_{\{\zeta(\eta)\le t\}}]\\ &= \mathbb{E}\big[1_{[0,t]}\big(\beta(\xi)+C_\xi^{(n)}\big)1_{[0,t]}\big(\beta(\eta)+C_\eta^{(n)}\big)\big]\\ &= \gamma^2\theta^{2n}\mathbb{E}\Bigg[\int_0^\infty\int_0^\infty 1_{[0,t]}(\beta(\xi)+u)\\ &\qquad\qquad\cdot 1_{[0,t]}(\beta(\eta)+v)e^{-\gamma\theta^n u}e^{-\gamma\theta^n v}\,du\,dv\Bigg]\\ &\le \gamma^2\theta^{2n}\mathbb{E}\Bigg[\int_0^\infty\int_0^\infty 1_{[0,t]}(\beta(\xi)+u)\\ &\qquad\qquad\cdot 1_{[0,t]}(\beta(\eta)+v)1_{\{x_\xi(\beta(\xi)+u)\ne\partial\}}1_{\{x_\eta(\beta(\eta)+u)\ne\partial\}}\,du\,dv\Bigg]\\ &= \gamma^2\theta^{2n}\mathbb{E}\Bigg[\Bigg(\int_{\beta(\xi)}^\infty 1_{[0,t]}(r)1_{\{x_\xi(r)\ne\partial\}}\,dr\Bigg)\Bigg(\int_{\beta(\eta)}^\infty 1_{[0,t]}(s)1_{\{x_\eta(s)\ne\partial\}}\,ds\Bigg)\Bigg]\\ &= \gamma^2\theta^{2n}\mathbb{E}\Bigg[\Bigg(\int_0^t 1_{[\beta(\xi),\zeta(\xi))}(r)\,dr\Bigg)\Bigg(\int_0^t 1_{[\beta(\eta),\zeta(\eta))}(s)\,ds\Bigg)\Bigg].\end{aligned}$$

Therefore

$$\begin{aligned}
&\sum_{\xi,\eta\in\Re,\,\xi\neq\eta}\mathbb{E}[1_{\{\zeta(\xi)\le t\}}\,1_{\{\zeta(\eta)\le t\}}]\\
&\le\sum_{\xi,\eta\in\Re}\gamma^2\theta^{2n}\mathbb{E}\left[\left(\int_0^t 1_{[\beta(\xi),\zeta(\xi))}(r)\,dr\right)\left(\int_0^t 1_{[\beta(\eta),\zeta(\eta))}(s)\,ds\right)\right]\\
&=\gamma^2\theta^{2n}\mathbb{E}\left[\left(\sum_{\xi\in\Re}\int_0^t 1_{[\beta(\xi),\zeta(\xi))}(r)\,dr\right)^2\right]\\
&=\gamma^2\theta^{4n}\mathbb{E}\left[\left(\int_0^t\langle 1,\,\hat\mu_r^{(n)}\rangle\,dr\right)^2\right].
\end{aligned}$$

It follows then

$$\begin{aligned}
\mathbb{E}[(\hat M_t^{(n)}(1))^4] &\le \frac{\sigma_n^4}{\theta^{4n}}\gamma^2\theta^{4n}\mathbb{E}\left[\left(\int_0^t\langle 1,\,\hat\mu_r^{(n)}\rangle\,dr\right)^2\right]+\frac{\gamma m_c^{(n)}}{\theta^{2n}}\mathbb{E}\left[\int_0^t\langle 1,\,\hat\mu_v^{(n)}\rangle\,dv\right]\\
&\le \gamma^2\sigma_n^4t^2\mathbb{E}\left[\sup_{r\in[0,t]}\langle 1,\,\hat\mu_r^{(n)}\rangle^2\right]+\frac{\gamma m_c^{(n)}}{\theta^{2n}}\langle 1,\,\hat\mu_0^{(n)}\rangle t\\
&\le \gamma^2\sigma_n^4t^2(8\gamma\sigma_n^2t\langle 1,\,\hat\mu_0^{(n)}\rangle+2\langle 1,\,\hat\mu_0^{(n)}\rangle^2)+\frac{\gamma m_c^{(n)}}{\theta^{2n}}\langle 1,\,\hat\mu_0^{(n)}\rangle t\\
&= 8\gamma^3\sigma_n^6t^3\langle 1,\,\hat\mu_0^{(n)}\rangle+2\gamma^2\sigma_n^4t^2\langle 1,\,\hat\mu_0^{(n)}\rangle^2+\frac{\gamma m_c^{(n)}}{\theta^{2n}}t\langle 1,\,\hat\mu_0^{(n)}\rangle.
\end{aligned}\tag{3.8}$$

By Doob's maximal inequality,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le s\le t}\langle 1,\,\hat\mu_s^{(n)}\rangle^4\right]\\
&\le\mathbb{E}\left[\sup_{0\le s\le t}(\langle 1,\,\hat\mu_s^{(n)}-\hat\mu_0^{(n)}\rangle+\langle 1,\,\hat\mu_0^{(n)}\rangle)^4\right]\\
&\le 8\mathbb{E}\left[\sup_{0\le s\le t}|\hat M_s^{(n)}(1)|^4\right]+8\langle 1,\,\hat\mu_0^{(n)}\rangle^4\\
&\le 8\left(\frac{4}{3}\right)^4\mathbb{E}[(\hat M_t^{(n)}(1))^4]+8\langle 1,\,\hat\mu_0^{(n)}\rangle^4\\
&\le\kappa\left(\gamma^3\sigma_n^6t^3\langle 1,\,\hat\mu_0^{(n)}\rangle+\gamma^2\sigma_n^4t^2\langle 1,\,\hat\mu_0^{(n)}\rangle^2+\frac{\gamma m_c^{(n)}}{\theta^{2n}}t\langle 1,\,\hat\mu_0^{(n)}\rangle+\langle 1,\,\hat\mu_0^{(n)}\rangle^4\right).
\end{aligned}$$

We know that $\hat\mu_0^{(n)}=\mu_0^{(n)}\in M_F(D)$. Therefore, the conclusion follows.

(iii) By Fouque's theorem (Fouque [16]), Theorem 4.5.4 in Dawson [5], and part (ii) above, which implies non-explosion in finite time, we only need to prove that, if we are given $\varepsilon>0$, $T>0$, $\phi\in\mathscr{D}(D)$, and a sequence of stopping times τ_n bounded by T, then $\forall\eta>0$, $\exists\delta$, n_0 such that $\sup_{n\ge n_0}\sup_{t\in[0,\delta]}\mathbb{P}\{|\mu_{\tau_n+t}^{(n)}(\phi)-\mu_{\tau_n}^{(n)}(\phi)|>\varepsilon\}\le\eta$.

We have by (3.3),

$$\begin{aligned}
&\mathbb{P}(|\mu^{(n)}_{\tau_n+t}(\phi) - \mu^{(n)}_{\tau_n}(\phi)| > \varepsilon) \\
&\le \frac{1}{\varepsilon^2}\mathbb{E}[(\mu^{(n)}_{\tau_n+t}(\phi) - \mu^{(n)}_{\tau_n}(\phi))^2] \\
&\le \frac{4}{\varepsilon^2}\mathbb{E}\bigg[\frac{1}{\theta^n}(U^{(n)}_{\tau_n+t}(\phi) - U^{(n)}_{\tau_n}(\phi))^2 + (X^{(n)}_{\tau_n+t}(\phi) - X^{(n)}_{\tau_n}(\phi))^2 \\
&\qquad\qquad + (Y^{(n)}_{\tau_{(n)}+t}(\phi) - Y^{(n)}_{\tau_n}(\phi))^2 + (M^{(n)}_{\tau_n+t}(\phi) - M^{(n)}_{\tau_n}(\phi))^2\bigg].
\end{aligned}$$

Note that by the independence of $\{B_\xi : \xi \in \Re\}$,

$$\begin{aligned}
&\mathbb{E}[(U^{(n)}_{\tau_n+t}(\phi) - U^{(n)}_{\tau_n}(\phi))^2] \\
&= \frac{1}{\theta^n}\sum_{\xi\in\Re}\sum_{p,i=1}^{d}\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+t} l_\xi(s)[\partial_p\phi(x_\xi(s))c_{pi}(x_\xi(s))]^2\,ds\bigg] \\
&= \sum_{p,i=1}^{d}\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+t}\langle(\partial_p\phi c_{pi})^2,\,\mu^{(n)}_s\rangle\,ds\bigg] \\
&\le t\sum_{p,i=1}^{d}\mathbb{E}\bigg[\sup_{s\le T+t}\langle(\partial_p\phi c_{pi})^2,\,\mu^{(n)}_s\rangle\bigg] \\
&\le t\sum_{p,i=1}^{d}\|\partial_p\phi c_{pi}\|^2_\infty\mathbb{E}\bigg[\sup_{s\le T+t}\langle 1,\,\mu^{(n)}_s\rangle\bigg],
\end{aligned}$$

while

$$\begin{aligned}
&\mathbb{E}[(X^{(n)}_{\tau_n+t}(\phi) - X^{(n)}_{\tau_n}(\phi))^2] \\
&= \sum_{p,q=1}^{d}\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+t}\int_{\mathbb{R}^d}\langle h_p(y-\,\cdot\,)\partial_p\phi(\,\cdot\,),\,\mu^{(n)}_s\rangle\langle h_q(y-\,\cdot\,)\partial_q\phi(\,\cdot\,),\,\mu^{(n)}_s\rangle\,dy\,ds\bigg] \\
&= \sum_{p,q=1}^{d}\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+t}\bigg[\int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\rho_{pq}(w,z)\,\partial_p\phi(w)\,\partial_q\phi(z)\,\mu^{(n)}_s(dw)\,\mu^{(n)}_s(dz)\bigg]\,ds\bigg] \\
&\le \sum_{p,q=1}^{d}\{\|\rho_{pq}\|_\infty\,\|\partial_p\phi\|_\infty\,\|\partial_q\phi\|_\infty\}\mathbb{E}\bigg[\int_{\tau_n}^{\tau_n+t}\langle 1,\,\mu^{(n)}_s\rangle^2\,ds\bigg] \\
&\le t\sum_{p,q=1}^{d}\{\|\rho_{pq}\|_\infty\,\|\partial_p\phi\|_\infty\,\|\partial_q\phi\|_\infty\}\mathbb{E}\bigg[\sup_{s\le T+t}\langle 1,\,\mu^{(n)}_s\rangle^2\bigg],
\end{aligned}$$

and

$$\mathbb{E}[(Y^{(n)}_{\tau_n+t}(\phi) - Y^{(n)}_{\tau_n}(\phi))^2] \le \left[\sum_{p,q=1}^{d}\left(\frac{1}{2}\|(a_{pq}+\rho_{pq})\,\partial_p\partial_q\phi\|_\infty\right)\right]^2 \mathbb{E}\left[\int_{\tau_n}^{\tau_n+t}\langle 1,\,\mu^{(n)}_s\rangle^2\,ds\right]$$
$$\le t\left[\sum_{p,q=1}^{d}\left(\frac{1}{2}\|(a_{pq}+\rho_{pq})\,\partial_p\partial_q\phi\|_\infty\right)\right]^2 \mathbb{E}\left[\sup_{s\le T+t}\langle 1,\,\mu^{(n)}_s\rangle^2\right].$$

Finally we have by part (i) of this lemma that

$$\mathbb{E}[(M^{(n)}_{\tau_n+t}(\phi) - M^{(n)}_{\tau_n}(\phi))^2] = \gamma\sigma_n^2\mathbb{E}\left[\int_{\tau_n}^{\tau_n+t}\langle\phi^2,\,\mu^{(n)}_s\rangle\,ds\right]$$
$$\le \gamma\sigma_n^2\|\phi\|_\infty^2 t\mathbb{E}\left[\sup_{s\le T+t}\langle 1,\,\mu^{(n)}_s\rangle\right].$$

Therefore by part (ii) of this lemma and Lemma 3.4 of Wang [32], we conclude that for every $\varepsilon > 0$, there is a constant $c > 0$ such that

$$\sup_{n\ge 1}\sup_{t\in[0,\delta]}\mathbb{P}(|\mu^{(n)}_{\tau_n+t}(\phi) - \mu^{(n)}_{\tau_n}(\phi)| > \varepsilon) \le c\delta \quad \text{for every} \quad \delta > 0,$$

which proves (iii). This completes the proof of the lemma. □

Theorem 3.3. *With the notation above, we have following conclusions*:
(i) $(\mu^{(n)}, U^{(n)}, Y^{(n)}, M^{(n)})$ *is tight on* $D([0,\infty), (\mathscr{D}'(D))^4)$.
(ii) (A Skorohod representation): *Suppose that the joint distribution of*

$$(\mu^{(n_m)}, U^{(n_m)}, Y^{(n_m)}, M^{(n_m)}, W)$$

converges weakly to the joint distribution of

$$(\mu^{(0)}, U^{(0)}, Y^{(0)}, M^{(0)}, W).$$

Then, there exist a probability space $(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *and* $D([0,\infty), \mathscr{D}'(D))$*-valued sequences* $\{\tilde\mu^{(n_m)}\}$, $\{\tilde U^{(n_m)}\}$, $\{\tilde Y^{(n_m)}\}$, $\{\tilde M^{(n_m)}\}$ *and a* $D([0,\infty), \mathcal{S}'(\mathbb{R}^d))$*-valued sequence* $\{\tilde W^{(n_m)}\}$ *defined on it, such that*

$$\mathbb{P}\circ(\mu^{(n_m)}, U^{(n_m)}, Y^{(n_m)}, M^{(n_m)}, W)^{-1}$$
$$= \tilde{\mathbb{P}}\circ(\tilde\mu^{(n_m)}, \tilde U^{(n_m)}, \tilde Y^{(n_m)}, \tilde M^{(n_m)}, \tilde W^{(n_m)})^{-1}$$

holds and, $\tilde{\mathbb{P}}$*-almost surely on* $D([0,\infty), (\mathscr{D}'(D))^4 \times \mathcal{S}'(\mathbb{R}^d))$,

$$(\tilde\mu^{(n_m)}, \tilde U^{(n_m)}, \tilde Y^{(n_m)}, \tilde M^{(n_m)}, \tilde W^{(n_m)}) \to (\tilde\mu^{(0)}, \tilde U^{(0)}, \tilde Y^{(0)}, \tilde M^{(0)}, \tilde W^{(0)})$$

as $m \to \infty$.

(iii) *There exists a dense subset* $\Xi \subset [0, \infty)$ *such that* $[0, \infty) \setminus \Xi$ *is at most countable and for each* $t \in \Xi$ *and each* $\phi \in \mathscr{D}(D)$ *and each* $\varphi \in \mathcal{S}(\mathbb{R}^d)$, *as* $\mathbb{R}^5$*-valued processes*

$$(\tilde{\mu}_t^{(n_m)}(\phi), \tilde{U}_t^{(n_m)}(\phi), \tilde{Y}_t^{(n_m)}(\phi), \tilde{M}_t^{(n_m)}(\phi), \tilde{W}_t^{(n_m)}(\varphi)) \\ \to (\tilde{\mu}_t^{(0)}(\phi), \tilde{U}_t^{(0)}(\phi), \tilde{Y}_t^{(0)}(\phi), \tilde{M}_t^{(0)}(\phi), \tilde{W}_t^{(0)}(\varphi))$$

in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ *as* $m \to \infty$. *Furthermore, let* $\tilde{\mathcal{F}}_t^{(0)}$ *be the* σ*-algebra generated by* $\tilde{\mu}_s^{(0)}(\phi)$, $\tilde{U}_s^{(0)}(\phi)$, $\tilde{Y}_s^{(0)}(\phi)$, $\tilde{M}_s^{(0)}(\phi)$, $\tilde{W}_s^{(0)}(\varphi)$ *for all* $\phi \in \mathscr{D}(D)$, *all* $\varphi \in \mathcal{S}(\mathbb{R}^d)$ *and* $s \le t$. $\tilde{M}_t^{(0)}(\phi)$ *is a continuous, square-integrable* $\tilde{\mathcal{F}}_t^{(0)}$*-martingale with quadratic variation process*

$$\langle \tilde{M}_t^{(0)}(\phi) \rangle = \gamma \sigma^2 \int_0^t \langle \phi^2, \tilde{\mu}_u^{(0)} \rangle \, du.$$

(iv) $\tilde{W}^{(0)}(dy, ds)$ *and* $\tilde{W}^{(n_m)}(dy, ds)$ *are Brownian sheets and for any* $\phi \in \mathscr{D}(D)$ *the continuous square integrable martingale*

$$\tilde{X}_t^{(n_m)}(\phi) := \sum_{p=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle h_p(y - \cdot) \, \partial_p \phi(\cdot), \tilde{\mu}_s^{(n_m)} \rangle \, \tilde{W}^{(n_m)}(dy, ds)$$

converges to

$$\tilde{X}_t^{(0)}(\phi) := \sum_{p=1}^{d} \int_0^t \int_{\mathbb{R}^d} \langle h_p(y - \cdot) \, \partial_p \phi(\cdot), \tilde{\mu}_s^{(0)} \rangle \, \tilde{W}^{(0)}(dy, ds)$$

in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$.

(v) $\tilde{\mu}^{(0)} = \{\tilde{\mu}_t^{(0)} : t \ge 0\}$ *is a solution to the* $(\mathcal{L}, \delta_{\mu_0})$*-martingale problem and* $\tilde{\mu}^{(0)}$ *is a continuous process and for any* $\phi \in \mathscr{D}(D)$ *we have*

$$\tilde{\mu}_t^{(0)}(\phi) - \tilde{\mu}_0^{(0)}(\phi) = \tilde{X}_t^{(0)}(\phi) + \int_0^t \int_{\mathbb{R}^d} \phi(x) \, \tilde{M}^{(0)}(dx, ds) \\ + \int_0^t \left\langle \sum_{p,q=1}^{d} \frac{1}{2} (a_{pq}(\cdot) + \rho_{pq}(\cdot, \cdot)) \, \partial_p \partial_q \phi(\cdot), \tilde{\mu}_s^{(0)} \right\rangle ds. \tag{3.9}$$

Proof. (i) By a theorem of Fouque [16], we only need to prove that, for any $\phi \in \mathscr{D}(D)$, the sequence of laws of

$$(\mu^{(n)}(\phi), U^{(n)}(\phi), Y^{(n)}(\phi), M^{(n)}(\phi))$$

is tight in $D([0, \infty), \mathbb{R}^4)$. This is equivalent to proving that each component and the sum of each pair of components are individually tight in $D([0, \infty), \mathbb{R})$. Since the same

idea works for each sequence, we only give the proof for $\{M^{(n)}(\phi)\}$. By Lemma 3.2 we have

$$\mathbb{P}(M_t^{(n)}(\phi) > k) \le \frac{\gamma\sigma_n^2}{k^2}\mathbb{E}\int_0^t \langle \phi^2, \mu_u^{(n)}\rangle \, du \le \frac{\gamma\sigma_n^2\|\phi\|_\infty^2 t}{k^2}\langle 1, \mu_0^{(n)}\rangle,$$

which yields the compact containment condition. Now we use the Kurtz tightness criterion (cf. Ethier-Kurtz [15] p. 137, Theorem 8.6) to prove the tightness of $\{M^{(n)}(\phi)\}$.

Let $\gamma_n^T(\delta) := \delta\gamma\sigma_n^2\|\phi\|_\infty^2 \sup_{0\le u\le T}\langle 1, \mu_u^{(n)}\rangle$, then for any $0 \le t+\delta \le T$,

$$\begin{aligned}\mathbb{E}[|M_{t+\delta}^{(n)}(\phi) - M_t^{(n)}(\phi)|^2 \mid \mathcal{F}_t^{(n)}] &= \mathbb{E}\left[\gamma\sigma_n^2\int_t^{t+\delta}\langle \phi^2, \mu_u^{(n)}\rangle \, du \;\middle|\; \mathcal{F}_t^{(n)}\right] \\ &\le \mathbb{E}\left[\gamma_n^T(\delta) \;\middle|\; \mathcal{F}_t^{(n)}\right].\end{aligned}$$

By Lemma 3.2, $\lim_{\delta\to 0}\sup_n \mathbb{E}[\gamma_n^T(\delta)] = 0$ holds, so $\{M^{(n)}(\phi)\colon n \ge 1\}$ is tight.

(ii) Let $E_c = \mathscr{D}(D)$ or $E_c = \mathcal{S}(\mathbb{R}^d)$, then $E_c' = \mathscr{D}'(D)$ or $E_c' = \mathcal{S}'(\mathbb{R}^d)$ respectively. Since E_c is separable and E_c' is a completely regular topological space (a nuclear space is separable, cf. Gel'fand-Vilenkin [17]), we can choose a countable dense subset $\{g_i\}_{i\in\mathbb{N}}$ of E_c and any enumeration $\{t_j\}_{j\in\mathbb{N}}$ of all the rational numbers, then use Theorem 1.7 of Jakubowski [20] to get that the countable family $\{f_{ij}\colon i, j \in \mathbb{N}\}$ of continuous functions separate points in $D([0,\infty), E_c')$ (with respect to Skorohod topology on $D([0,\infty), E_c')$), where

$$f_{ij}\colon x \in D([0,\infty), E_c') \to f_{ij}(x) := \arctan\langle g_i, x(t_j)\rangle \in [-\pi, \pi].$$

This proves that the space $D([0,\infty), E_c')$, and thus the space $D([0,\infty), (\mathscr{D}'(D))^4 \times \mathcal{S}'(\mathbb{R}^d))$ satisfy the basic assumption for a version of the Skorohod representation theorem due to Jakubowski [21].

(iii) For each $t \in \Xi$ and each $\phi \in \mathscr{D}(D)$ and each $\varphi \in \mathcal{S}(\mathbb{R}^d)$, from Lemma 3.2 we obtain the uniform integrability of $\{\tilde{\mu}_t^{(n_m)}(\phi)^2, \tilde{U}_t^{(n_m)}(\phi)^2, \tilde{Y}_t^{(n_m)}(\phi)^2, \tilde{M}_t^{(n_m)}(\phi)^2, \tilde{W}_t^{(n_m)}(\varphi))^2\}$. So (ii) implies their convergence in $L^2(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{\mathbb{P}})$ as $m \to \infty$. For each $\phi \in \mathscr{D}(D)$, $\varphi \in \mathcal{S}(\mathbb{R}^d)$ and $G_i \in C_b(\mathbb{R}^5)$, and any $0 < t_1 \le t_2 \le \cdots \le t_n = s < t$ with t_i, $t \in \Xi$, $i = 1, \ldots, n$, let

$$f^{(n_m)}(t_1, \ldots, t_n) := \prod_{i=1}^n G_i(\tilde{\mu}_{t_i}^{(n_m)}(\phi), \tilde{U}_{t_i}^{(n_m)}(\phi), \tilde{Y}_{t_i}^{(n_m)}(\phi), \tilde{M}_{t_i}^{(n_m)}(\phi), \tilde{W}_{t_i}^{(n_m)}(\varphi))).$$

Then, we have

$$\tilde{\mathbb{E}}[(\tilde{M}_t^{(n_m)}(\phi) - \tilde{M}_s^{(n_m)}(\phi)) f^{(n_m)}(t_1, \ldots, t_n)] = 0 \tag{3.10}$$

and

$$\tilde{\mathbb{E}}\Bigg[\Bigg(\tilde{M}_t^{(n_m)}(\phi)^2 - \gamma\sigma_{n_m}^2\int_0^t \langle\phi^2,\,\tilde{\mu}_u^{(n_m)}\rangle\,du - \tilde{M}_s^{(n_m)}(\phi)^2 + \gamma\sigma_{n_m}^2\int_0^s \langle\phi^2,\,\tilde{\mu}_u^{(n_m)}\rangle\,du\Bigg) f^{(n_m)}(t_1,\ldots,t_n)\Bigg] = 0. \tag{3.11}$$

By the convergence in $L^2(\tilde{\Omega},\tilde{\mathcal{F}},\tilde{\mathbb{P}})$ above, this implies that $\tilde{M}_t^{(0)}$ and

$$\tilde{M}_t^{(0)}(\phi)^2 - \gamma\sigma^2\int_0^t \langle\phi^2,\,\tilde{\mu}_u^{(0)}\rangle\,du$$

are $\tilde{\mathcal{F}}_t^{(0)}$-martingales. Let $K = \sup_{x\in D}\phi^2(x)$. Using (3.8) we can get

(3.12)

$$\begin{aligned}
&\tilde{\mathbb{E}}[(\tilde{M}_t^{(n)}(\phi) - \tilde{M}_s^{(n)}(\phi))^4]\\
&= \mathbb{E}[(M_t^{(n)}(\phi) - M_s^{(n)}(\phi))^4]\\
&= \mathbb{E}\Bigg[\sum_{\xi,\eta\in\mathfrak{R},\,\xi\neq\eta}\Bigg(\frac{\left(O_\xi^{(n)}-1\right)^2}{\theta^{2n}}\phi^2(x_\xi(\zeta(\xi)-))1_{\{s<\zeta(\xi)\le t\}}\\
&\qquad\qquad\cdot\frac{\left(O_\eta^{(n)}-1\right)^2}{\theta^{2n}}\phi^2(x_\eta(\zeta(\eta)-))1_{\{s<\zeta(\eta)\le t\}}\Bigg)\Bigg]\\
&\quad+\mathbb{E}\Bigg[\sum_{\xi\in\mathfrak{R}}\Bigg(\frac{\left(O_\xi^{(n)}-1\right)^4}{\theta^{4n}}\phi^4(x_\xi(\zeta(\xi)-))1_{\{s<\zeta(\xi)\le t\}}\Bigg)\Bigg]\\
&\le K^2\mathbb{E}\Bigg[\sum_{\xi,\eta\in\mathfrak{R},\,\xi\neq\eta}\Bigg(\frac{\left(O_\xi^{(n)}-1\right)^2}{\theta^{2n}}1_{\{s<\zeta(\xi)\le t\}}\frac{\left(O_\eta^{(n)}-1\right)^2}{\theta^{2n}}1_{\{s<\zeta(\eta)\le t\}}\Bigg)\Bigg]\\
&\quad+K^2\mathbb{E}\Bigg[\sum_{\xi\in\mathfrak{R}}\Bigg(\frac{\left(O_\xi^{(n)}-1\right)^4}{\theta^{4n}}1_{\{s<\zeta(\xi)\le t\}}\Bigg)\Bigg]\\
&\le K^2\Bigg(8\gamma^3\sigma_n^6(t-s)^3\langle 1,\,\mu_0^{(n)}\rangle + 2\gamma^2\sigma_n^4(t-s)^2\langle 1,\,\mu_0^{(n)}\rangle^2 + \frac{\gamma m_c^{(n)}}{\theta^{2n}}(t-s)\langle 1,\,\mu_0^{(n)}\rangle\Bigg).
\end{aligned}$$

In particular, for any $m \ge 1$ we have

(3.13)

$$\begin{aligned}
&\tilde{\mathbb{E}}[(\tilde{M}_t^{(n_m)}(\phi) - \tilde{M}_s^{(n_m)}(\phi))^4]\\
&\le K^2\Bigg(8\gamma^3\sigma_{n_m}^6(t-s)^3\langle 1,\,\mu_0^{(n_m)}\rangle + 2\gamma^2\sigma_{n_m}^4(t-s)^2\langle 1,\,\mu_0^{(n_m)}\rangle^2 + \frac{\gamma m_c^{(n_m)}}{\theta^{2n_m}}(t-s)\langle 1,\,\mu_0^{(n_m)}\rangle\Bigg).
\end{aligned}$$

Let $m\to\infty$ we get

$$\tilde{\mathbb{E}}[(\tilde{M}_t^{(0)}(\phi) - \tilde{M}_s^{(0)}(\phi))^4] \le K^2(8\gamma^3\sigma^6(t-s)^3\langle 1,\,\mu_0\rangle + 2\gamma^2\sigma^4(t-s)^2\langle 1,\,\mu_0\rangle^2). \tag{3.14}$$

Thus, $\tilde{M}_t^{(0)}$ has a continuous modification according to the Kolmogorov continuity criterion and

$$\langle \tilde{M}_t^{(0)}(\phi)\rangle = \gamma\sigma^2 \int_0^t \langle \phi^2, \tilde{\mu}_u^{(0)}\rangle \, du.$$

(iv) Since W, $\tilde{W}^{(0)}$ and $\tilde{W}^{(n_m)}$ have the same distribution, $\tilde{W}^{(0)}$ and $\tilde{W}^{(n_m)}$ are Brownian sheets. The conclusion follows from (ii) and Theorem 2.1 of Cho [4].

(v) Since $\tilde{U}_t^{(n_m)}(\phi)^2$ is uniformly integrable, we have $\tilde{\mathbb{P}}$-a.s. and in $L^2(\tilde{\mathbb{P}})$

$$\lim_{m\to\infty} \frac{1}{\sqrt{\theta^{n_m}}} \tilde{U}_t^{(n_m)}(\phi) = 0.$$

By taking $n \to \infty$ along the subsequence $\{n_m : m \geq 1\}$ in (3.3), we have

$$\tilde{\mu}_t^{(0)}(\phi) - \tilde{\mu}_0^{(0)}(\phi) = \tilde{X}_t^{(0)}(\phi) + \tilde{Y}_t^{(0)}(\phi) + \tilde{M}_t^{(0)}(\phi) \quad \text{for every} \quad \phi \in \mathscr{D}(D) \quad \text{and} \quad t \geq 0.$$

As

$$\tilde{Y}_t^{(n)}(\phi) = \int_0^t \left\langle \sum_{p,q=1}^d \frac{1}{2}(a_{pq}(\,\cdot\,) + \rho_{pq}(\,\cdot\,,\,\cdot\,))\, \partial_p\partial_q\phi(\,\cdot\,),\, \tilde{\mu}_s^{(n)} \right\rangle ds$$

and $\tilde{M}_t^{(n)}(\phi) = \int_0^t \int_D \phi(x)\, \tilde{M}^{(n)}(dx, ds)$, we see from (ii) above that

$$\tilde{Y}_t^{(0)}(\phi) = \int_0^t \left\langle \sum_{p,q=1}^d \frac{1}{2}(a_{pq}(\,\cdot\,) + \rho_{pq}(\,\cdot\,,\,\cdot\,))\, \partial_p\partial_q\phi(\,\cdot\,),\, \tilde{\mu}_s^{(0)} \right\rangle ds$$

and $\tilde{M}_t^{(0)}(\phi) = \int_0^t \int_D \phi(x)\, \tilde{M}^{(0)}(dx, ds)$. So, $\tilde{\mu}_t^{(0)}$ satisfies (3.9).

By Itô's formula, we see that $\{\tilde{\mu}_t^{(0)} : t \geq 0\}$ is a solution to the martingale problem for $(\mathcal{L}, \delta_{\mu_0})$. The continuity of $\tilde{\mu}^{(0)}$ follows from the result of Bakry-Emery [2]. □

We see from the theorem above that $\tilde{\mu}^{(0)} = \{\tilde{\mu}_t^{(0)} : t \geq 0\}$ is a solution to the martingale problem for $(\mathcal{L}, \delta_{\mu_0})$. For uniqueness, we will use a duality argument due to Dawson-Kurtz [7]. Before we start the discussion of uniqueness of the martingale problem for $\mathcal{L}$, in the following we will rewrite $\mathcal{L}$ into an equivalent form. Recall

$$\mathcal{L}F(\mu) := \mathcal{A}F(\mu) + \mathcal{B}F(\mu), \tag{3.15}$$

$$\mathcal{B}F(\mu) := \frac{1}{2}\gamma\sigma^2 \int_D \frac{\delta^2 F(\mu)}{\delta\mu(x)^2}\, \mu(dx), \tag{3.16}$$

and

$$\begin{aligned}\mathcal{A}F(\mu) := {} & \frac{1}{2}\sum_{p,q=1}^{d}\int_D (a_{pq}(x)+\rho_{pq}(x,x))\left(\frac{\partial^2}{\partial x_p\,\partial x_q}\right)\frac{\delta F(\mu)}{\delta\mu(x)}\,\mu(dx)\\ & + \frac{1}{2}\sum_{p,q=1}^{d}\int_D\int_D \rho_{pq}(x,y)\left(\frac{\partial}{\partial x_p}\right)\left(\frac{\partial}{\partial y_q}\right)\frac{\delta^2 F(\mu)}{\delta\mu(x)\,\delta\mu(y)}\,\mu(dx)\,\mu(dy).\end{aligned} \tag{3.17}$$

Let $C_0^2(D^m)$ be the collection of functions in $C^2(D^m)$ vanishing on the boundary and outside of D^m. Therefore, for $\forall f \in C_0^2(D^m)$ and $\forall x \in (D^m)^c := (\mathbb{R}^d)^m \setminus D^m$ we have $f(x) = 0$. For $f \in \bigcup_{m=1}^{\infty} C_0^2(D^m)$, we define $N(f)$ to be m if $f \in C_0^2(D^m)$ and define

$$F_\mu(f) := F_f(\mu) := \int_{\mathbb{R}^d}\cdots\int_{\mathbb{R}^d} f(x_1,\ldots,x_m)\,\mu(dx_1)\cdots\mu(dx_m) \quad \text{for} \quad \mu \in M_F(\mathbb{R}^d).$$

Such a function F_f is called a monomial function on the space $M_F(\mathbb{R}^d)$. Note that for such a monomial function F_f,

$$\frac{\partial F_f(\mu)}{\partial\mu(x)} = \sum_{j=1}^{N(f)}\int_{(\mathbb{R}^d)^{N(f)-1}} f(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_{N(f)})\prod_{l=1,\,l\neq j}^{N(f)}\mu(dx_l),$$

$$\begin{aligned}&\frac{\partial^2 F_f(\mu)}{\partial\mu(y)\,\partial\mu(x)}\\ &= \sum_{j,k=1,\,j\neq k}^{N(f)}\int_{(\mathbb{R}^d)^{N(f)-2}} f(x_1,\ldots,x_{j-1},x,x_{j+1},\ldots,x_{k-1},y,x_{k+1},\ldots,x_{N(f)})\prod_{l=1,\,l\neq j,k}^{N(f)}\mu(dx_l).\end{aligned}$$

For $f \in C_0^2(D^m)$, $x = (x_1,\ldots,x_m) \in (\mathbb{R}^d)^m$, we define

$$G_m f(x) := \frac{1}{2}\sum_{i,j=1}^{m}\sum_{p,q=1}^{d}\Gamma_{pq}^{ij}(x_1,\ldots,x_m)\frac{\partial^2}{\partial x_{ip}\,\partial x_{jq}} f(x_1,\ldots,x_m) \tag{3.18}$$

where $x_i = (x_{i1},\ldots,x_{id}) \in \mathbb{R}^d$ for $1 \le i \le m$ and Γ_{pq}^{ij} is defined by (1.8). Then by (1.2), we have for any monomial function F_f on $M_F(\mathbb{R}^d)$ with $N(f) = m$,

$$\begin{aligned}\mathcal{L}F_f(\mu) &= \mathcal{A}F_f(\mu) + \mathcal{B}F_f(\mu)\\ &= F_{G_m f}(\mu) + \frac{\gamma\sigma^2}{2}\sum_{j,k=1,\,j\neq k}^{m} F_{\Phi_{jk}f}(\mu)\\ &= F_{G_m f}(\mu) + \frac{\gamma\sigma^2}{2}\sum_{j,k=1,\,j\neq k}^{m}(F_{\Phi_{jk}f}(\mu) - F_f(\mu)) + \frac{\gamma\sigma^2}{2}m(m-1)F_f(\mu)\end{aligned}$$

$$= F_\mu(G_m f) + \frac{\gamma\sigma^2}{2} \sum_{j,k=1,\ j\neq k}^{m} (F_\mu(\Phi_{jk} f) - F_\mu(f)) + \frac{\gamma\sigma^2}{2} m(m-1) F_\mu(f)$$

$$:= \mathcal{L}^* F_\mu(f) + \frac{1}{2}\gamma\sigma^2 m(m-1) F_\mu(f).$$

Here for $j < k$, $\Phi_{jk} f$ is a function on $(\mathbb{R}^d)^{m-1}$ defined by, for $y = (y_1, \ldots, y_{m-1}) \in (\mathbb{R}^d)^{m-1}$,

$$\Phi_{jk} f(y) := \Phi_{kj}(y) := f(y_1, \ldots, y_j, \ldots, y_{k-1}, y_j, y_k, \ldots, y_{m-1}).$$

We know that (1.1) has a unique strong solution. For any positive integer m, let $\{z_i(t)\colon t \geq 0,\ 1 \leq i \leq m\}$ be a sequence of strong solutions to (1.1) for the given standard $\mathbb{R}^d$-valued Brownian motions $\{B_i(t)\colon t \geq 0,\ 1 \leq i \leq m\}$ and W, a Brownian sheet on $\mathbb{R}^d$, defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. Let $Z_m(t) := (z_1(t), \ldots, z_m(t))$ and let $\{P_t^m\colon t \geq 0\}$ be the transition semigroup of $Z_m(t)$ killed upon leaving D^m. Since the coefficients of (1.1) satisfy the Lipschitz condition, P_t^m is a Feller semigroup and maps $C_0^2(D^m)$ to $C_0^2(D^m)$.

For any given integer $m \geq 1$, the law of Z_m with initial point $x \in D^m$ will be denoted by $\mathbb{P}_x$, and expectation with respect to $\mathbb{P}_x$ will be denoted by $\mathbb{E}_x$. Let τ^m be the first exit time of Z_m from D^m.

Theorem 3.4. *Assume that D is a bounded regular domain in $\mathbb{R}^d$, $c \in \mathrm{Lip}_b(\mathbb{R}^d)$, $h \in L^2(\mathbb{R}^d) \cap L^1(\mathbb{R}^d) \cap \mathrm{Lip}_b(\mathbb{R}^d)$ and the diffusion matrix $(a_{pq})_{1\leq p,q\leq d}$ defined by* (1.5) *is uniformly elliptic, bounded on $\mathbb{R}^d$. Let $\{P_t^m\colon t \geq 0\}$ be the transition semigroup for $Z_m(t)$ killed upon leaving D^m, that is*

$$P_t^m f(x) := \mathbb{E}_x[f(Z_m(t));\ t < \tau^m] \quad \textit{for} \quad t \geq 0 \quad \textit{and} \quad f \in \mathcal{B}_b(D^m).$$

Then for every $f \in C_0^2(D^m)$ and $t > 0$, $P_t^m f(x)$ as a function of x belongs to $C_0^2(D^m)$. Therefore, $C_0^2(D^m)$ is invariant under P_t^m for every $t > 0$ and $m \geq 1$.

Proof. To apply Theorem 1.6 in Part II of [14], we only need to check the uniform ellipticity of (Γ_{pq}^{ij}). Let $\xi_i = (\xi_{i1}, \ldots, \xi_{id})^T$ denote an arbitrary column vector in

$\mathbb{R}^d$ and $\Gamma := (\Gamma^{ij}_{pq}(x_1, \ldots, x_m))_{1\le i,j\le m,\ 1\le p,q\le d}$. Since

$$
\begin{aligned}
&(\xi_1^T, \ldots, \xi_m^T)\Gamma \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_m \end{pmatrix} \\
&= \sum_{i,j=1}^{m} \sum_{p,q=1}^{d} \xi_{ip}\Gamma^{ij}_{pq}(x_1, \ldots, x_m)\xi_{jq} \\
&= \sum_{i=1}^{m} \sum_{p,q=1}^{d} [\xi_{ip}(a_{pq}(x_i) + \rho_{pq}(x_i, x_i))\xi_{iq}] + \sum_{i,j=1,\ i\ne j}^{m} \sum_{p,q=1}^{d} \xi_{ip}\rho_{pq}(x_i, x_j)\xi_{jq} \qquad (3.19) \\
&= \sum_{i=1}^{m} \left[\sum_{r=1}^{d} \left(\sum_{p=1}^{d} \xi_{ip}c_{pr}(x_i) \right)^2 + \int_E \left(\sum_{p=1}^{d} \xi_{ip}h_p(u - x_i) \right)^2 du \right] \\
&\quad + \sum_{i,j=1,\ i\ne j}^{m} \int_E \left(\sum_{p=1}^{d} \xi_{ip}h_p(u - x_i) \right) \left(\sum_{p=1}^{d} \xi_{jq}h_q(u - x_j) \right) du \\
&= \sum_{i=1}^{m} \sum_{r=1}^{d} \left(\sum_{p=1}^{d} \xi_{ip}c_{pr}(x_i) \right)^2 + \int_E \left[\sum_{i=1}^{m} \left(\sum_{p=1}^{d} \xi_{ip}h_p(u - x_i) \right) \right]^2 du \ge 0
\end{aligned}
$$

and by the uniform ellipticity assumption of $(a_{pq})_{1\le p,q\le d}$ there exists a positive real number $\epsilon > 0$ such that for each $1 \le i \le m$

$$
\sum_{r=1}^{d} \left(\sum_{p=1}^{d} \xi_{ip}c_{pr}(x_i) \right)^2 = \sum_{p,q=1}^{d} [\xi_{ip}a_{pq}(x_i)\xi_{iq}] \ge \epsilon |\xi_i|^2, \qquad (3.20)
$$

where $|\xi_i| = \sqrt{\xi_{i1}^2 + \cdots + \xi_{id}^2}$, the uniform ellipticity of Γ follows. The assumption of $h \in L^1(\mathbb{R}^d)$ implies that $\rho(x_i, x_j) \in \mathrm{Lip}(\mathbb{R}^d \times \mathbb{R}^d)$ for $i, j = 1, 2, \ldots, m$, and therefore $\Gamma^{i,j}_{p,q} \in \mathrm{Lip}((\mathbb{R}^d)^m)$. By Theorem 1.6 in Part II of [14], we have for every $f \in C_0(D^m)$, there is a $u \in C_b^2(D^m)$ such that

$$
\frac{\partial u}{\partial s} - G_m u = 0 \quad \text{in} \quad (0, t) \times D^m, \qquad (3.21)
$$

and

$$
u(0, x) = f(x), \quad u(s, x)|_{(0,t)\times \partial D^m} = 0, \qquad (3.22)
$$

where G_m is the differential operator given by (3.18). Let D_n^m, $n \ge 1$ be a sequence of bounded smooth domains such that $\overline{D_n^m} \subset D^m$ and $D_n^m \uparrow D^m$ as $n \uparrow \infty$, and let τ_n^m

be the first exit time from D_n^m. Applying Itô's formula to $s \mapsto u(t-s, Z_m(s))$ (see, e.g. the calculation for (2.2)), we see that $L_s := u(t-s, Z_m(s))$ is a local martingale on $[0, t \wedge \tau^m)$, i.e., for any fixed $m \geq 1$ and any fixed $n \geq 1$, $(L_{s\wedge t\wedge \tau_n^m},\ s \geq 0)$ is a martingale. Since L_s is bounded and $\tau^m < \infty$ a.s. P_x for any $x \in D^m$, L_s converges to a limit as $s \to t \wedge \tau^m$. Since u is continuous on $[0, t] \times \overline{D^m}$ and satisfies the boundary condition (3.22), the limit must be $f(Z_m(t))I_{(t<\tau^m)}$. Thus

$$u(t, x) = \mathbb{E}_x[\, f(Z_m(t));\ t < \tau^m] = P_t^m f(x).$$

This proves the theorem. □

Let $\mathcal{S} = D \cup \left(\bigcup_{k=1}^{\infty} C_0^2(D^k)\right)$ (disjoint union). We see from the proof of Theorem 3.4 that G_m coincides on $C_0^2(D^m)$ with the infinitesimal generator of the strong Markov process Z_m for the motion of m particles given by (2.1) killed upon exiting D^m. Thus $\mathcal{L}^*$ has the structure of the infinitesimal generator of an $\mathcal{S}$-valued strong Markov process Y, whose dynamics contains the following two mechanisms:
(a) Jumping mechanism: Let $\{J_t\colon t \geq 0\}$ be a nonnegative integer-valued càdlàg Markov process with $J_0 = m$ and transition intensities $\{q_{i,j}\}$ such that

$$q_{i,i-1} = -q_{i,i} = \frac{\gamma\sigma^2}{2} i(i-1) \quad \text{and} \quad q_{i,j} = 0 \quad \text{for all other pairs} \quad (i, j).$$

Thus, $\{J_t,\ t \geq 0\}$ is just the well-known Kingman's coalescent process. Let $\tau_0 = 0$, $\tau_{J_0+1} = \infty$ and $\{\tau_k\colon 1 \leq k \leq J_0\}$ be the sequence of jump times of $\{J_t\colon t \geq 0\}$. Let $\{S_k\colon 1 \leq k \leq M_0\}$ be a sequence of random operators which are conditionally independent given $\{J_t\colon t \geq 0\}$ and satisfy

$$\mathbb{P}\{S_k = \Phi_{i,j} \mid J(\tau_k-) = l\} = \frac{1}{l(l-1)}, \quad 1 \leq i \neq j \leq l.$$

(b) Spatial jump-diffusion semigroup: Let $\boldsymbol{B}$ denote the topological union of $\{L^\infty((D)^m)\colon m = 1, 2, \dots\}$ endowed with pointwise convergence on each $L^\infty((D)^m)$. Then

$$Y_t = P_{t-\tau_k}^{J_{\tau_k}} S_k P_{\tau_k-\tau_{k-1}}^{J_{\tau_{k-1}}} S_{k-1} \cdots P_{\tau_2-\tau_1}^{J_{\tau_1}} S_1 P_{\tau_1}^{J_0} Y_0, \quad \tau_k \leq t < \tau_{k+1},\ 0 \leq k \leq J_0,$$

defines a Markov process $Y := \{Y_t\colon t \geq 0\}$ with $Y_0 \in C_0^2(D^m)$. By Theorem 3.4, process Y takes values in $\mathcal{S} \subset \boldsymbol{B}$. Clearly, $\{(J_t, Y_t)\colon t \geq 0\}$ is also a Markov process.

The duality relationship can be described as follows. Let $\mathcal{D}(\mathcal{L})$ be the set of all functions of the form $F_{m,f}(\mu) = \langle f, \mu^m\rangle$ with $f \in C_0^2(D^m)$. If $\{X_t\colon t \geq 0\}$ is a solution to the $(\mathcal{L}, \mathcal{D}(\mathcal{L}))$-martingale problem with $X_0 = \mu_0$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, then, by Feynman-Kac formula (see [7]), we have

$$\mathbb{E}[\langle f, X_t^m\rangle] = \mathbb{E}_{m,f}\left[\langle Y_t, \mu_0^{J_t}\rangle \exp\left(\frac{\gamma\sigma^2}{2}\int_0^t J_s(J_s-1)\,ds\right)\right] \tag{3.23}$$

for any $t \geq 0$, $f \in C_0^2(D^m)$ and integer $m \geq 1$, where the right hand side is the expectation taken on the probability space where the dual process is defined with giving $J_0 = m$ and $Y_0 = f \in C_0^2(D^m)$. From this, we see that the marginal distribution of X is uniquely determined and hence the law of X is unique (see, e.g. [15, Theorem 4.4.2]). This proves the uniqueness of the martingale problem for $\mathcal{L}$.

We summarize these results in the following theorem.

Theorem 3.5. *Assume that D is a bounded regular domain in $\mathbb{R}^d$, $c \in \mathrm{Lip}_b(\mathbb{R}^d)$ and $h \in L^2(\mathbb{R}^d) \cap \mathrm{Lip}(\mathbb{R}^d)$, and the diffusion matrix $(a_{pq})_{1\leq p,q\leq d}$ defined by* (1.5) *is uniformly elliptic and bounded on $\mathbb{R}^d$. For the uniqueness for the martingale problem below, assume further that c is bounded below by a strictly positive constant and $h \in L^1(\mathbb{R}^d)$. For any measure $\mu \in M_F(D)$ with compact support, the $(\mathcal{L}, \delta_\mu)$-martingale problem has a unique solution μ_t, which is a diffusion process and satisfies*

$$\begin{aligned}\mu_t(\phi) - \mu_0(\phi) = X_t(\phi) &+ \int_0^t \int_{\mathbb{R}^d} \phi(x)\, M(dx, ds) \\ &+ \int_0^t \left\langle \sum_{p,q=1}^d \frac{1}{2}(a_{pq}(\,\cdot\,) + \rho_{pq}(\,\cdot\,,\,\cdot\,))\, \partial_p\partial_q\phi(\,\cdot\,),\, \mu_s \right\rangle ds\end{aligned} \tag{3.24}$$

for every $t > 0$ and $\phi \in \mathscr{D}(D)$, where W is a Brownian sheet,

$$X_t(\phi) := \sum_{p=1}^d \int_0^t \int_{\mathbb{R}^d} \langle h_p(y - \,\cdot\,)\, \partial_p\phi(\,\cdot\,),\, \mu_s\rangle\, W(dy, ds)$$

and M is a square-integrable martingale measure with

$$\langle M(\phi)\rangle_t = \gamma\sigma^2 \int_0^t \langle \phi^2, \mu_u\rangle\, du \quad \text{for every} \quad t > 0 \quad \text{and} \quad \phi \in \mathscr{D}(D).$$

Here

$$M_t(\phi) := \int_0^t \int_{\mathbb{R}^d} \phi(y)\, M(ds, dy)$$

is a square-integrable, continuous $\{\mathcal{F}_t\}$-martingale, where $\mathcal{F}_t := \sigma\{\mu_s(f), M_s(f), X_s(f)\colon f \in C^1(D),\ s \leq t\}$. Moreover $X_t(\phi)$, $M_t(\phi)$ are orthogonal square-integrable, $\{\mathcal{F}_t\}$-martingales for every $\phi \in \mathscr{D}(D)$.

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Yan-Xia Ren
LMAM School of Mathematical Sciences
Peking University
Beijing, 100871
P.R. China
e-mail: yxren@math.pku.edu.cn

Renming Song
Department of Mathematics
The University of Illinois
Urbana, IL 61801
U.S.A.
e-mail: rsong@math.uiuc.edu

Hao Wang
Department of Mathematics
The University of Oregon
Eugene OR 97403–1222
U.S.A.
e-mail: haowang@uoregon.edu