

Joint continuity of the solutions to a class of nonlinear SPDEs

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Abstract For a one-dimensional superprocess in random environment, a nonlinear SPDE was derived by Dawson et al. (*Ann Inst Henri Poincaré Probab Stat* 36(2):167–180, 2000) for its density process. The time-space joint continuity of the density process was left as an open problem. In this paper we give an affirmative answer to this problem.

Keywords Superprocess · Random environment · Stochastic partial differential equation

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1 Introduction

Suppose that in a system of k_n particles each particle has an independent exponential clock with parameter n . Before any of these exponential times is up, the particles with initial locations $(x_1^n, \dots, x_{k_n}^n) \in \mathbb{R}^{k_n}$ move according to the following system of stochastic differential equations (SDE):

$$x_i^n(t) = x_i^n + B^i(t) + \int_0^t \int_{\mathbb{R}} h(y - x_i^n(s)) W(ds dy), \quad i = 1, 2, \dots, k_n, \quad (1.1)$$

where $h \in L^2(\mathbb{R})$ and (B^1, \dots, B^{k_n}) is a k_n -dimensional Brownian motions independent of the Brownian sheet W on $\mathbb{R}_+ \times \mathbb{R}$. The W can be regarded as the random environment for the particle system. For convenience we assume that

$$\rho(0) \equiv \int_{\mathbb{R}} h(x)^2 dx = 1.$$

When its clock rings the particle either splits into two or dies with equal probabilities. The new particles will inherit their mother’s position together with new independent exponential clocks. This pattern of motion-splitting/dying then continues as before.

In general, each particle in the system can be denoted by a multi-index $\alpha = \alpha_1 \alpha_2 \dots \alpha_k$ with $\alpha_1 = 1, \dots, k_n$ and $\alpha_i = 1, 2$ for $i \geq 2$. For example, $\alpha = (3, 1)$ represents the oldest daughter of the third particle in the first generation. Write $\alpha \sim t$ if particle α is alive at time t . For each n we define a measure-valued stochastic process

$$X_t^n = \frac{1}{n} \sum_{\alpha \sim t} \delta_{x_\alpha^n(t)}, \quad t \geq 0.$$

This model is first studied by Wang [15, 16]. Write $\mathcal{M}_F(\mathbb{R})$ for the space of finite measures on \mathbb{R} with the topology of weak convergence. Under suitable conditions, it is proved by Wang [16] and Dawson et al. [2] that as $n \rightarrow \infty$, X^n converges weakly in $D([0, T], \mathcal{M}_F(\mathbb{R}))$ to the unique solution X of the following martingale problem (MP):

$$M_t^\phi \equiv \langle X_t, \phi \rangle - \langle \mu, \phi \rangle - \int_0^t \langle X_s, \Delta \phi \rangle ds, \quad \forall \phi \in C_b^2(\mathbb{R}) \quad (1.2)$$

is a continuous martingale with quadratic variation process

$$\langle M^\phi \rangle_t = \int_0^t \langle X_s, \phi^2 \rangle ds + \int_0^t \int_{\mathbb{R}^2} \rho(x - y) \phi'(x) \phi'(y) X_s(dx) X_s(dy) ds, \tag{1.3}$$

where $\mu \in \mathcal{M}_F(\mathbb{R})$ is the initial measure and

$$\rho(x - y) = \int_{\mathbb{R}} h(z - x) h(z - y) dz. \tag{1.4}$$

Here $\Delta\phi \equiv \phi''$ is the second derivative of ϕ . Similarly, we shall use both $\nabla\phi$ and ϕ' to denote the first derivative of ϕ .

It is proved by Dawson et al. [3] and Wang [15] that X_t is absolutely continuous with respect to Lebesgue measure and its density, denoted by $X_t(x)$, solves SPDE

$$\begin{aligned} X_t(x) = & \mu(x) + \int_0^t \Delta X_s(x) ds - \int_0^t \int_{\mathbb{R}} \nabla_x(h(y - x) X_s(x)) W(ds dy) \\ & + \int_0^t \sqrt{X_s(x)} \frac{B(ds dx)}{dx}, \end{aligned} \tag{1.5}$$

where B is a Brownian sheet on $\mathbb{R}_+ \times \mathbb{R}$ independent of W . The joint continuity of $(t, x) \mapsto X_t(x)$ is left as an open problem in [3].

When the third term on the RHS of (1.5) is replaced by $\int_0^t \nabla(h(x) X_s(x)) d\tilde{W}(s)$ with a real-valued Brownian motion \tilde{W} , the SPDE is satisfied by the density process of a measure-valued process for a related model studied by Skoulakis and Adler [14]. For that model, Lee et al. [11] proves the continuity in x for Lebesgue almost all fixed t using Krylov’s (cf. Krylov [8]) L_p theory for linear SPDE.

The goal of this paper is to prove the joint continuity of $X_t(x)$ in Theorem 1.1. For $k \in \mathbb{R}$, $p \geq 1$ the space H_p^k with norm $\|\cdot\|_{k,p}$ will be introduced in Sect. 2. We always make the following assumption (I) on the initial measure X_0 .

Assumption (I): X_0 has a bounded density $\mu \in H_2^1$.

Theorem 1.1 *Suppose that $h \in H_2^2$, $\|h\|_{1,2}^2 < 2$ and X_0 satisfies the condition (I). Then the measure-valued process X_t has a density $X_t(x)$ which is almost surely jointly Hölder continuous. Furthermore, for fixed t its Hölder exponent in x is in $(0, 1/2)$; for fixed x its Hölder exponent in t is in $(0, 1/10)$.*

We now describe the major difficulties and sketch our approaches for the main result. When $h = 0$, X becomes the well known Dawson–Watanabe process with the joint continuity for its density studied by Konno and Shiga [6] and Reimers [13] via

a convolution technique. If we adopt the same technique here, then the density can be represented as

$$\begin{aligned}
 X_t(x) = & \int \varphi_t(x - y)\mu(y)dy + \int_0^t \int_{\mathbb{R}} \sqrt{X_s(y)}\varphi_{t-s}(x - y)B(dsdy) \\
 & + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} h(y - z)X_s(z)\partial_z\varphi_{t-s}(x - z)dzW(dsdy), \tag{1.6}
 \end{aligned}$$

where φ is the heat kernel with generator Δ . However, the third term on the RHS of the above equation is (for some suitable function g) roughly equal to

$$\int_0^t \int_{\mathbb{R}} (t - s)^{-1/2}g(z)W(dsdz),$$

which does *not* converge. Therefore, the convolution argument of Konno and Shiga fails in our model. It actually means that the SPDE (1.5) does not have a *mild* solution.

Since it is the term containing W that causes the problem, we want to absorb it to the kernel by considering a stochastic transition function. For this purpose let $p^W(s, x; t, y)$ be the conditional transition function of a single particle with W given (to be made precise in Sect. 3). We will prove that

$$X_t(y) = \int_{\mathbb{R}} p^W(0, x; t, y)\mu(x)dx + \int_0^t \int_{\mathbb{R}} \sqrt{X_s(x)}p^W(s, x; t, y)B(dsdx).$$

The first term in the above equation is easy to deal with. So we focus on the second term. We will apply Kolmogorov’s criteria to obtain the joint continuity. To this end, we need the following estimates: for any $y_1, y_2 \in \mathbb{R}$,

$$\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} |p^W(s, x, t, y_1) - p^W(s, x, t, y_2)|^2 dx ds \right|^p \leq K|y_1 - y_2|^{2+\epsilon} \tag{1.7}$$

and for $y \in \mathbb{R}$ and $t_1 < t_2$,

$$\mathbb{E} \left| \int_0^{t_1} \int_{\mathbb{R}} |p^W(s, x, t_2, y) - p^W(s, x, t_1, y)|^2 dx ds \right|^p \leq K|t_1 - t_2|^{2+\epsilon}, \tag{1.8}$$

for some $\epsilon > 0$ and suitable $p > 0$.

To obtain (1.7) we fix t and let $u_s(x) = p^W(t - s, x, t, y_1) - p^W(t - s, x, t, y_2)$. Then u satisfies the following linear SPDE

$$u_t(x) = u_0(x) + \int_0^t \Delta u_s(x) ds + \int_0^t \int_{\mathbb{R}} \nabla u_s(x) h(y - x) \tilde{W}(ds dy) \tag{1.9}$$

with initial condition $u_0 = \delta_{y_1} - \delta_{y_2}$, where \tilde{W} is a Brownian sheet defined by W with its time reversed (to be made precise later). We shall derive an estimate of u_s in terms of u_0 in the spirit of Kurtz and Xiong [10] and obtain (1.7).

For (1.8), we note that $\tilde{u}_s(x) = p^W(t_1 - s, x, t_2, y) - p^W(t_1 - s, x, t_1, y)$ is a solution to the linear SPDE (1.9) with initial condition $\tilde{u}_0 = p^W(t_1, \cdot, t_2, y) - \delta_y$. The LHS of (1.8) is then bounded by $\mathbb{E} \|\tilde{u}_0\|_{-1,2}^{2p}$, where $\|\cdot\|_{-1,2}$ is a Sobolev norm to be defined later. To estimate this quantity, we further define $v_t(x) = p^W(t_2 - t, x, t_2, y)$ which solves SPDE (1.9) with initial $v_0(x) = \delta_y(x)$, and then estimate $\mathbb{E} \|v_{t_2-t_1} - \delta_y\|_{-1,2}^{2p}$. Similar to what we mentioned above for the convolution (1.6), we cannot directly apply the convolution with kernel φ_t to (1.9). We shall use a partial convolution by kernel φ_{t^α} where $\alpha \in (0, 1)$ is a constant to be decided later. Then

$$\begin{aligned} v_t(z) &= \varphi_{t^\alpha}(z - y) + \int_0^t \int_{\mathbb{R}} \Delta v_{t-r}(x) \varphi_{r^\alpha}(z - x) dx dr \\ &\quad + \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \nabla v_{t-r}(x) h(y - x) \varphi_{r^\alpha}(z - x) dx \tilde{W}(dr dy) \\ &\quad - \alpha \int_0^t \int_{\mathbb{R}} \Delta v_{t-r}(x) \varphi_{r^\alpha}(z - x) dx r^{\alpha-1} dr. \end{aligned} \tag{1.10}$$

The main difficulty now lies in the fourth term because, due to the integrability, we can not apply integration by parts to move Δ completely to φ . Instead, we have to transform a fraction Δ^β of Δ to φ with $\beta < 1$ to be decided (together with α).

The novelty of this article is as follows. Firstly, to the best of our knowledge the joint continuity for the solution to SPDE was only previously studied when the mild solution for that equation can be defined. The current paper appears to be the first attempt for such a problem when the SPDE does not allow a mild solution. Secondly, a fractional integration by parts technique is introduced to obtain estimates for the solution to SPDE. We believe this technique will be useful in studying other SPDEs. Thirdly, a stochastic convolution technique is implemented, which provides the solution to SPDE with a ‘‘conditionally mild’’ representation. This technique will be applicable to other SPDEs arising from particle systems in random environments.

Besides the continuity of the solution, mild representation has been used by many authors to derive various properties for the solution of the SPDE. For example, Foondun and Khoshnevisan [4] use this representation to study the intermittency.

We believe that the methods we develop in this paper can be applied to study other properties of the SPDEs for which the mild representations are not available.

The rest of the paper is organized as follows. In Sect. 2 we establish some estimates for the solutions to a class of linear SPDEs. Then in Sect. 3 we derive a representation of the density $X_t(x)$ in terms of a random transition function. Based on this representation, we estimate the spatial-increments of $X_t(x)$ in Sect. 4 and the time-increments in Sect. 5. We conclude the proof of Theorem 1.1 in Sect. 5.

The following conventions will be used throughout the paper. We use K to represent a positive constant whose value can be different from place to place. We use I or J with a subscript to represent a term in the quantity to be evaluated. Again, what I_1 stands for can be different from place to place.

2 Two SPDE estimates

In this section we study the SPDE (1.9) where u_0 is either a real or a generalized function for different purposes. To this end, we need to introduce some notation taken from Krylov [8]. For $\alpha \in (0, 1)$ and generalized function u on \mathbb{R} , let

$$(I - \Delta)^\alpha u = c(\alpha) \int_0^\infty \frac{e^{-t} T_t u - u}{t^{\alpha+1}} dt, \quad (2.1)$$

and

$$(I - \Delta)^{-\alpha} u = d(\alpha) \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt, \quad (2.2)$$

where $c(\alpha)$ and $d(\alpha)$ are two constants and T_t is the Brownian semigroup. As being indicated by Krylov [8], (2.1) and (2.2) are sufficient to define $(I - \Delta)^{n/2}$ consistently for any $n \in \mathbb{R}$ (cf. Krasnoselskii et al. [7]). In particular, $(I - \Delta)^\alpha (I - \Delta)^\beta = (I - \Delta)^{\alpha+\beta}$ for any $\alpha, \beta \in \mathbb{R}$. In this paper we only need it for $n \in [-1, 1]$.

Let H_p^n be the spaces of Bessel potentials with norms

$$\|u\|_{n,p} \equiv \|(I - \Delta)^{n/2} u\|_p \quad (2.3)$$

where $\|\cdot\|_p$ is the norm in L_p . Note that for $n = 1$ and $p = 2$, $\|u\|_{1,2}$ coincides with the usual Sobolev norm on H_2^1 .

The existence and uniqueness of the solution to (1.9) has been studied by Krylov [8] in suitable Banach spaces. In the remaining of this section, we assume this equation has a solution (the existence will be evident from the applications in later sections), and the aim of this section is to prove that, with the appropriate initial condition, the solution actually lies in the spaces which will be useful for our purpose.

Let $\beta \in [0, 1)$ and $u_0 \in H_2^{\beta-1}$. For $f \in C_0^\infty(\mathbb{R})$, i.e., f is infinitely differentiable with compact support, we have

$$\langle u_r, f \rangle = \langle u_0, f \rangle + \int_0^r \langle \Delta u_s, f \rangle ds + \int_0^r \int_{\mathbb{R}} \langle \nabla u_s h(y - \cdot), f \rangle \tilde{W}(dsdy) \tag{2.4}$$

where $\langle u, f \rangle$ stands for the duality between the Hilbert spaces H_2^{-n} and H_2^n . Applying Itô's formula to $\langle u_r, f \rangle^2$ and summing up f over a complete orthonormal system of $H_2^{1-\beta}$, by (2.4) we get

$$\begin{aligned} \|u_r\|_{\beta-1,2}^2 &= \|u_0\|_{\beta-1,2}^2 + \int_0^r 2 \langle u_s, \Delta u_s \rangle_{\beta-1,2} ds + \int_0^r \int_{\mathbb{R}} \|\nabla u_s h(y - \cdot)\|_{\beta-1,2}^2 dy ds \\ &\quad + \int_0^r \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{\beta-1,2} \tilde{W}(dsdy). \end{aligned} \tag{2.5}$$

We first apply (2.5) for $\beta = 0$. The following lemmas will be used in Theorem 2.3.

Lemma 2.1 *If $h \in H_2^1$, then for any $u \in H_2^0$,*

$$\int_{\mathbb{R}} \|\nabla u h(y - \cdot)\|_{-1,2}^2 dy \leq \|h\|_{1,2}^2 \|u\|_{0,2}^2.$$

Proof Note that

$$\begin{aligned} \int_{\mathbb{R}} \|\nabla u h(y - \cdot)\|_{-1,2}^2 dy &= \int_{\mathbb{R}} \sup_{\|f\|_{1,2} \leq 1} \langle \nabla u, h(y - \cdot) f \rangle^2 dy \\ &= \int_{\mathbb{R}} \sup_{\|f\|_{1,2} \leq 1} \langle u, f' h(y - \cdot) - f h'(y - \cdot) \rangle_{0,2}^2 dy \\ &\leq \int_{\mathbb{R}} \sup_{\|f\|_{1,2} \leq 1} (\|u h(y - \cdot)\|_{0,2} \|f'\|_{0,2} \\ &\quad + \|u h'(y - \cdot)\|_{0,2} \|f\|_{0,2})^2 dy \\ &\leq \int_{\mathbb{R}} (\|u h(y - \cdot)\|_{0,2}^2 + \|u h'(y - \cdot)\|_{0,2}^2) dy \\ &= \|h\|_{1,2}^2 \|u\|_{0,2}^2, \end{aligned}$$

where the first inequality follows from the definition of the norm using duality. □

Lemma 2.2 *If $h \in H_2^2$, then for any $u \in H_2^{-1}$,*

$$\int_{\mathbb{R}} \langle u, \nabla u h(y - \cdot) \rangle_{-1,2}^2 dy \leq K \|u\|_{-1,2}^2.$$

Proof Let $g = (I - \Delta)^{-1}u$. By integration by parts, we get

$$\begin{aligned} \langle u, f \nabla u \rangle_{-1,2} &= \langle (I - \Delta)^{-1}u, f \nabla u \rangle_{0,2} = -\langle f'g, g \rangle_{0,2} - \langle (f + f'')g, g \rangle_{0,2} \\ &\quad + \langle f'g', g' \rangle_{0,2} - \langle f'g', g' \rangle_{0,2}. \end{aligned} \tag{2.6}$$

By Lemma 3.2 in [9], we get

$$\int_{\mathbb{R}} \langle h(y - \cdot)g', g'' \rangle_{0,2}^2 dy \leq K \|h'\|_{0,2}^2 \|g'\|_{0,2}^2.$$

Applying Cauchy–Schwartz inequality to the other terms of (2.6) with f replaced by $h(y - \cdot)$, we have

$$\int_{\mathbb{R}} \langle u, \nabla u h(y - \cdot) \rangle_{-1,2}^2 dy \leq K \|h\|_{2,2}^2 \|g'\|_{0,2}^2. \tag{2.7}$$

As

$$\|g'\|_{0,2} = \|\nabla(I - \Delta)^{-1}u\|_{0,2} \leq K \|(I - \Delta)^{-\frac{1}{2}}u\|_{0,2} = K \|u\|_{-1,2},$$

the conclusion of the lemma then follows from (2.7), where the last inequality follows from the boundedness of the operator $\nabla(I - \Delta)^{-\frac{1}{2}}$ (cf. [8]). □

Theorem 2.3 *For $p \geq 1$, $u_0 \in H_2^{-1}$, $h \in H_2^2$ and $\|h\|_{1,2}^2 < 2$, we have*

$$\mathbb{E} \sup_{t \leq T} \|u_t\|_{-1,2}^{2p} + \mathbb{E} \left(\int_0^T \|u_t\|_{0,2}^2 dt \right)^p \leq K \|u_0\|_{-1,2}^{2p}. \tag{2.8}$$

Proof In this proof we adapt the arguments of Kurtz and Xiong [9] for the norm given by (2.3). Using a smoothing technique as in [9] if necessary, we may assume that $u_t \in H_2^0$. Note that for $u \in H_2^0$, by (2.3) we have

$$\langle u, \Delta u \rangle_{-1,2} = \langle u, \Delta u - u \rangle_{-1,2} + \|u\|_{-1,2}^2 = -\|u\|_{0,2}^2 + \|u\|_{-1,2}^2. \tag{2.9}$$

By Lemma 2.1 we then have

$$2 \langle u, \Delta u \rangle_{-1,2} + \int_{\mathbb{R}} \|\nabla u h(y - \cdot)\|_{-1,2}^2 dy \leq - \left(2 - \|h\|_{1,2}^2\right) \|u\|_{0,2}^2 + 2\|u\|_{-1,2}^2. \tag{2.10}$$

It follows from Lemma 2.2 that

$$\begin{aligned} & \mathbb{E} \sup_{t \leq r} \left| \int_0^t \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{-1,2} \tilde{W}(ds dy) \right|^p \\ & \leq K \mathbb{E} \left(\int_0^r \int_{\mathbb{R}} \langle u_s, \nabla u_s h(y - \cdot) \rangle_{-1,2}^2 ds dy \right)^{p/2} \\ & \leq K \mathbb{E} \int_0^r \|u_s\|_{-1,2}^{2p} ds. \end{aligned} \tag{2.11}$$

Using (2.5) with $\beta = 0$, together with (2.10) and (2.11) we have

$$\mathbb{E} \sup_{s \leq r} \|u_s\|_{-1,2}^{2p} + \mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p \leq K \mathbb{E} \|u_0\|_{-1,2}^{2p} + K \int_0^r \mathbb{E} \|u_s\|_{-1,2}^{2p} ds. \tag{2.12}$$

Removing the second term on the LHS of (2.12), we get

$$\mathbb{E} \sup_{s \leq r} \|u_s\|_{-1,2}^{2p} \leq K_1 \|u_0\|_{-1,2}^{2p} + K_2 \int_0^r \mathbb{E} \|u_s\|_{-1,2}^{2p} ds.$$

It follows from Gronwall’s inequality that

$$\mathbb{E} \sup_{s \leq r} \|u_s\|_{-1,2}^{2p} \leq K_1 \|u_0\|_{-1,2}^{2p} e^{K_2 r}.$$

Removing the first term on the LHS of (2.12), we get

$$\mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p \leq K_1 \|u_0\|_{-1,2}^{2p} + K_2 \int_0^r K_1 \|u_0\|_{-1,2}^{2p} e^{K_2 r} ds \leq K_3 \|u_0\|_{-1,2}^{2p}.$$

□

In most applications, we shall take $u_0 = \delta_y$ or $u_0 = \delta_{y_1} - \delta_{y_2}$. It is well known that for any $y \in \mathbb{R}$, $\delta_y \in H_2^{-\alpha}$ for any $\alpha > \frac{1}{2}$ (cf. Example 1 of Section 5.2 in the book of Barros-Neto [1]). This justifies the applicability of the last theorem.

We will prove a stronger version of Theorem 2.3 which is useful in estimating time-increment of the random field $X_t(y)$. To this end, we need the following two lemmas.

Lemma 2.4 For $h \in H_2^1$ and for any $\alpha \in (0, \frac{1}{2}]$ there exists a constant K such that

$$\int_{\mathbb{R}} \|\nabla u h(y - \cdot)\|_{-2\alpha, 2}^2 dy \leq \frac{3}{2} \|\nabla u\|_{-2\alpha, 2}^2 + K \|u\|_{0, 2}^2.$$

Proof Note that

$$\begin{aligned} I(t, x, y) &\equiv T_t(u'h(y - \cdot))(x) - T_t(u')(x)T_t h(y - \cdot)(x) \\ &= \frac{1}{2} \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u'(z_1) - u'(z_2)) (h(y - z_1) - h(y - z_2)) \\ &\quad \times \varphi_t(x - z_1)\varphi_t(x - z_2) \\ &= \frac{1}{2} \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1) - u(z_2)) (h'(y - z_1) - h'(y - z_2)) \\ &\quad \times \varphi_t(x - z_1)\varphi_t(x - z_2) \\ &\quad + \frac{1}{2} \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1) - u(z_2)) (h(y - z_1) - h(y - z_2)) \\ &\quad \times \frac{z_1 - z_2}{t} \varphi_t(x - z_1)\varphi_t(x - z_2). \end{aligned}$$

Then

$$\begin{aligned} &\int_{\mathbb{R}} dx \int_{\mathbb{R}} dy |I(t, x, y)|^2 \\ &\leq K \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1) - u(z_2))^2 (h'(y - z_1) - h'(y - z_2))^2 \\ &\quad \times \varphi_t(x - z_1)\varphi_t(x - z_2) \\ &\quad + K \int_{\mathbb{R}} dx \int_{\mathbb{R}} dy \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1) - u(z_2))^2 \left| \frac{1}{z_1 - x} \int_x^{z_1} h'(y - z) dz \right|^2 \\ &\quad \times \frac{|z_1 - z_2|^2}{t} \varphi_t(x - z_1)\varphi_t(x - z_2) \\ &\leq K \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)^2 + u(z_2)^2) \varphi_{2t}(z_1 - z_2) \end{aligned}$$

$$\begin{aligned}
 &+K \int_{\mathbb{R}} dz_1 \int_{\mathbb{R}} dz_2 (u(z_1)^2 + u(z_2)^2) \varphi_{(2+\epsilon)t}(z_1 - z_2) \\
 &= K \|u\|_{0,2}^2,
 \end{aligned}$$

where the constant K depends on $\|h'\|_{0,2}^2$ and we used the inequality

$$\begin{aligned}
 \frac{x^2}{t} \varphi_t(x) \varphi_{(1+\epsilon)t}(x)^{-1} &= \frac{x^2}{t} \sqrt{1+\epsilon} \exp\left(\frac{x^2}{2(1+\epsilon)t} - \frac{x^2}{4t}\right) \\
 &= \sqrt{1+\epsilon} \frac{x^2}{t} \exp\left(-\frac{\epsilon x^2}{4t(1+\epsilon)}\right) \\
 &\leq \sqrt{1+\epsilon} \sup_{y \geq 0} \left(y \exp\left(-\frac{\epsilon}{2(1+\epsilon)}y\right)\right) \equiv K(\epsilon).
 \end{aligned}$$

On the other hand,

$$\begin{aligned}
 &\int_{\mathbb{R}} \int_{\mathbb{R}} dx dy \left| \int_0^\infty t^{\alpha-1} e^{-t} T_t(u')(x) T_t h(y - \cdot)(x) dt \right|^2 \\
 &= \int_{\mathbb{R}} \int_{\mathbb{R}} dx dy \int_0^\infty \int_0^\infty ds dt (ts)^{\alpha-1} e^{-(t+s)} T_s(u')(x) T_t(u')(x) T_t h(y - \cdot)(x) \\
 &\quad \times T_s h(y - \cdot)(x) \\
 &\leq \rho(0) \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty ds dt (ts)^{\alpha-1} e^{-(t+s)} T_s(u')(x) T_t(u')(x) \\
 &= \|\nabla u\|_{-2\alpha,2}^2.
 \end{aligned}$$

By the triangular inequality, we have

$$\begin{aligned}
 \left(\int_{\mathbb{R}} \|\nabla u h(y - \cdot)\|_{-2\alpha,2}^2 dy \right)^{1/2} &\leq \left(\int_{\mathbb{R}} \int_{\mathbb{R}} dx dy \int_0^\infty t^{\alpha-1} e^{-t} I(t, x, y)^2 dt \right)^{1/2} \\
 &\quad + \|\nabla u\|_{-2\alpha,2} \\
 &\leq K \|u\|_{0,2} + \|\nabla u\|_{-2\alpha,2}.
 \end{aligned}$$

The conclusion then follows from the elementary inequality $(a + b)^2 \leq \frac{3}{2}a^2 + 3b^2$. □

Lemma 2.5 For $h \in H_2^1$, there exists a constant K such that for any $0 \leq u \in H_2^0$,

$$\int_{\mathbb{R}} \langle u, \nabla u h(y - \cdot) \rangle_{-2\alpha,2}^2 dy \leq K \|u\|_{-2\alpha,2}^2 \|u\|_{0,2}^2.$$

Proof Note that

$$\begin{aligned} & \left(\int_{\mathbb{R}} \langle u, \nabla u h(y - \cdot) \rangle_{-2\alpha, 2}^2 dy \right)^{1/2} \\ &= \left(\int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_s (u' h(y - \cdot))(x) \right)^2 dy \right)^{1/2} \\ &\leq \sqrt{I_1} + \sqrt{I_2}, \end{aligned}$$

where

$$I_1 = \int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_s u'(x) h(y - x) \right)^2 dy$$

and

$$\begin{aligned} I_2 &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds \right. \\ &\quad \left. \times T_t u(x) (T_s (u' h(y - \cdot))(x) - T_s u'(x) h(y - x)) \right)^2 dy. \end{aligned}$$

By integration by parts and changing the order of T_s and ∇ , we get

$$\begin{aligned} & \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_s u'(x) h(y - x) \\ &= \frac{1}{2} \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) h'(y - x) T_s u(x). \end{aligned}$$

Thus,

$$\begin{aligned} I_1 &\leq K \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_s u(x) \\ &\quad \times \int_{\mathbb{R}} dx' \int_0^\infty \int_0^\infty (t's')^{\alpha-1} e^{-(t'+s')} dt' ds' T_{t'} u(x') T_{s'} u(x') \\ &= K \|u\|_{-2\alpha, 2}^4 \leq K \|u\|_{-2\alpha, 2}^2 \|u\|_{0, 2}^2. \end{aligned}$$

Note that we used the non-negativity of $u(x)$ in the inequality above.

Now we estimate I_2 . Note that

$$I_2 = \int_{\mathbb{R}^2} d(x, x') \int_{(0, \infty)^4} d(t, s, t', s') (ts t' s')^{\alpha-1} e^{-(t+s+t'+s')} T_t u(x) T_{t'} u(x') J(x, x'),$$

where

$$\begin{aligned} J(x, x') &= \int_{\mathbb{R}} dy \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) (h(y-z) - h(y-x)) u'(z) \\ &\quad \times \varphi_{s'}(x'-z') (h(y-z') - h(y-x')) u'(z') \\ &= \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \varphi_{s'}(x'-z') u'(z) u'(z') \\ &\quad \times (\rho(z-z') - \rho(x-z') - \rho(z-x') + \rho(x-x')). \end{aligned}$$

By integration by parts again, we can continue with

$$\begin{aligned} J(x, x') &\equiv \int_{\mathbb{R}^2} dz dz' \nabla_z \varphi_s(x-z) \nabla_{z'} \varphi_{s'}(x'-z') u(z) u(z') \\ &\quad \times (\rho(z-z') - \rho(x-z') - \rho(z-x') + \rho(x-x')) \\ &\quad + \int_{\mathbb{R}^2} dz dz' \nabla_z \varphi_s(x-z) \varphi_{s'}(x'-z') (\rho'(x-z') - \rho'(z-z')) u(z) u(z') \\ &\quad + \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \nabla_{z'} \varphi_{s'}(x'-z') (\rho'(z-z') - \rho'(z-x')) u(z) u(z') \\ &\quad - \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \varphi_{s'}(x'-z') \rho''(z-z') u(z) u(z') \\ &\equiv J_1 + J_2 + J_3 + J_4. \end{aligned}$$

Note that for some $\epsilon > 0$ we have

$$J_2 \leq K \int_{\mathbb{R}^2} dz dz' \varphi_s(x-z) \frac{|x-z|^2}{s} \varphi_{s'}(x'-z') u(z) u(z') \leq K_1 T_{(1+\epsilon)s} u(x) T_{s'} u(x').$$

Let $\mathcal{G}u(x) = \int_0^\infty s^{\alpha-1} e^{-s} T_{(1+\epsilon)s} u(x) ds$. Then the corresponding term of J_2 in I_2 is bounded (up to a constant multiplication) by

$$\|u\|_{-2\alpha, 2}^2 \int_{\mathbb{R}} dx \int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds T_t u(x) T_{(1+\epsilon)s} u(x)$$

$$\begin{aligned}
 &= \|u\|_{-2\alpha,2}^2 \langle (I - \Delta)^{-\alpha} u, \mathcal{G}u \rangle_{0,2} \\
 &\leq \|u\|_{-2\alpha,2}^3 \|\mathcal{G}u\|_{0,2} \\
 &\leq \|u\|_{-2\alpha,2}^3 \left(\int_0^\infty \int_0^\infty (ts)^{\alpha-1} e^{-(t+s)} dt ds \|u\|_{0,2}^2 \right)^{1/2} \\
 &\leq K \|u\|_{-2\alpha,2}^3 \|u\|_{0,2} \leq K \|u\|_{-2\alpha,2}^2 \|u\|_{0,2}^2.
 \end{aligned}$$

The other terms can be estimated similarly. □

The following estimate will be used in the arguments in Sect. 5.

Theorem 2.6 *Suppose that the conditions of Theorem 2.3 hold. Then for $\beta \in [0, 1/2]$, $u_0 = \delta_z$ and $p \geq 1$ we have*

$$\mathbb{E} \sup_{t \leq T} \|u_t\|_{\beta-1,2}^{2p} + \mathbb{E} \left(\int_0^T \|u_t\|_{\beta,2}^2 dt \right)^p \leq K \|\delta_z\|_{\beta-1,2}^{2p}. \tag{2.13}$$

Proof Similar to Theorem 2.3 we may assume that $u_t \in H_2^\beta$ a.s.. Further, using a stopping argument if necessary we may and will assume that the LHS of (2.13) is finite. Denote $1 - \beta = 2\alpha$ for simplicity. By (2.5) and Lemma 2.4 we get

$$\begin{aligned}
 \|u_r\|_{-2\alpha,2}^2 &\leq \|u_0\|_{-2\alpha,2}^2 - \frac{1}{2} \int_0^r \|\nabla u_s\|_{-2\alpha,2}^2 ds + 3 \int_0^r \|u_s\|_{0,2}^2 ds \\
 &\quad + \int_0^r \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{\beta-1,2} \tilde{W}(ds dy) \\
 &\leq \|u_0\|_{-2\alpha,2}^2 - \frac{1}{2} \int_0^r \|u_s\|_{1-2\alpha,2}^2 ds + \frac{7}{2} \int_0^r \|u_s\|_{0,2}^2 ds \\
 &\quad + \int_0^r \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{\beta-1,2} \tilde{W}(ds dy),
 \end{aligned}$$

where the last inequality follows from

$$\begin{aligned}
 \|\nabla u\|_{-2\alpha,2}^2 &= d(\alpha)^2 \left\langle \int_0^\infty t^{\alpha-1} e^{-t} T_t u' dt, \int_0^\infty t^{\alpha-1} e^{-t} T_t u' dt \right\rangle_{0,2} \\
 &= -d(\alpha)^2 \left\langle \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt, \Delta \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt \right\rangle_{0,2}
 \end{aligned}$$

$$\begin{aligned}
 &= d(\alpha)^2 \left\langle \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt, (I - \Delta) \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt \right\rangle_{0,2} \\
 &\quad - d(\alpha)^2 \left\langle \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt, \int_0^\infty t^{\alpha-1} e^{-t} T_t u dt \right\rangle_{0,2} \\
 &= \|(I - \Delta)^{\frac{1}{2}} (I - \Delta)^{-\alpha} u\|_2^2 - \|(I - \Delta)^{-\alpha} u\|_2^2 \\
 &= \|u\|_{1-2\alpha,2}^2 - \|u\|_{-2\alpha,2}^2 \geq \|u\|_{1-2\alpha,2}^2 - \|u\|_{0,2}^2.
 \end{aligned}$$

Thus,

$$\begin{aligned}
 &\mathbb{E} \sup_{t \leq r} \|u_t\|_{\beta-1,2}^{2p} + \mathbb{E} \left(\int_0^r \|u_t\|_{\beta,2}^2 dt \right)^p \\
 &\leq K \|u_0\|_{-2\alpha,2}^{2p} + K \mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p + K \mathbb{E} \left(\int_0^r \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{\beta-1,2}^2 dy ds \right)^{p/2} \\
 &\leq K_1 \|u_0\|_{-2\alpha,2}^{2p} + K \mathbb{E} \left(\int_0^r \int_{\mathbb{R}} 2 \langle u_s, \nabla u_s h(y - \cdot) \rangle_{\beta-1,2}^2 dy ds \right)^{p/2},
 \end{aligned}$$

where the last inequality follows from Theorem 2.3 and the fact $\|u_0\|_{-1,2} \leq \|u_0\|_{-2\alpha,2}$.
 By Lemma 2.5, we get

$$\begin{aligned}
 &\mathbb{E} \sup_{s \leq r} \|u_s\|_{\beta-1,2}^{2p} + \mathbb{E} \left(\int_0^r \|u_s\|_{\beta,2}^2 ds \right)^p \\
 &\leq K \|\delta_z\|_{\beta-1,2}^{2p} + K \mathbb{E} \left(\int_0^r \|u_s\|_{\beta-1,2}^2 \|u_s\|_{0,2}^2 ds \right)^{p/2} \\
 &\leq K \|\delta_z\|_{\beta-1,2}^{2p} + K \mathbb{E} \left(\sup_{s \leq r} \|u_s\|_{\beta-1,2}^2 \int_0^r \|u_s\|_{0,2}^2 ds \right)^{p/2} \\
 &\leq K \|\delta_z\|_{\beta-1,2}^{2p} + \frac{1}{2} \mathbb{E} \sup_{s \leq r} \|u_s\|_{\beta-1,2}^{2p} + 8K^2 \mathbb{E} \left(\int_0^r \|u_s\|_{0,2}^2 ds \right)^p \\
 &\leq K \|\delta_z\|_{\beta-1,2}^{2p} + \frac{1}{2} \mathbb{E} \sup_{s \leq r} \|u_s\|_{\beta-1,2}^{2p} + K_1 \|\delta_z\|_{-1,2}^{2p}.
 \end{aligned}$$

The conclusion then follows from easy calculations. □

3 A convolution representation

In this section, we establish a convolution representation for the density $X_t(x)$ in terms of a random transition function. We first define the random transition function by considering the spatial motion of a typical particle in the system, which satisfies

$$\xi_t = \xi_0 + B_t + \int_0^t \int_{\mathbb{R}} h(y - \xi_s) W(dsdy).$$

For $r \leq t$ and $x \in \mathbb{R}$ fixed, we define the conditional transition probability

$$p_t^{r,x,W}(\cdot) \equiv p^W(r, x; t, \cdot) \equiv P^W(\xi_t \in \cdot | \xi_r = x).$$

Then for r and x fixed $p_t^{r,x,W}$ can be regarded as the optimal filter with vanishing observation function. Thus, it is a $\mathcal{P}(\mathbb{R})$ -valued process satisfying the Zakai equation

$$\left\langle p_t^{r,x,W}, f \right\rangle = f(x) + \int_r^t \left\langle p_s^{r,x,W}, \Delta f \right\rangle ds + \int_r^t \int_{\mathbb{R}} \left\langle p_s^{r,x,W}, \nabla f h(y - \cdot) \right\rangle W(dsdy), \tag{3.1}$$

where $\mathcal{P}(\mathbb{R})$ is the space of Borel probability measures on \mathbb{R} . We refer the reader to the books of Kallianpur [5] and Xiong [17] for an introduction to nonlinear filtering and the related Zakai equation.

Next, we consider the dual equation on $C_b(\mathbb{R})$:

$$T_{r,t}(x) = f(x) + \int_r^t \Delta T_{s,t}(x) ds + \int_r^t \int_{\mathbb{R}} \nabla T_{s,t}(x) h(y - x) W(\hat{d}sdy), \tag{3.2}$$

where $\hat{d}s$ stands for the backward Itô integral. We refer to Li et al. [12] for the definition of the backward Itô integral. We also denote $T_{r,t}(x)$ by $T_{r,t}^f(x)$ to indicate the dependence on f . Similar to Corollary 6.22 in Xiong [17] it is easy to show that

$$T_{s,t}^f(x) = \int_{\mathbb{R}} f(y) p^W(s, x; t, dy) = \mathbb{E}_{s,x}^W f(\xi_t), \tag{3.3}$$

where $\mathbb{E}_{s,x}^W$ denotes the conditional expectation given W and $\xi_s = x$.

The following convolution representation is the key in proving the joint continuity of $X_t(y)$. We shall denote $Z(dsdx) \equiv \sqrt{X_s(x)} B(dsdx)$.

Lemma 3.1 *Suppose that X_0 satisfies condition (I) and $f \in C_b^2(\mathbb{R})$. Then we have*

$$\langle X_t, f \rangle = \langle X_0, T_{0,t} \rangle + \int_0^t \int_{\mathbb{R}} T_{s,t}(x) Z(dsdx). \tag{3.4}$$

Proof Similar to Theorem 2.1 in Lee et al. [11], we can prove that $X_t \in H_2^0$ and

$$\sup_{s \leq t} \mathbb{E} \|X_s\|_{0,2}^2 < \infty.$$

Denote the RHS of (3.4) by $\langle Y_t, f \rangle$. It is easy to show that Y_t is an H_2^0 -valued process. Note that for $f \in C_b^2(\mathbb{R})$ we have

$$\begin{aligned} & \langle Y_t, f \rangle - \langle X_0, f \rangle - \int_0^t \langle Y_s, \Delta f \rangle ds - \int_0^t \int_{\mathbb{R}} \langle Y_s, h(y - \cdot) \nabla f \rangle W(dsdy) \\ &= \left\langle X_0, T_{0,t}^f \right\rangle + \int_0^t \int_{\mathbb{R}} T_{s,t}^f(x) Z(dsdx) - \langle X_0, f \rangle \\ & \quad - \int_0^t \left\{ \left\langle X_0, T_{0,s}^{\Delta f} \right\rangle + \int_0^s \int_{\mathbb{R}} T_{r,s}^{\Delta f}(x) Z(dr dx) \right\} ds \\ & \quad - \int_0^t \int_{\mathbb{R}} \left\{ \left\langle X_0, T_{0,s}^{h(y-\cdot)\nabla f} \right\rangle + \int_0^s \int_{\mathbb{R}} T_{r,s}^{h(y-\cdot)\nabla f}(x) Z(dr dx) \right\} W(dsdy) \\ &= \left\langle X_0, T_{0,t}^f - f - \int_0^t T_{0,s}^{\Delta f} ds - \int_0^t \int_{\mathbb{R}} T_{0,s}^{h(y-\cdot)\nabla f} W(dsdy) \right\rangle \\ & \quad + \int_0^t \int_{\mathbb{R}} \mathbb{E}_{s,x}^W f(\xi_t) Z(dsdx) - \int_0^t \int_{\mathbb{R}} \int_r^t \mathbb{E}_{r,x}^W \Delta f(\xi_s) ds Z(dr dx) \\ & \quad - \int_0^t \int_{\mathbb{R}} \int_r^t \int_{\mathbb{R}} \mathbb{E}_{r,x}^W (h(y - \xi_s) \nabla f(\xi_s)) W(dsdy) Z(dr dx) \\ &= \int_{\mathbb{R}} X_0(dx) \mathbb{E}_{0,x}^W \left(f(\xi_t) - f(x) - \int_0^t \Delta f(\xi_s) ds - \int_0^t \int_{\mathbb{R}} h(y - \xi_s) \nabla f(\xi_s) W(dsdy) \right) \end{aligned}$$

$$\begin{aligned}
 & + \int_0^t \int_{\mathbb{R}} Z(dsdx) \mathbb{E}_{s,x}^W \left\{ f(\xi_t) - \int_s^t \Delta f(\xi_r) dr - \int_s^t \int_{\mathbb{R}} h(y - \xi_r) \nabla f(\xi_r) W(dr dy) \right\} \\
 & = \int_0^t \int_{\mathbb{R}} f(x) Z(dsdx).
 \end{aligned}$$

Let $\tilde{X}_t = X_t - Y_t$. By (1.5) \tilde{X} is an H_2^0 -valued solution to the following linear SDE

$$\langle \tilde{X}_t, f \rangle = \int_0^t \langle \tilde{X}_s, \Delta f \rangle ds + \int_0^t \int_{\mathbb{R}} \langle \tilde{X}_s, h(y - \cdot) \nabla f \rangle W(ds dy). \tag{3.5}$$

By Theorem 3.5 in Kurtz and Xiong [9] we have that $\tilde{X} = 0$. □

4 An estimate in spatial increment

In this section we estimate spatial increment of the density $X_t(y)$. As a consequence, we shall see that for $t > 0$ fixed, $X_t(y)$ is Hölder continuous with exponent $1/2 - \epsilon$.

Applying Theorem 2.3 to (3.1), we see that $p^W(s, x; t, \cdot)$ has a density, denote it by $p^W(s, x; t, y)$. By Lemma 3.1, $X_t(y)$ can be represented as

$$X_t(y) = \int_{\mathbb{R}} \mu(x) p^W(0, x; t, y) dx + \int_0^t \int_{\mathbb{R}} p^W(s, x; t, y) Z(dsdx) \equiv X_t^1(y) + X_t^2(y). \tag{4.1}$$

To prove the joint continuity by Kolmogorov’s criteria, we need the following estimate.

Lemma 4.1 *Suppose that Condition (I) holds. Then $\forall p \geq 1$,*

$$\begin{aligned}
 & \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2)) Z(dsdx) \right|^{2p} \\
 & \leq K \left(\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 dx ds \right|^{2p-1} \right)^{\frac{p}{2p-1}}. \tag{4.2}
 \end{aligned}$$

Proof By BDG inequality, we have

$$\begin{aligned}
 L &\equiv \mathbb{E} \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2)) Z(ds dx) \right|^{2p} \\
 &\leq K \mathbb{E} \mathbb{E}^W \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 X_s(x) dx ds \right|^p.
 \end{aligned}$$

For $2 = (2p - 1)/p + 1/p$, applying the Cauchy–Schwarz inequality we have

$$\begin{aligned}
 L &\leq K \mathbb{E} \left(\left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 dx ds \right|^{\frac{2p-1}{2}} \right. \\
 &\quad \times \left. \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 X_s(x)^{2p} dx ds \right|^{\frac{1}{2}} \right) \\
 &\leq K \left(\mathbb{E} \left| \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 dx ds \right|^{2p-1} \right)^{\frac{1}{2}} \\
 &\quad \times \left(\mathbb{E} \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 X_s(x)^{2p} dx ds \right)^{\frac{1}{2}} \\
 &\equiv KI \times J.
 \end{aligned}$$

Since μ is bounded, it is easy to show that

$$\sup_{t,x} \langle \mu, \varphi_t(x - \cdot) \rangle < \infty. \tag{4.3}$$

It then follows from the same arguments as in the proof of Lemma 3.1 of Lee et al. [11] that $\mathbb{E} X_s(x)^{2p}$ is bounded. Therefore,

$$J \leq K \left(\mathbb{E} \int_0^t \int_{\mathbb{R}} (p^W(s, x; t, y_1) - p^W(s, x; t, y_2))^2 dx ds \right)^{\frac{1}{2}} \leq KI^{1/(2p-1)}.$$

Thus, $L \leq KI^{2p/(2p-1)}$ which coincides with the RHS of (4.2). □

As a consequence of Theorem 2.3, we get

Proposition 4.2 *Suppose the conditions of Theorem 1.1 hold. Let $t \in [0, T]$ and $p \geq 1$ be fixed. Then, there exists a constant $K = K(p, T)$ such that*

$$\mathbb{E}|X_t^2(y_1) - X_t^2(y_2)|^{2p} \leq K|y_1 - y_2|^p, \quad \forall y_1, y_2 \in \mathbb{R}. \tag{4.4}$$

Consequently, for $t > 0$ fixed X_t^2 is Hölder continuous with exponent $1/2 - \epsilon$ for any $\epsilon > 0$.

Proof Let $u_s(x) = p^W(t - s, x, t, y_1) - p^W(t - s, x, t, y_2)$. Then u solves equation (1.9) with $u_0 = \delta_{y_1} - \delta_{y_2}$. For any $f \in H^1_2$ we have

$$|\langle u_0, f \rangle| = |f(y_1) - f(y_2)| = \left| \int_{y_1}^{y_2} f'(s) ds \right| \leq \sqrt{|y_2 - y_1|} \|f\|_{1,2}.$$

Thus,

$$\|u_0\|_{-1,2} \leq \sqrt{|y_2 - y_1|}. \tag{4.5}$$

By Theorem 2.3 we get

$$\mathbb{E} \left(\int_0^t \int_{\mathbb{R}} |p^W(s, x, t, y_1) - p^W(s, x, t, y_2)|^2 dx ds \right)^p \leq K|y_1 - y_2|^p.$$

Inequality (4.4) then follows from Lemma 4.1. □

Finally, we consider $X_t^1(y)$.

Proposition 4.3 *Suppose the conditions of Theorem 1.1 hold. Let $t \in [0, T]$. Then, for $p \geq 1$, there exists a constant $K = K(p, T)$ such that*

$$\mathbb{E}|X_t^1(y_1) - X_t^1(y_2)|^{2p} \leq K|y_1 - y_2|^p.$$

Proof Note that

$$\begin{aligned} \mathbb{E}|X_t^1(y_1) - X_t^1(y_2)|^{2p} &= \mathbb{E} \left| \int_{\mathbb{R}} \left(p^W(0, x; t, y_1) - p^W(0, x; t, y_2) \right) \mu(x) dx \right|^{2p} \\ &\leq \mathbb{E} \|p^W(0, \cdot; t, y_1) - p^W(0, \cdot; t, y_2)\|_{-1,2}^{2p} \|\mu\|_{1,2}^{2p} \\ &\leq K_1 \|\delta_{y_1} - \delta_{y_2}\|_{-1,2}^{2p} \|\mu\|_{1,2}^{2p}. \end{aligned}$$

The conclusion then follows from (4.5). □

5 Estimates in time increment

In this section we consider time-increments of the types of

$$\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s, x; t_2, y) - p^W(s, x; t_1, y) \right) Z(dsdx) \tag{5.1}$$

and

$$\int_{t_1}^{t_2} \int_{\mathbb{R}} p^W(s, x; t_2, y) Z(dsdx). \tag{5.2}$$

For the type of (5.1), we first use Theorem 2.3 to obtain a preliminary estimate by $\mathbb{E}\|u_{t_2-t_1} - \delta_y\|_{-1,2}^{2p}$, where u_t is a solution to SDE (1.9) with $u_0 = \delta_y$. To further estimate this quantity, we need to develop two major techniques, i.e., the partial convolution by kernel φ_{r^α} and the partial integration by parts introduced in Sect. 1. For the type of (5.2), we will use a technique developed by Xiong and Zhou [18].

Lemma 5.1 *For any $t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$\begin{aligned} & \mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s, x; t_2, y) - p^W(s, x; t_1, y) \right) Z(dsdx) \right)^{2p} \\ & \leq K \mathbb{E} \|p^W(t_1, \cdot; t_2, y) - \delta_y\|_{-1,2}^{2p}. \end{aligned}$$

Proof Note that $p^W(t_1 - s, x; t_2, y) - p^W(t_1 - s, x; t_1, y)$ is the solution of SPDE (1.9) with initial condition $p^W(t_1, \cdot; t_2, y) - \delta_y$ and hence,

$$\begin{aligned} & \mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s, x; t_2, y) - p^W(s, x; t_1, y) \right) Z(dsdx) \right)^{2p} \\ & \leq K \left(\mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s, x; t_2, y) - p^W(s, x; t_1, y) \right)^2 dsdx \right)^{2p-1} \right)^{\frac{p}{2p-1}} \\ & \leq K \mathbb{E} \|p^W(t_1, \cdot; t_2, y) - \delta_y\|_{-1,2}^{2p}. \end{aligned}$$

□

Let $u_s(x) = p^W(t_2 - s, x; t_2, y)$. Then u solves (1.9) with $u_0 = \delta_y$. As Δu_s is not in H_2^{-1} we cannot use (1.9) directly to get an estimate on $\mathbb{E}\|u_{t_2-t_1} - \delta_y\|_{-1,2}^{2p}$. Instead, fixing t and taking differential of $\int_{\mathbb{R}} u_{t-r}(x) \varphi_{r^\alpha}(z - x) dx$ with respect to r , and then

taking integral we get (1.10). Denote the second and the third term on the RHS by I_2 and I_3 , respectively. Write the fourth term by $I_4 - I_5$ with

$$I_4 = \alpha \int_0^t \int_{\mathbb{R}} (I - \Delta)u_{t-r}(x)\varphi_{r^\alpha}(z - x)dxr^{\alpha-1}dr$$

and

$$I_5 = \alpha \int_0^t \int_{\mathbb{R}} u_{t-r}(x)\varphi_{r^\alpha}(z - x)dxr^{\alpha-1}dr.$$

Then

$$u_t(z) - \delta_y(z) = I_1 + I_2 + I_3 + I_4 - I_5.$$

We now estimate $I_j, j = 1, 2, \dots, 5$, separately. Although the following result can be implied directly from the analyticity of Δ on $L^2(\mathbb{R})$, we give a brief and elementary proof for the convenience of the reader.

Lemma 5.2 For $\beta \in (0, 1)$ there is a constant such that for $r \in (0, T)$ we have

$$\int_{\mathbb{R}} |(I - \Delta)^\beta \varphi_r(x)| dx \leq Kr^{-\beta}. \tag{5.3}$$

Proof Note that the integral in the definition of $(I - \Delta)^\beta$ can be split up into two parts: I_1 denotes the part from 0 to r and I_2 from r to ∞ . Then

$$\int_{\mathbb{R}} |I_2(x)|dx \leq \int_r^\infty \frac{e^{-t} + 1}{t^{1+\beta}} dt \leq \frac{2}{\beta} r^{-\beta}.$$

For $t \leq r$, we have

$$\begin{aligned} & |e^{-t}\varphi_{t+r}(x) - \varphi_r(x)|\varphi_{t+r}(x)^{-1} \\ &= \left| e^{-t} - \sqrt{\frac{t+r}{r}} \exp\left(-\frac{x^2}{2r} + \frac{x^2}{2(t+r)}\right) \right| \\ &\leq |e^{-t} - 1| + \left| 1 - \sqrt{\frac{t+r}{r}} \right| + \sqrt{\frac{t+r}{r}} \left| 1 - \exp\left(-\frac{tx^2}{2r(t+r)}\right) \right| \\ &\leq \sqrt{2} \left(\frac{tx^2}{2r(r+t)} + t + \frac{t}{r} \right). \end{aligned}$$

Multiplying both sides by $\varphi_{t+r}(x)$ and taking integral we see that $\int_{\mathbb{R}} |I_1(x)| dx \leq Kr^{-\beta}$. □

Now we estimate I_4 . Note that

$$\begin{aligned} \|I_4\|_{-1,2} &\leq \alpha \int_0^t \left\| \int_{\mathbb{R}} (I - \Delta)^{\frac{1+\beta}{2}} u_{t-r}(\cdot - x) (I - \Delta)^{\frac{1-\beta}{2}} \varphi_{r^\alpha}(x) dx \right\|_{-1,2} r^{\alpha-1} dr \\ &\leq K \int_0^t \left\| (I - \Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2} \int_{\mathbb{R}} \left| (I - \Delta)^{\frac{1-\beta}{2}} \varphi_{r^\alpha}(x) \right| dx r^{\alpha-1} dr \\ &\leq K \int_0^t \left\| (I - \Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2} r^{-\frac{\alpha}{2}(1-\beta)} r^{\alpha-1} dr \\ &\leq K \left(\int_0^t \left\| (I - \Delta)^{\frac{1+\beta}{2}} u_{t-r} \right\|_{-1,2}^2 dr \right)^{\frac{1}{2}} \left(\int_0^t r^{\alpha(1+\beta)-2} dr \right)^{\frac{1}{2}} \\ &= K \left(\int_0^t \|u_r\|_{\beta,2}^2 dr \right)^{\frac{1}{2}} t^{(\alpha(1+\beta)-1)/2} \end{aligned}$$

where $\beta \in (0, 1/2)$ is chosen such that $\alpha(1 + \beta) > 1$. Thus, $\mathbb{E}\|I_4\|_{-1,2}^{2p} \leq Kt^{\alpha(1+\beta)-1}p$. I_2 and I_5 can be estimated similarly (easier).

Next, we estimate I_3 . Note that

$$\begin{aligned} &\int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} \nabla u_{t-r}(x) h(y - x) \varphi_{r^\alpha}(\cdot - x) dx \right\|_{-1,2}^2 dr dy \\ &= \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} u_{t-r}(x) \nabla h(y - x) \varphi_{r^\alpha}(\cdot - x) dx \right\|_{-1,2}^2 dr dy \\ &\quad + \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} u_{t-r}(x) h(y - x) \nabla \varphi_{r^\alpha}(\cdot - x) dx \right\|_{-1,2}^2 dr dy \equiv I_{31} + I_{32}. \end{aligned}$$

We calculate

$$I_{32} = \int_0^t \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} u_{t-r}(\cdot - x) h(y + x - \cdot) \nabla \varphi_{r^\alpha}(x) dx \right\|_{-1,2}^2 dr dy$$

$$\begin{aligned}
 &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \langle u_{t-r}(\cdot - x)h(y + x - \cdot), u_{t-r}(\cdot - x')h(y + x' - \cdot) \rangle_{-1,2} \\
 &\quad \times \nabla \varphi_{r^\alpha}(x) \nabla \varphi_{r^\alpha}(x') dx dx' dy dr \\
 &= \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_0^\infty \int_0^\infty (uv)^{-1/2} e^{-(u+v)} \\
 &\quad \times \int_{\mathbb{R}} \int_{\mathbb{R}} \varphi_u(z - z_1) u_{t-r}(z_1 - x) h(y + x - z_1) dz_1 \\
 &\quad \times \int_{\mathbb{R}} \varphi_v(z - z_2) u_{t-r}(z_2 - x') h(y + x' - z_2) dz_2 dz du dv \\
 &\quad \times \nabla \varphi_{r^\alpha}(x) \nabla \varphi_{r^\alpha}(x') dx dx' dy dr \\
 &\leq K \int_0^t \int_{\mathbb{R}} \int_{\mathbb{R}} \langle u_{t-r}(\cdot - x), u_{t-r}(\cdot - x') \rangle_{-1,2} |\nabla \varphi_{r^\alpha}(x)| |\nabla \varphi_{r^\alpha}(x')| dx dx' dr \\
 &\leq K \int_0^t \left(\int_{\mathbb{R}} \|u_{t-r}(\cdot - x)\|_{-1,2} |\nabla \varphi_{r^\alpha}(x)| dx \right)^2 dr \\
 &\leq K \sup_{r \leq t} \|u_r\|_{-1,2}^2 \int_0^t r^{-\alpha} dr \leq K \sup_{r \leq t} \|u_r\|_{-1,2}^2 t^{1-\alpha},
 \end{aligned}$$

where in the first inequality we used the identity (1.4) and $\rho(x) \leq 1$. I_{31} can be estimated similarly. Estimation for I_1 is easy. To summarize, we get

Proposition 5.3 *For $p \geq 1, \alpha \in (0, 1)$ and $\beta \in (0, 1/2)$ satisfying $\alpha(1 + \beta) > 1$, there exists a constant K such that $\forall t_1 < t_2$, we have*

$$\begin{aligned}
 &\mathbb{E} \left(\int_0^{t_1} \int_{\mathbb{R}} \left(p^W(s, x; t_2, y) - p^W(s, x; t_1, y) \right)^2 Z(ds dx) \right)^p \\
 &\leq K \max \left(|t_2 - t_1|^{(\alpha(1+\beta)-1)p}, |t_2 - t_1|^{(1-\alpha)p} \right).
 \end{aligned}$$

Finally, we estimate

$$\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}} p^W(s, x, t_2, y)^2 Z(ds dx) \right)^{2p}.$$

Similar to Sect. 4, the above moment is bounded by

$$\left(\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}} p^W(s, x, t_2, y)^2 dx ds \right)^{2p-1} \right)^{\frac{p}{2p-1}}$$

which we shall estimate using the method of Xiong and Zhou [18].

The key identity proved in [18] is given in the following lemma. We sketch the proof for convenience of the reader since [18] is not easily accessible.

Lemma 5.4 *For any $k \in \mathbb{N}$, $s < t$ and $x, y \in \mathbb{R}^k$, we have*

$$\mathbb{E} \Pi_{i=1}^k p^W(s, x_i, t, y_i) = P_k(t - s, x, y),$$

where P_k is the transition function of the k -dimensional Markov process consisting of the motion of k particles of the branching particles system introduced in Sect. 1.

Sketch of the proof Let t and y be fixed. We define $u_r^i(x_i) = p^W(t - r, x^i, t, y)$, $i = 1, 2, \dots, k$. Then u^i is a solution to (1.9) with initial δ_y . Applying Itô's formula to the product and taking expectation, we get

$$\frac{d}{dr} \mathbb{E} \Pi_{i=1}^k u_r^i(x^i) = A_k \mathbb{E} \Pi_{i=1}^k u_r^i(x^i)$$

where A_k is the generator of the k -dimensional Markov process consisting of the motion of k particles of the branching particles system. The conclusion of the lemma then follows easily. □

Lemma 5.5 *For any integer $n \geq 1$, we have*

$$\mathbb{E} \left(\int_{t_1}^{t_2} \int_{\mathbb{R}} p^W(s, x, t_2, y)^2 dx ds \right)^n \leq K |t_2 - t_1|^{n/2}. \tag{5.4}$$

Proof Let $t_1 = 0$ and $t_2 = t$ for simplicity. The LHS of (5.4) is estimated as follows.

$$\begin{aligned} L &\equiv n! \mathbb{E} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_1 \cdots dx_n \Pi_{i=1}^n p^W(s_i, x_i, t, y)^2 \\ &= n! \mathbb{E} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_1 \cdots dx_n \Pi_{i=2}^n p^W(s_i, x_i, t, y)^2 \\ &\quad \times \int_{\mathbb{R}} p^W(s_1, x_1, s_2, x_{11}) p^W(s_2, x_{11}, t, y) dx_{11} \int_{\mathbb{R}} p^W(s_1, x_1, s_2, x_{12}) p^W \\ &\quad \times (s_2, x_{12}, t, y) dx_{12} \end{aligned}$$

$$\begin{aligned}
 &= n! \mathbb{E} \int_0^t ds_1 \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_1 \cdots dx_n \Pi_{i=2}^n p^W(s_i, x_i, t, y)^2 \\
 &\quad \times \int_{\mathbb{R}} \int_{\mathbb{R}} P_2(s_2 - s_1, (x_1, x_1), (x_{11}, x_{12})) p^W(s_2, x_{11}, t, y) p^W(s_2, x_{12}, t, y) dx_{11} \\
 &\quad \times dx_{12},
 \end{aligned}$$

where the last equality follows from Lemma 5.4. Note that

$$P_2(s_2 - s_1, (x_1, x_1), (x_{11}, x_{12})) \leq \frac{K}{\sqrt{s_2 - s_1}} \varphi_{s_2 - s_1}(x_1 - x_{11}).$$

We now continue the estimate with

$$\begin{aligned}
 L &\leq K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_{11} dx_{12} dx_2 \cdots dx_n \\
 &\quad \times p^W(s_2, x_{11}, t, y) p^W(s_2, x_{12}, t, y) \Pi_{i=2}^n p^W(s_i, x_i, t, y)^2 \\
 &= K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_{11} dx_{12} dx_2 \cdots dx_n \\
 &\quad \times \int_{\mathbb{R}} p^W(s_2, x_{11}, s_3, x'_{11}) p^W(s_3, x'_{11}, t, y) dx'_{11} \\
 &\quad \times \int_{\mathbb{R}} p^W(s_2, x_{12}, s_3, x'_{12}) p^W(s_3, x'_{12}, t, y) dx'_{12} \\
 &\quad \times \int_{\mathbb{R}} p^W(s_2, x_2, s_3, x_{21}) p^W(s_3, x_{21}, t, y) dx_{21} \\
 &\quad \times \int_{\mathbb{R}} p^W(s_2, x_2, s_3, x_{22}) p^W(s_3, x_{22}, t, y) dx_{22} \Pi_{i=3}^n p^W(s_i, x_i, t, y)^2 \\
 &= K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t ds_2 \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_{11} dx_{12} dx_2 \cdots dx_n \\
 &\quad \times \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} \int_{\mathbb{R}} dx'_{11} dx'_{12} dx_{21} dx_{22} P_4(s_3 - s_2, (x_{11}, x_{12}, x_2, x_2), (x'_{11}, x'_{12}, x_{21}, x_{22})) \\
 &\quad \times p^W(s_3, x'_{11}, t, y) p^W(s_3, x'_{12}, t, y) p^W(s_3, x_{21}, t, y) p^W(s_3, x_{22}, t, y) \Pi_{i=3}^n p^W(s_i, x_i, t, y)^2,
 \end{aligned}$$

where the last equality follows again from Lemma 5.4. Note that

$$\begin{aligned}
 &P_4(s_3 - s_2, (x_{11}, x_{12}, x_2, x_2), (x'_{11}, x'_{12}, x_{21}, x_{22})) \\
 &\leq \frac{K}{\sqrt{s_3 - s_2}} \varphi_{s_3 - s_2}(x'_{11} - x_{11}) \varphi_{s_3 - s_2}(x'_{12} - x_{12}) \varphi_{s_3 - s_2}(x_{21} - x_2).
 \end{aligned}$$

Finally, we continue to estimate the LHS of (5.4) with

$$L \leq K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t \frac{ds_2}{\sqrt{s_3 - s_2}} \cdots \int_{s_{n-1}}^t ds_n \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx'_{11} dx'_{12} dx_{21} dx_{22} dx_3 \cdots dx_n \\ \times p^W(s_3, x'_{11}, t, y) p^W(s_3, x'_{12}, t, y) p^W(s_3, x_{21}, t, y) p^W(s_3, x_{22}, t, y) \prod_{i=3}^n p^W(s_i, x_i, t, y)^2.$$

Continue this procedure, we see that

$$L \leq K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t \frac{ds_2}{\sqrt{s_3 - s_2}} \cdots \int_{s_{n-1}}^t \frac{ds_n}{\sqrt{t - s_n}} \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} dx_{11} dx_{12} \cdots dx_{n1} dx_{n2} \\ \times \prod_{i=1}^n p^W(s_n, x_{i1}, t, y) p^W(s_n, x_{i2}, t, y) \\ \leq K \mathbb{E} \int_0^t \frac{ds_1}{\sqrt{s_2 - s_1}} \int_{s_1}^t \frac{ds_2}{\sqrt{s_3 - s_2}} \cdots \int_{s_{n-1}}^t \frac{ds_n}{\sqrt{t - s_n}} \\ \leq K t^{n/2}.$$

Thus we finish the proof by replacing t by $t_2 - t_1$. □

To summarize, we get

Proposition 5.6 *Suppose the conditions of Theorem 1.1 hold. Then, there exist integer $p \geq 1$ and real numbers $\epsilon > 0$ and $K > 0$ such that $\forall t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$\mathbb{E}|X_{t_1}^2(y) - X_{t_2}^2(y)|^{2p} \leq K |t_1 - t_2|^{2+\epsilon}. \tag{5.5}$$

Proof Choose $p \geq 2$, $\alpha \in (0, 1)$ and $\beta \in (0, \frac{1}{2})$ such that

$$\min \left\{ (\alpha(1 + \beta) - 1)p, (1 - \alpha)p, \frac{p}{2} \right\} \geq 2 + \epsilon.$$

By Proposition 5.3 and Lemma 5.5, we see that (5.5) holds. □

Note that

$$\mathbb{E}|X_{t_1}^1(y) - X_{t_2}^1(y)|^{2p} = \mathbb{E} \left| \int_{\mathbb{R}} \left(p^W(0, x; t_2, y) - p^W(0, x; t_1, y) \right) \mu(x) dx \right|^{2p} \\ \leq \mathbb{E} \| p^W(0, \cdot; t_2, y) - p^W(0, \cdot; t_1, y) \|_{-1,2}^{2p} \| \mu \|_{1,2}^{2p}.$$

Similar to the proof for $X_t^2(y)$, we get

Proposition 5.7 *Suppose the conditions of Theorem 1.1 hold. Then, there exist integer $p \geq 1$ and real numbers $\epsilon > 0$ and $K > 0$ such that $\forall t_1 < t_2$ and $y \in \mathbb{R}$, we have*

$$\mathbb{E}|X_{t_1}^1(y) - X_{t_2}^1(y)|^{2p} \leq K|t_1 - t_2|^{2+\epsilon}.$$

Remark 5.8 It is conjectured by Yaozhong Hu and David Nualart that for x fixed, $X_t(x)$ should be Hölder continuous in t with exponent $1/4 - \epsilon$. However, the method in this paper cannot confirm this conjecture. Instead, it follows from Proposition 5.3 that $X_t(x)$ is Hölder continuous in t with exponent $\min(\alpha(1 + \beta) - 1, 1 - \alpha)/2 - \epsilon$. Since $\alpha < 1$ and $\beta < 1/2$, the best Hölder exponent we can get here is $1/10 - \epsilon$.

Proof of Theorem 1.1 Combining Propositions 4.2, 4.3, 5.6 and 5.7, we get

$$\mathbb{E}|X_{t_1}(y_1) - X_{t_2}(y_2)|^{2p} \leq K|(t_1, y_1) - (t_2, y_2)|^{2+\epsilon}.$$

The joint continuity then follows from Kolmogorov's criteria. \square

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References

1. Barros-Neto, J.: An Introduction to the Theory of Distributions. Marcel Dekker, New York (1973)
2. Dawson, D.A., Li, Z., Wang, H.: Superprocesses with dependent spatial motion and general branching densities. Electron. J. Probab. **6**, 1–33 (2001)
3. Dawson, D.A., Vaillancourt, J., Wang, H.: Stochastic partial differential equations for a class of interacting measure-valued diffusions. Ann. Inst. Henri Poincaré Probab. Stat. **36**(2), 167–180 (2000)
4. Foondum, M., Khoshnevisan, D.: Intermittence and nonlinear parabolic stochastic partial differential equations. Electron. J. Probab. **14**, 548–568 (2009)
5. Kallianpur, G.: Stochastic Filtering Theory. Springer, Berlin (1980)
6. Konno, N., Shiga, T.: Stochastic partial differential equations for some measure-valued diffusions. Probab. Theory Relat. Fields **79**, 201–225 (1988)
7. Krasnoselskii, M.A., Pustyl'nik, E.I., Sobolevski, P.E., Zabrejko, P.P.: Integral Operators in Spaces of Summable Functions. Nauka, Moscow (1966, in Russian); English translation: Noordhoff International Publishing, Leyden (1976)
8. Krylov, N.V.: An analytic approach to SPDEs. Stochastic partial differential equations: six perspectives. In: Math. Surveys Monogr., vol. 64, pp. 185–242. Amer. Math. Soc., Providence (1999)
9. Kurtz, T., Xiong, J.: Particle representations for a class of nonlinear SPDEs. Stoch. Process. Appl. **83**, 103–126 (1999)
10. Kurtz, T., Xiong, J.: A stochastic evolution equation arising from the fluctuation of a class of interacting particle systems. Commun. Math. Sci. **2**, 325–358 (2004)
11. Lee, K.J., Mueller, C., Xiong, J.: Some properties for superprocess over a stochastic flow. Ann. Inst. Henri Poincaré Probab. Stat. **45**, 477–490 (2009)
12. Li, Z., Wang, H., Xiong, J.: Conditional log-Laplace functionals of superprocesses with dependent spatial motion. Acta Appl. Math. **88**, 143–175 (2005)
13. Reimers, M.: One-dimensional stochastic partial differential equations and the branching measure diffusion. Probab. Theory Relat. Fields **81**, 319–340 (1989)
14. Skoulakis, G., Adler, R.J.: Superprocesses over a stochastic flow. Ann. Appl. Probab. **11**, 488–543 (2001)
15. Wang, H.: State classification for a class of measure-valued branching diffusions in a Brownian medium. Probab. Theory Relat. Fields **109**, 39–55 (1997)

16. Wang, H.: A class of measure-valued branching diffusions in a random medium. *Stoch. Anal. Appl.* **16**, 753–786 (1998)
17. Xiong, J.: An introduction to stochastic filtering theory. In: *Oxford Graduate Texts in Mathematics*, vol. 18. Oxford University Press, Oxford (2008)
18. Xiong, J., Zhou, X.: Superprocess over a stochastic flow with superprocess catalyst. *Int. J. Pure Appl. Math.* **17**, 353–382 (2004)