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An almost sure scaling limit theorem for Dawson–Watanabe superprocesses

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Abstract

We establish a scaling limit theorem for a large class of Dawson–Watanabe superprocesses whose underlying spatial motions are symmetric Hunt processes, where the convergence is in the sense of convergence in probability. When the underling process is a symmetric diffusion with C^1_b-coefficients or a symmetric Lévy process on $\mathbb{R}^d$ whose Lévy exponent $\Psi(\eta)$ is bounded from below by $c|\eta|^\alpha$ for some $c>0$ and $\alpha\in(0,2)$ when $|\eta|$ is large, a stronger almost sure limit theorem is established for the superprocess. Our approach uses the principal eigenvalue and the ground state for some associated Schrödinger operator. The limit theorems are established under the assumption that an associated Schrödinger operator has a spectral gap.

Keywords: Dawson–Watanabe superprocess; Scaling limit theorem; Symmetric Hunt process; Dirichlet form; Symmetric diffusion; Symmetric Lévy process; Feller property; Feller generator; Feynman–Kac transform; Schrödinger semigroup; Ground state; Spectral gap; h-Transform; Ito's formula; Martingale

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1. Introduction

Let $A(x) = (a_{ij}(x))_{1 \leqslant i,j \leqslant d}$ be a symmetric matrix-valued function on $\mathbb{R}^d$ that is uniformly elliptic and bounded:

$$\Lambda_1 |v|^2 \leqslant \sum_{i,j=1}^{d} a_{i,j}(x) v_i v_j \leqslant \Lambda_2 |v|^2 \quad \text{for all } v \in \mathbb{R}^d \text{ and } x \in \mathbb{R}^d,$$

for some positive constants $0 < \Lambda_1 \leqslant \Lambda_2 < \infty$, and its entries $\{a_{ij}(x), 1 \leqslant i, j \leqslant d\}$ are $C^{1,\gamma}$-smooth on $\mathbb{R}^d$ for some $\gamma \in (0,1)$. Assume that $\mathbf{b}(x) = (b_1(x), \ldots, b_d(x))$ is a $C^{1,\gamma}$-smooth $\mathbb{R}^d$-valued function on $\mathbb{R}^d$ and define

$$\mathcal{A} = \nabla \cdot A \nabla + \mathbf{b} \cdot \nabla \quad \text{on } \mathbb{R}^d.$$

Consider a super-diffusion $\{X_t, t \geqslant 0\}$ corresponding to the operator $\mathcal{L}u := \mathcal{A}u + \beta u - \kappa u^2$ on $\mathbb{R}^d$, where β and $\kappa \geqslant 0$ are C^γ-functions. Denote by λ_c the generalized principal eigenvalue for operator $\mathcal{A} + \beta$ on $\mathbb{R}^d$, i.e.,

$$\lambda_c = \inf\{\lambda \in \mathbb{R}\colon\ \mathcal{A} + \beta - \lambda \text{ possesses a Green's function}\}$$

(see [17,18], for example). Let $\widetilde{\mathcal{A}}$ denote the formal adjoint operator of $\mathcal{A}$. The positive eigenfunction of the operator $\mathcal{A} + \beta$ corresponding to λ_c, if it exists, will be denoted by ϕ, and the positive eigenfunction of the operator $\widetilde{\mathcal{A}} + \beta$ corresponding to λ_c will be denoted by $\widetilde{\phi}$. Pinsky [18] proved that if $\mathcal{A} + \beta - \lambda_c$ is critical and if $\int \phi(x)\widetilde{\phi}(x)\,dx < \infty$, then

$$\lim_{t \uparrow \infty} e^{-\lambda_c t} \mathbb{P}_\mu \langle X_t, g \rangle = \langle \mu, \phi \rangle (\widetilde{\phi}, g) \quad \text{for any } g \in C_c^+\big(\mathbb{R}^d\big), \tag{1.1}$$

where $(f, g) := \int_{\mathbb{R}^d} f(x) g(x)\,dx$, and ϕ and $\widetilde{\phi}$ are normalized to have $\int \phi(x)\widetilde{\phi}(x)\,dx = 1$. Here $C_c^+(\mathbb{R}^d)$ denotes the space of non-negative continuous functions on $\mathbb{R}^d$ having compact support. Engländer and Turaev [11] proved that if $\lambda_c > 0$, $\kappa\phi$ is bounded and the initial state μ is such that $\langle \mu, \phi \rangle < \infty$, then for $g \in C_c^+(\mathbb{R}^d)$,

$$\lim_{t \to \infty} e^{-\lambda_c t} \langle X_t, g \rangle = N_\mu (\widetilde{\phi}, g) \quad \text{in distribution,}$$

where the limiting non-negative non-degenerate random variable N_μ was identified with the help of a certain invariant curve. Later, Engländer and Winter [12] improved the above result to show that the above convergence holds in probability. Very recently Engländer [10] further extended the above convergence in probability result to some superdiffusions without assuming that $\int \phi(x)\widetilde{\phi}(x)\,dx < \infty$.

The present paper is devoted to establish that the above convergence in probability result holds for a large class of Dawson–Watanabe superprocesses whose underlying spatial motions are symmetric Hunt processes which can have discontinuous sample paths. Moreover, when the underlying spatial motion is a symmetric diffusion on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{A} = \rho(x)^{-1} \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(\rho(x) a_{ij}(x) \frac{\partial}{\partial x_j} \right), \tag{1.2}$$

where $A(x) = (a_{ij}(x))_{ij}$ is uniformly elliptic and bounded with $a_{ij} \in C_b^1(\mathbb{R}^d)$ and the function $\rho \in C_b^1(\mathbb{R}^d)$ is bounded between two positive constants, or a symmetric Lévy process on $\mathbb{R}^d$ whose Lévy exponent $\Psi(\eta)$ is bounded from below by $c|\eta|^\alpha$ for some $c > 0$ and $\alpha \in (0, 2)$ when $|\eta|$ is large, we can establish an almost sure scaling limit theorem for superprocesses. Here $C_b^1(\mathbb{R}^d)$ is the space of bounded continuous functions on $\mathbb{R}^d$ that have bounded continuous first derivatives. Although almost sure limit theorem for branching Markov processes has been studied by several authors during the past 40 years (see [6] and the references therein), as far as we know, this is the first time that an almost sure scaling limit theorem has been established for superprocesses.

Our approach is quite different from that of Pinsky [18], Engländer and Turaev [11] and Engländer and Winter [12]. Motivated from Chen and Shiozawa [6], where an almost sure limit theorem is established for branching symmetric Hunt processes, we use the principal eigenvalue and the ground state of an associated Schrödinger operator to establish a scaling limit theorem in the sense of convergence in probability for a large class of superprocesses. More specifically, let ξ be a symmetric Hunt process on state space E with symmetrizing measure m, which can have discontinuous sample paths. Let $\mathcal{A}$ denote the infinitesimal generator of ξ. Consider an $(\mathcal{A}, \beta, \kappa)$-superprocess X, where β and κ are related to mean and variance of the offspring distribution for the particle model of the superprocess. Under some Kato class condition on β, we show that $\mathcal{A} + \beta$ has a non-negative L^2-eigenfunction h with first eigenvalue $-\lambda_1(\beta)$. (Observe that $-\lambda_1(\beta)$ is the same as the generalized principal eigenvalue λ_c mentioned above.) Under the assumption of $\lambda_1(\beta) < 0$, we show that for every bounded $L^2(E, m)$-integrable function f,

$$\lim_{t\to\infty} e^{\lambda_1(\beta)t}\langle f, X_t\rangle = M_\infty^h \int_E f(x)h(x)\,m(dx) \quad \text{in probability with respect to } \mathbb{P}_{\delta_x}.$$

Here M_∞^h is the limit of the non-negative martingale $M_t := e^{\lambda_1(\beta)t}\langle h, X_t\rangle$. When ξ is a diffusion process in Euclidean space, it is known (see [18]) that $\lambda_1(\beta) < 0$ is equivalent to the superprocess X of no local extinction.

The proof of the limit theorem for superprocesses in the sense of convergence in probability is very similar to that in Chen and Shiozawa [6], which is applicable to a large class of superprocesses. However there is significant new difficulty in obtaining an almost sure scaling limit theorem for superprocesses. One of the main steps in [6] to establish the almost sure limit theorem for branching symmetric Hunt processes is first to obtain the almost sure limit result at discrete times and then extend it to all times. But we are unable to carry out this strategy for superprocesses when the underling spatial motion is a general symmetric Hunt process. When the underling spatial motion is a symmetric diffusion mentioned above with C_b^1-smooth coefficients or a symmetric Lévy process on $\mathbb{R}^d$ whose Lévy exponent $\Psi(\eta)$ is bounded from below by $c|\eta|^\alpha$ for some $c > 0$ and $\alpha \in (0, 2)$ when $|\eta|$ is sufficiently large, we can establish an almost sure scaling limit theorem for superprocesses with the help of an Ito's formula for superprocesses developed by Perkins [16].

The remainder of this paper is organized as follows. We start Section 2.1 with a review on definitions and basic properties of Kato class functions for symmetric Hunt processes as well as of symmetric Lévy processes. We then show that the working hypothesis of this paper (Assumption 2.1 and condition (2.4)) are satisfied by a large class of symmetric diffusions and symmetric Lévy processes. The latter extends [6, Remark 2.6] and has its own independent interests, as these conditions are also the working hypothesis for paper [6] on scaling limit theorems for branching

Markov processes. Section 2.2 recalls some results established in [6] on Schrödinger semigroups that will play a crucial role in our approach to scaling limit theorems for superproceses. Basic properties of Dawson–Watanabe superprocesses and the Ito type formula for superprocesses established by Perkins are reviewed in Section 2.3. The proof of scaling limit theorems for superprocesses (both convergence in probability version and almost sure convergence version) are given in Section 3.

Throughout this paper, we will use $C_c(\mathbb{R}^d)$ and $C_\infty(\mathbb{R}^d)$ to denote the space of continuous functions on $\mathbb{R}^d$ having compact support and the space of continuous functions on $\mathbb{R}^d$ that vanish at infinity. The space of functions in $C_c(\mathbb{R}^d)$ that has continuous first derivatives (respectively, continuous derivatives up to second order) will be denoted as $C_c^1(\mathbb{R}^d)$ (respectively $C_c^2(\mathbb{R}^d)$). For a real number $a \in \mathbb{R}$, we define $a^+ := \max\{a, 0\}$ and $a^- := \max\{-a, 0\}$.

2. Preliminaries

2.1. Symmetric Hunt processes and Kato classes

Let E be a locally compact separable metric space and $E_\Delta := E \cup \{\Delta\}$ its one point compactification. Denote by $\mathcal{B}(E)$ and $\mathcal{B}(E_\Delta)$ the Borel σ-fields on E and E_Δ, respectively. Let $\mathcal{B}_b(E)$ (respectively, $\mathcal{B}^+(E)$) denote the set of all bounded (respectively, non-negative) $\mathcal{B}(E)$-measurable functions on E. The space of continuous (respectively, bounded continuous) functions on E will be denoted as $C(E)$ (respectively $C_b(E)$) and for $f \in C_b(E)$, $\|f\|_\infty := \sup_E |f(x)|$. Let m be a positive Radon measure on E with full support. Let $\xi = (\Omega^0, \mathcal{G}^0, \mathcal{G}_t^0, \theta_t, \xi_t, \Pi_x, \zeta)$ be an m-symmetric Hunt process on E, where $\{\mathcal{G}_t^0\}_{t \geqslant 0}$ is the minimal admissible filtration of ξ, $\{\theta_t, t \geqslant 0\}$ is the time-shift operator satisfying $\xi_t \circ \theta_s = \xi_{t+s}$ identically for $s, t \geqslant 0$, and $\zeta = \inf\{t > 0\colon\ \xi_t = \Delta\}$ the lifetime. The L^2-infinitesimal generator of ξ will be denoted as $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. The same notation $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ will also be used to denote the Feller generator of ξ when ξ is a Feller process. When $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ is used to denote the Feller generator of a Feller process ξ, we will say so explicitly. Let $\{P_t, t \geqslant 0\}$ be the Markovian transition semigroup of ξ:

$$P_t f(x) = \Pi_x\big[f(\xi_t)\big] \quad \text{for } f \in \mathcal{B}^+(E),$$

where Π_x is the distributional law of ξ starting from x. The symmetric Dirichlet form on $L^2(E; m)$ generated by ξ will be denoted as $(\mathcal{E}, \mathcal{F})$ (cf. [13]):

$$\mathcal{F} := \left\{u \in L^2(E, m)\colon\ \lim_{t \to 0} \frac{1}{t} \int_E \big(u(x) - P_t u(x)\big) u(x)\, m(dx) < \infty\right\},$$

$$\mathcal{E}(u, v) := \lim_{t \to 0} \frac{1}{t} \int_E \big(u(x) - P_t u(x)\big) v(x)\, m(dx) \quad \text{for } u, v \in \mathcal{F}.$$

The Dirichlet form $(\mathcal{E}, \mathcal{F})$ is known to be quasi-regular [15] and hence is quasi-homeomorphic to a regular Dirichlet form on a locally compact separable metric space [8]. But we will not use these properties in this paper.

Throughout this paper we assume the following.

Assumption 2.1. For each $t > 0$, the process ξ has a bounded symmetric transition density function $p(t, x, y)$ with respect to the measure m that is continuous in x for every fixed $y \in E$:

$$P_t f(x) = \int_E p(t, x, y) f(y)\, m(dy) \quad \text{for every } f \in L^1(E, m) \text{ and } x \in E.$$

It can be shown that Assumption 2.1 is equivalent to the following two properties:

(i) (*Strong Feller property*) For any $f \in \mathcal{B}_b(E)$, $P_t f$ is a bounded and continuous function on E.
(ii) (*Ultracontractivity*) For any $t > 0$, it holds that $\|P_t\|_{1,\infty} < \infty$, where $\|\cdot\|_{p,q}$ denotes the operator norm from $L^p(E, m)$ to $L^q(E, m)$.

Clearly Assumption 2.1 implies (i) and (ii). Conversely, using Chapman–Kolmogorov equation and Riesz representation theorem for the dual of $L^1(E, m)$, it is easy to see that (i) and (ii) also imply Assumption 2.1.

For $\alpha \geqslant 0$, let $G_\alpha(x, y)$ be the α-resolvent kernel of ξ defined by

$$G_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y)\, dt.$$

When $\alpha = 0$, we omit the subscript 0 and denote $G_0(x, y)$ by $G(x, y)$. We call $G(x, y)$ the Green function of ξ. If ξ is transient, its Green function $G(x, y)$ is finite on $E \times E$ off the diagonal.

Definition 2.2. (i) A measurable function q is said to be in the Kato class $\mathbf{K}(\xi)$ if

$$\lim_{\alpha\to\infty} \sup_{x\in E} \int_E G_\alpha(x, y)\big|q(y)\big|\, dy = 0.$$

(ii) If ξ is transient, a measurable function $q \in \mathbf{K}(\xi)$ is said to be in the class $\mathbf{K}_\infty(\xi)$, if for any $\varepsilon > 0$, there is a compact set $K \subset E$ and a positive constant $\delta > 0$ such that

$$\sup_{x\in E} \int_{E\setminus K} G(x, y)\big|q(y)\big|\, m(dy) + \sup_{\substack{B\subset K \\ m(B)<\delta}} \sup_{x\in E} \int_B G(x, y)\big|q(y)\big|\, m(dy) < \varepsilon.$$

(iii) If ξ is recurrent, we define $\mathbf{K}_\infty(\xi) := \bigcap_{\alpha>0} \mathbf{K}_\infty(\xi^\alpha)$. Here for $\alpha > 0$, ξ^α denotes the α-subprocess of ξ killed at rate α. Note that for $0 < \alpha_1 < \alpha_2$, $\mathbf{K}_\infty(\xi^{\alpha_1}) \subset \mathbf{K}_\infty(\xi^{\alpha_2})$.

For $\beta \in \mathbf{K}(\xi)$, put

$$e_\beta(t) := \exp\left(\int_0^t \beta(\xi_s)\, ds\right), \quad t \geqslant 0, \tag{2.1}$$

which is well defined since for every $x \in E$ and $t > 0$, $\int_0^t |\beta(\xi_s)|\,ds < \infty$ Π_x-a.s. In fact by Khasminskii's inequality (cf. [7, Proposition 2.3]), for every $\beta \in \mathbf{K}(\xi)$ and $t > 0$,

$$\sup_{x \in E, s \in [0,t]} \Pi_x\big[e_\beta(s)\big] < \infty. \tag{2.2}$$

Let $\{P_t^\beta, t \geqslant 0\}$ be the Feynman–Kac semigroup given by

$$P_t^\beta f(x) := \Pi_x\big[e_\beta(t) f(\xi_t)\big], \quad f \in \mathcal{B}(E).$$

It is known that the quadratic form associated with the semigroup $\{P_t^\beta, t \geqslant 0\}$ in $L^2(E, m)$ is $(\mathcal{E}^\beta, \mathcal{F})$, where

$$\mathcal{E}^\beta(u,u) := \mathcal{E}(u,u) - \int_E u^2(x)\beta(x)\,m(dx) \quad \text{for } u \in \mathcal{F}.$$

The function β is said to be gaugeable if

$$\sup_{x \in E} \Pi_x\big[e_\beta(\zeta)\big] < \infty.$$

Theorem 2.3. *Suppose that $\beta \in \mathbf{K}_\infty(\xi)$. Then the following are equivalent*:

(i) *β is gaugeable*;

(ii) $\inf\left\{\mathcal{E}(u,u) + \int_E u(x)^2\beta^-(x)\,m(dx);\ u \in \mathcal{F}, \int_E u(x)^2\beta^+(x)\,m(dx) = 1\right\} > 1$;

(iii) $\sup_{x \in E} \Pi_x\Big[\sup_{0 \leqslant t \leqslant \zeta} e_\beta(t)\Big] < \infty$.

Moreover, if one of the above holds, then

(iv) $\sup_{x \in E} \Pi_x\left[\int_0^\zeta \kappa(\xi_t) e_\beta(t)\,dt\right] < \infty$ *for any* $0 \leqslant \kappa \in \mathbf{K}_\infty(\xi)$.

Proof. Parts (i)–(iii) are proved in Chen [5, Corollary 2.9 and Theorem 5.2]. For $0 \leqslant \kappa \in \mathbf{K}_\infty(\xi)$, define $\beta_1 := \beta^+ + \kappa$ and $\beta_2 := \beta^- + \kappa$. Then $\beta_1, \beta_2 \in \mathbf{K}_\infty(\xi)$ and $\beta = \beta_1 - \beta_2$. Assume now that (ξ, β) is gaugeable and define

$$\tau_t := \inf\left\{s \geqslant 0\colon \int_0^s \big(|\beta| + \kappa\big)(\xi_r)\,dr \geqslant t\right\}, \quad t \geqslant 0.$$

Using the above definition for τ_t and with $\int_0^{\cdot} \beta_1(\xi_s)\,ds$ and $\int_0^{\cdot} \beta_2(\xi_s)\,ds$ playing the role for A^+ and A^- there, the same proof for [5, Theorem 2.8] yields that

$$\sup_{x\in E} \mathbb{E}_x\left[\int_0^{\zeta} e_\beta(t)\big(|\beta|+\kappa\big)(\xi_t)\,dt\right] < \infty.$$

This in particular implies (iv). □

For $\beta \in \mathbf{K}_\infty(\xi)$, define

$$\lambda_1(\beta) = \inf\left\{\mathcal{E}^\beta(u,u):\ u \in \mathcal{F} \text{ with } \int_E u^2(x)\,m(dx) = 1\right\},$$

and

$$\lambda_0 := \inf\left\{\mathcal{E}(u,u):\ u \in \mathcal{F} \text{ with } \int_E u^2(x)\,m(dx) = 1\right\}.$$

If

$$\text{the embedding of } (\mathcal{F}, \mathcal{E}_1) \text{ into } L^2\big(E;\beta^+\big) \text{ is compact,} \tag{2.3}$$

where $\mathcal{E}_1(u,u) := \mathcal{E}(u,u) + \int_E u^2(x)\,m(dx)$, then by the Friedrichs theorem, the spectrum of $\sigma(\mathcal{E}^\beta)$ less than λ_0 consists of only isolated eigenvalues with finite multiplicities. Note that if $\mathcal{A}$ is of the form (1.2), then the generalized principal eigenvalue λ_c for operator $\mathcal{A}+\beta$ on $\mathbb{R}^d$ equals $-\lambda_1(\beta)$. Let h be the normalized positive L^2-eigenfunction of $\mathcal{A}+\beta$ corresponding to $\lambda_1 := \lambda_1(\beta)$ with $\int_E h^2(x)\,m(dx) = 1$. Let $\lambda_2(\beta)$ denote the second bottom of the spectrum of $\sigma(\mathcal{E}^\beta)$:

$$\lambda_2(\beta) = \inf\left\{\mathcal{E}^\beta(u,u):\ u \in \mathcal{F},\ \int_E u^2(x)\,m(dx) = 1,\ \int_E u(x)h(x)\,m(dx) = 0\right\}.$$

Then $\lambda_2(\beta) - \lambda_1(\beta) > 0$ if $\lambda_1(\beta) < \lambda_0$.

In the remainder of this paper, we fix $\beta \in \mathbf{K}_\infty(\xi)$ and a nonnegative function $\kappa \in \mathbf{K}_\infty(\xi)$. We assume that

$$\text{the embedding of } (\mathcal{F}, \mathcal{E}_1) \text{ into } L^2\big(E;\beta^+\big) \text{ is compact,} \tag{2.4}$$

and

$$\lambda_1(\beta) < 0. \tag{2.5}$$

Note that, since $\lambda_0 \geqslant 0$, the condition that $\lambda_1(\beta) < \lambda_0$ is automatically satisfied if $\lambda_1(\beta) < 0$.

Now we give some concrete sufficient conditions on symmetric process ξ so that the Assumption 2.1 is satisfied and that condition (2.4) holds for every $\beta \in \mathbf{K}_\infty(\xi)$ with non-trivial β^+.

Recall that a Lévy process Z taking values in $\mathbb{R}^d$ is a process that has stationary independent increments. Its one-dimensional distribution and hence the process itself can be characterized by its characteristic exponent Ψ:

$$\mathbb{E}_x\big[\exp\big(-i\eta\cdot(Z_t-Z_0)\big)\big]=e^{-t\Psi(\eta)}\quad\text{for every } t>0 \text{ and } \eta\in\mathbb{R}^d.$$

It is known that every Lévy process is a Feller process, that is, for every $f\in C_\infty(\mathbb{R}^d)$, $P_tf\in C_\infty(\mathbb{R}^d)$ for every $t>0$ and $\lim_{t\to 0}P_tf=f$ uniformly (see, e.g., [3, Proposition I.5]). Here $C_\infty(\mathbb{R}^d)$ is the space of continuous functions on $\mathbb{R}^d$ that converges to 0 at infinity and $\{P_t, t\geqslant 0\}$ is the transition semigroup of the Lévy process. If for every $t>0$, $e^{-t\Psi(-\eta)}$ is L^1-integrable on $\mathbb{R}^d$, then Z has bounded continuous transition density function $p(t,x,y)$ given by (cf. [3, p. 24])

$$p(t,x,y):=(2\pi)^{-d}\int_{\mathbb{R}^d}e^{-i(x-y)\cdot\eta}e^{-t\Psi(-\eta)}\,d\eta. \tag{2.6}$$

We say a Lévy process Z is symmetric if $-(Z-Z_0)$ has the same distribution as that of $Z-Z_0$. This is equivalent to say that the transition semigroup $\{P_t, t\geqslant 0\}$ of Z is symmetric in $L^2(\mathbb{R}^d,dx)$. It is easy to see that a Lévy process Z is symmetric if and only if its characteristic exponent Ψ is an even function, that is, $\Psi(-\eta)=\Psi(\eta)$ for every $\eta\in\mathbb{R}^d$. It follows from the Lévy–Khintchine formula (cf. [3, p. 3]) that for symmetric Lévy process Z, its characteristic exponent can be expressed as

$$\Psi(\eta)=\frac{1}{2}\sum_{i,j=1}^d a_{ij}\eta_i\eta_j+\int_{\mathbb{R}^d}\big(1-\cos(\eta\cdot x)\big)\,J(dx),\quad \eta=(\eta_1,\dots,\eta_d)\in\mathbb{R}^d, \tag{2.7}$$

where $(a_{ij})_{1\leqslant i,j\leqslant d}$ is a symmetric semi-positive definite constant matrix and $J(dx)$ is a non-negative measure on $\mathbb{R}^d\setminus\{0\}$ with

$$\int_{\mathbb{R}^d}\big(1\wedge|x|^2\big)\,J(dx)<\infty. \tag{2.8}$$

This implies in particular that the characteristic exponent Ψ of a symmetric Lévy process takes non-negative real values. The measure J in (2.7) describes the jumps of the Lévy process Z and is called the Lévy measure of Z. If Ψ has the property that

$$\Psi(\eta)=|\eta|^\alpha\Psi\big(\eta/|\eta|\big)\quad\text{for every } \eta\in\mathbb{R}^d\setminus\{0\},$$

where $0<\alpha<2$, the Lévy process is called an α-stable process. The Lévy exponent Ψ of a general α-stable process has the form

$$\Psi(\xi)=\int_0^\infty\int_{S^{d-1}}\big(1-e^{i\xi\cdot re^{i\theta}}+i\xi\cdot re^{i\theta}\mathbf{1}_{\{r\leqslant 1\}}\big)r^{-1-\alpha}\,dr\,\nu(d\theta),$$

where S^{d-1} is the $(d-1)$-dimensional unit sphere in $\mathbb{R}^d$ and ν is a finite measure on S^{d-1}. An isotropically symmetric α-stable process corresponds to the case where ν is the constant multiple of the surface measure on S^{d-1}. This corresponds to $\Psi(\eta) = c|\eta|^\alpha$, where $c > 0$. In this case, the expression for Ψ in (2.7) has $a_{ij} = 0$ and $J(dy) = c(d,\alpha)|y|^{-d-\alpha}\,dy$.

For a complex number $z = a + ib$ with $a, b \in \mathbb{R}$, set $\bar{z} := a - ib$. For $u \in L^2(\mathbb{R}^d)$, let

$$\widehat{u}(x) := (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{ix\cdot y} u(y)\,dy$$

denote the Fourier transform of u. It is known that (see [13, pp. 30, 31]) the Dirichlet form $(\mathcal{E}, \mathcal{F})$ of a symmetric Lévy process Z with characteristic exponent Ψ is given by

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \widehat{u}(x)\bar{\widehat{v}}(x)\Psi(x)\,dx. \tag{2.9}$$

Thus,

$$\begin{aligned}\mathcal{E}(u,u) &= \int_{\mathbb{R}^d} \big|\widehat{u}(x)\big|^2 \Psi(x)\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}^d} \sum_{i,j=1}^d a_{ij} \frac{\partial u(x)}{\partial x_i}\frac{\partial u(x)}{\partial x_j}\,dx + \frac{1}{2}\int_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \big(u(x+y)-u(x)\big)^2 J(dy)\,dx,\end{aligned} \tag{2.10}$$

$$\mathcal{F} = \big\{u \in L^2\big(\mathbb{R}^d, dx\big)\colon\ \mathcal{E}(u,u) < \infty\big\}. \tag{2.11}$$

Clearly, $C_c^1(\mathbb{R}^d) \subset \mathcal{F}$. For $\alpha > 0$, we define

$$\mathcal{E}_\alpha(u,u) := \mathcal{E}(u,u) + \alpha \int_{\mathbb{R}^d} u(x)^2\,dx \quad \text{for } u \in \mathcal{F}.$$

For $\alpha \in (0,2)$, let $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ denote the Dirichlet form for isotropically symmetric α-stable process in $\mathbb{R}^d$ with characteristic exponent $\Psi(\eta) = |\eta|^\alpha$. We see from above that

$$\mathcal{E}^{(\alpha)}(u,u) = \int_{\mathbb{R}^d} \big|\widehat{u}(\eta)\big|^2 |\eta|^\alpha\,d\eta = c(d,\alpha) \int_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \frac{(u(x+y)-u(x))^2}{|y|^{d+\alpha}}\,dx\,dy, \tag{2.12}$$

$$\mathcal{F}^{(\alpha)} = \big\{u \in L^2\big(\mathbb{R}^d, dx\big)\colon\ \mathcal{E}(u,u) < \infty\big\}, \tag{2.13}$$

where $c(d,\alpha)$ is a positive constant that depends only on dimension d and α. It is well known that $C_c^\infty(\mathbb{R}^d)$, the space of smooth functions with compact support on $\mathbb{R}^d$, is $\mathcal{E}_1^{(\alpha)}$-dense in $\mathcal{F}^{(\alpha)}$. The Hilbert space $(\mathcal{F}^{(\alpha)}, \mathcal{E}_1^{(\alpha)})$ is the Sobolev space (or Bessel potential space) $W^{\alpha/2,2}(\mathbb{R}^d)$ of fractional order. The following compact embedding theorem is known to experts. However we are unable to find a direct, explicit reference so we record it here for future reference.

Theorem 2.4. *Let $D \subset \mathbb{R}^d$ be a bounded open set. For every $\alpha \in (0, 2 \wedge d)$ and $1 < p < \frac{2d}{d-\alpha}$, the map $u \mapsto \mathbf{1}_D u$ is a compact embedding from $(\mathcal{F}^{(\alpha)}, \mathcal{E}_1^{(\alpha)})$ into $L^p(D, dx)$.*

Proof. For $s > 0$, define the Bessel kernel

$$g_s(x) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{-ix\cdot\eta} \big(1 + |\eta|^2\big)^{-s/2} \, d\eta, \quad x \in \mathbb{R}^d,$$

which is a decreasing function in $|x|$ (see [1, (1.2.11)]). The Bessel potential space $\mathcal{L}^{s,2}(\mathbb{R}^d)$ is defined to be

$$\mathcal{L}^{s,2}\big(\mathbb{R}^d\big) := \big\{u\colon\ u = g_s * f \text{ with } f \in L^2\big(\mathbb{R}^d, dx\big)\big\},$$

with norm $\|u\|_{s,2} := \|f\|_{L^2(\mathbb{R}^d,dx)}$. Here for two functions f, g, $f * g(x) := \int_{\mathbb{R}^d} f(x-y)g(y)\,dy$ whenever it is defined. For $u = g_s * f$, since

$$\widehat{u}(\eta) = \widehat{g_s}(\eta)\widehat{f}(\eta) = \big(1 + |\eta|^2\big)^{-s/2} \widehat{f}(\eta),$$

by Plancherel's identity,

$$\|u\|_{s,2}^2 = \int_{\mathbb{R}^d} \big|f(x)\big|^2 \, dx = \int_{\mathbb{R}^d} \big|\widehat{u}(\eta)\big|^2 \big(1 + |\eta|^2\big)^s \, d\eta.$$

So for $\alpha \in (0, 2)$, $\mathcal{F}^{(\alpha)} = \mathcal{L}^{\alpha/2,2}(\mathbb{R}^d)$ and there is a constant $c_1 > 1$ such that

$$c_1^{-1}\|u\|_{\alpha/2,2} \leqslant \mathcal{E}_1^{(\alpha)}(u, u)^{1/2} \leqslant c_1\|u\|_{\alpha/2,2} \quad \text{for every } u \in \mathcal{F}^{(\alpha)}.$$

Let Cap_α be the 1-capacity of Dirichlet form $(\mathcal{E}^{(\alpha)}, \mathcal{F}^{(\alpha)})$ (cf. [13]). That is, for every Borel set F,

$$\mathrm{Cap}_\alpha(F) := \inf\big\{\mathcal{E}_1^{(\alpha)}(u, u)\colon\ u \in \mathcal{F}^{(\alpha)} \text{ with } u \geqslant 1 \text{ q.e. on } F\big\}.$$

Here q.e. is the abbreviation for $\mathcal{E}$-quasi-everywhere, which means that everywhere except for a set that having zero $\mathcal{E}_1$-capacity (cf. [13]). From the norm comparison above, we have

$$c_1^{-1}\mathrm{Cap}_{\alpha/2,2}(F) \leqslant \mathrm{Cap}_\alpha(F) \leqslant c_1\mathrm{Cap}_{\alpha/2,2}(F) \quad \text{for every Borel set } F \subset \mathbb{R}^d.$$

Here $\mathrm{Cap}_{\alpha/2,2}$ denotes the Bessel capacity of order $(\alpha/2, 2)$ defined by

$$\mathrm{Cap}_{\alpha/2,2}(F) := \inf\big\{\|u\|_{\alpha/2,2}\colon\ u \in \mathcal{L}^{\alpha/2,2}(\mathbb{R}^d) \text{ with } u \geqslant 1 \text{ q.e. on } F\big\}$$

for every Borel set $F \subset \mathbb{R}^d$. Let $\mu(dx) := 1_D(dx)\,dx$. By Theorem 7.3.1 in [1], the restriction map $u \mapsto \mathbf{1}_D u$ is a compact embedding from $(\mathcal{F}^{(\alpha)}, \mathcal{E}_1^{(\alpha)})$ into $L^p(D, dx)$ if and only if the following two conditions are satisfied:

$$\lim_{r\to 0} \sup_{\substack{K\subset\mathbb{R}^d\\ \mathrm{diam}(K)\leqslant r}} \frac{\mu(K)}{\mathrm{Cap}_\alpha(K)^{p/2}} = 0 \tag{2.14}$$

and

$$\lim_{\rho\to\infty} \sup_{K\subset\mathbb{R}^d\setminus B(0,\rho)} \frac{\mu(K)}{\mathrm{Cap}_\alpha(K)^{p/2}} = 0. \tag{2.15}$$

Here $B(0,\rho)$ denotes the ball centered at 0 with radius ρ. Since μ is compactly supported, condition (2.15) is automatically satisfied for every $p \geqslant 0$. Let m denote the Lebesgue measure on $\mathbb{R}^d$. It is clear that both m and the capacity Cap_α are translation invariant. For (2.14) to hold, it suffices to show

$$\lim_{r\to 0} \sup_{K\subset B(0,r)} \frac{m(K)}{\mathrm{Cap}_\alpha(K)^{p/2}} = 0. \tag{2.16}$$

For $0 < r \leqslant 1$, with $u_r(x) := u(rx)$,

$$\mathcal{E}^{(\alpha)}(u_r,u_r) = r^{\alpha-d} c(d,\alpha) \int\limits_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \frac{(u(x+y)-u(x))^2}{|y|^{d+\alpha}}\,dx\,dy = r^{\alpha-d}\mathcal{E}(u,u).$$

This combined with [1, Proposition 5.1.4] yields that there is a constant $c_2 \geqslant 1$ such that for every $0 < r \leqslant 1$ and Borel set $K \subset \mathbb{R}^d$ with $\mathrm{diam}(K) \leqslant 2r$,

$$c_2^{-1} r^{d-\alpha} \mathrm{Cap}_\alpha\big(r^{-1}K\big) \leqslant \mathrm{Cap}_\alpha(K) \leqslant c_2 r^{d-\alpha} \mathrm{Cap}_\alpha\big(r^{-1}K\big).$$

Hence

$$\sup_{K\subset B(0,r)} \frac{m(K)}{\mathrm{Cap}_\alpha(K)^{p/2}} \leqslant c_2 \sup_{K\subset B(0,r)} \frac{r^d m(r^{-1}K)}{(r^{d-\alpha}\mathrm{Cap}_\alpha(r^{-1}K))^{p/2}} \leqslant c_2 r^{d-\frac{(d-\alpha)p}{2}} \sup_{F\subset B(0,1)} \frac{m(F)}{\mathrm{Cap}_\alpha(F)^{p/2}}.$$

By the Sobolev embedding theorem (see [1, Theorem 1.2.4]), for every $p \in (1, \frac{2d}{d-\alpha}]$, there is a constant $c_3 > 0$ such that

$$\|u\|_{L^p(B(0,1),dx)} \leqslant c_3 \mathcal{E}_1^{(\alpha)}(u,u)^{1/2} \quad \text{for every } u \in \mathcal{F}^{(\alpha)}.$$

It follows that for every $F \subset B(0,1)$,

$$m(F) \leqslant c_3^p \,\mathrm{Cap}_\alpha(F)^{p/2}.$$

Therefore for $1 < p < \frac{2d}{d-\alpha}$, (2.16) holds and so the map $u \mapsto \mathbf{1}_D u$ is a compact embedding from $(\mathcal{F}^{(\alpha)}, \mathcal{E}_1^{(\alpha)})$ into $L^p(D,dx)$. $\square$

Lemma 2.5. *Let $(\mathcal{E},\mathcal{F})$ be the Dirichlet form of a symmetric Lévy process on $\mathbb{R}^d$. Suppose $\phi \in C_c^1(\mathbb{R}^d)$ and $u \in \mathcal{F}$. Then $u\phi \in \mathcal{F}$ and there is a constant $c > 0$ independent of u such that*

$$\mathcal{E}(u\phi,u\phi) \leqslant c\,\mathcal{E}_1(u,u).$$

Proof. By (2.8) and Cauchy–Schwarz inequality,

$$
\begin{aligned}
&\mathcal{E}(u\phi, u\phi) \\
&\quad = \frac{1}{2}\int\limits_{\mathbb{R}^d} \sum_{i,j=1}^{d} a_{ij} \frac{\partial(u\phi)}{\partial x_i}\frac{\partial(u\phi)}{\partial x_j}\,dx + \frac{1}{2}\int\limits_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \big((u\phi)(x+y)-(u\phi)(x)\big)^2 J(dy)\,dx \\
&\quad \leqslant \int\limits_{\mathbb{R}^d} u(x)^2 \sum_{i,j=1}^{d} a_{ij}\frac{\partial\phi}{\partial x_i}\frac{\partial\phi}{\partial x_j}\,dx + \int\limits_{\mathbb{R}^d}\phi(x)^2 \sum_{i,j=1}^{d} a_{ij}\frac{\partial u}{\partial x_i}\frac{\partial u}{\partial x_j}\,dx \\
&\qquad + \int\limits_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} \phi(x+y)^2\big(u(x+y)-u(x)\big)^2 J(dy)\,dx \\
&\qquad + \int\limits_{\mathbb{R}^d\times(\mathbb{R}^d\setminus\{0\})} u(x)^2\big(\phi(x+y)-\phi(x)\big)^2 J(dy)\,dx \\
&\quad \leqslant c\int\limits_{\mathbb{R}^d} u(x)^2\,dx + 2\|\phi\|_\infty \mathcal{E}(u,u) + \int\limits_{\mathbb{R}^d} u(x)^2\Bigg(\int\limits_{\{y\in\mathbb{R}^d:\ 0<|y|\leqslant 1\}} \big(\phi(x+y)-\phi(x)\big)^2 J(dy)\Bigg)dx \\
&\qquad + \int\limits_{\mathbb{R}^d} u(x)^2\Bigg(\int\limits_{\{y\in\mathbb{R}^d:\ |y|>1\}} \big(\phi(x+y)-\phi(x)\big)^2 J(dy)\Bigg)dx \\
&\quad \leqslant c\int\limits_{\mathbb{R}^d} u(x)^2\,dx + 2\|\phi\|_\infty \mathcal{E}(u,u) + \int\limits_{\mathbb{R}^d} u(x)^2\Bigg(\int\limits_{\{y\in\mathbb{R}^d:\ 0<|y|\leqslant 1\}} \|\nabla\phi\|_\infty^2 |y|^2 J(dy)\Bigg)dx \\
&\qquad + \int\limits_{\mathbb{R}^d} u(x)^2\Bigg(\int\limits_{\{y\in\mathbb{R}^d:\ |y|>1\}} 4\|\phi\|_\infty^2 J(dy)\Bigg)dx \\
&\quad \leqslant c\mathcal{E}_1(u,u),
\end{aligned}
$$

where in the last inequality we used (2.8). This proves that $u\phi\in\mathcal{F}$. Note that the constant $c>0$ varies from line to line but it is independent of $u\in\mathcal{F}$. □

Recall that for a symmetric Markov process ξ and $\alpha>0$, we denote by ξ^α the α-subprocess of ξ. As we noted previously, $\mathbf{K}_\infty(\xi)\subset\mathbf{K}_\infty(\xi^\alpha)$. The following theorem extends [22, Theorem 2.7], where the result is established for isotropically symmetric α-stable processes on $\mathbb{R}^d$.

Theorem 2.6. *Let ξ be a symmetric Lévy processes on $\mathbb{R}^d$ with characteristic exponent Ψ. Assume that there are constants $c_0>0$, $R_0>0$ and $\alpha_0>0$ so that*

$$\Psi(\eta)\geqslant c_0|\eta|^{\alpha_0} \quad \textit{for } |\eta|\geqslant R_0. \tag{2.17}$$

Then Assumption 2.1 *holds for ξ and the condition* (2.4) *holds for every $\beta\in\bigcup_{\alpha>0}\mathbf{K}_\infty(\xi^\alpha)$ with non-trivial β^+.*

Proof. As for every $t > 0$, $e^{-t\Psi(\eta)}$ is $L^1(\mathbb{R}^d)$-integrable, the Lévy process ξ has a continuous transition density function

$$p(t,x,y) = (2\pi)^{-d} \int_{\mathbb{R}^d} e^{-i(x-y)\cdot\eta} e^{-t\Psi(-\eta)}\, d\eta$$

with respect to the Lebesgue measure on $\mathbb{R}^d$ (see Breiman [4, Theorem 8.39]), which is bounded in (x,y) for each $t > 0$. Thus Assumption 2.1 holds for ξ. The proof that condition (2.4) holds for every $\beta \in \bigcup_{\alpha>0} \mathbf{K}_\infty(\xi^\alpha)$ with non-trivial β^+ is similar to that for [22, Theorem 2.7] except that we use Lemma 2.5 above instead of [22, Lemma 2.6]. For reader's convenience, we spell out the details of the proof.

Without loss of generality, assume that β is a non-negative non-trivial function in $\mathbf{K}_\infty(\xi^{\alpha_1})$ for some $\alpha_1 > 0$. Note that the Green's function of ξ^{α_1} is $G_{\alpha_1}(x,y)$, the α_1-resolvent kernel of ξ. Since $\beta \in \mathbf{K}_\infty(\xi^{\alpha_1}) \subset \mathbf{K}(\xi^{\alpha_1})$, by [19], for every $\alpha > 0$, we have

$$\int_{\mathbb{R}^d} u(x)^2 \beta(x)\, dx \leqslant \|G_\alpha \beta\|_\infty \mathcal{E}_\alpha(u,u) \quad \text{for every } \alpha \geqslant \alpha_1 \text{ and } u \in \mathcal{F}. \tag{2.18}$$

As $\lim_{\alpha\to\infty} \|G_\alpha\beta\|_\infty = 0$, it follows that for every $\varepsilon > 0$, there is a constant $M_\varepsilon > 0$ so that

$$\int_{\mathbb{R}^d} u(x)^2 \beta(x)\, dx \leqslant \varepsilon \mathcal{E}_1(u,u) + M_\varepsilon \int_{\mathbb{R}^d} u(x)^2\, dx \quad \text{for every } u \in \mathcal{F}. \tag{2.19}$$

By decreasing the value of α_0 in (2.17) if necessary, we may and do assume that $\alpha_0 \in (0,d)$. Denote by $(\mathcal{E}^{(\alpha_0)}, \mathcal{F}^{(\alpha_0)})$ the Dirichlet form of an isotropically symmetric α_0-stable process on $\mathbb{R}^d$. It follows from condition (2.17) that there is a constant $c > 0$ so that

$$1 + \Psi(\eta) \geqslant c\big(1 + |\eta|^{\alpha_0}\big) \quad \text{for every } \eta \in \mathbb{R}^d.$$

Thus by (2.10), (2.11),

$$\mathcal{F} \subset \mathcal{F}^{(\alpha_0)} \quad \text{and} \quad \mathcal{E}_1(u,u) \geqslant c\mathcal{E}_1^{(\alpha_0)}(u,u) \quad \text{for every } u \in \mathcal{F}.$$

In other words, Hilbert space $(\mathcal{F}, \mathcal{E}_1)$ embeds continuously into $(\mathcal{F}^{(\alpha_0)}, \mathcal{E}_1^{(\alpha_0)})$. On the other hand, by Theorem 2.4, for every $k \geqslant 1$, the restriction map $u \mapsto \mathbf{1}_{B_k} u$ is a compact embedding from $(\mathcal{F}^{(\alpha_0)}, \mathcal{E}_1^{(\alpha_0)})$ into $L^2(B_k, dx)$. Here B_k is the ball in $\mathbb{R}^d$ centered at the origin with radius k. Therefore the restriction map $u \mapsto \mathbf{1}_{B_k} u$ is a compact embedding from $(\mathcal{F}, \mathcal{E}_1)$ into $L^2(B_k, dx)$ for every $k \geqslant 1$.

Let $\{u_n, n \geqslant 1\} \subset \mathcal{F}$ with $\sup_{n\geqslant 1} \mathcal{E}_1(u_n,u_n) < \infty$. Taking a subsequence if necessary, we may and do assume that u_n converges weakly to some u in $(\mathcal{F}, \mathcal{E}_1)$ and $\frac{1}{n}\sum_{j=1}^n u_j$ converges to some v in $(\mathcal{F}, \mathcal{E}_1)$. Clearly $u = v \in \mathcal{F}$. Furthermore by taking a subsequence if necessary, we may and do assume that

$$\text{for every } k \geqslant 1,\ u_n \text{ converges strongly to } u \text{ in } L^2(B_k, dx) \text{ as } n \to \infty. \tag{2.20}$$

Let $\phi_k \in C_c^1(\mathbb{R}^d)$ so that $0 \leqslant \phi_k \leqslant 1$ on $\mathbb{R}^d$ and $\phi_k = 1$ on B_k. Then by (2.19) and Lemma 2.5, for every $\varepsilon > 0$,

$$\begin{aligned}
&\int_{B_k} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx \\
&\quad\leqslant \int_{\mathbb{R}^d} \left|u_n(x)\phi_k(x) - u(x)\phi_k(x)\right|^2 \beta(x)\,dx \\
&\quad\leqslant \varepsilon \mathcal{E}_1\big((u_n - u)\phi_k, (u_n - u)\phi_k\big) + M_\varepsilon \int_{\mathbb{R}^d} \left|u_n(x) - u(x)\right|^2 \phi_k(x)^2\,dx \\
&\quad\leqslant \varepsilon c_k \sup_{n \geqslant 1} \mathcal{E}_1(u_n, u_n) + M_\varepsilon \int_{\mathbb{R}^d} \left|u_n(x) - u(x)\right|^2 \phi_k(x)^2\,dx.
\end{aligned}$$

It follows from (2.20) that

$$\limsup_{n \to \infty} \int_{B_k} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx \leqslant \varepsilon c_k \sup_{n \geqslant 1} \mathcal{E}_1(u_n, u_n).$$

On the other hand, by (2.18),

$$\begin{aligned}
\int_{B_k^c} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx &\leqslant \left\| G_{\alpha_1}(\mathbf{1}_{B_k^c}\beta)\right\|_\infty \mathcal{E}_{\alpha_1}(u_n - u, u_n - u) \\
&\leqslant 2\left\| G_{\alpha_1}(\mathbf{1}_{B_k^c}\beta)\right\|_\infty \sup_{n \geqslant 1} \mathcal{E}_{\alpha_1}(u_n, u_n).
\end{aligned}$$

By the definition of $\beta \in \mathbf{K}_\infty(\xi^{\alpha_1})$, for every $\varepsilon > 0$, there is some $k_0 \geqslant 1$ such that

$$2\left\| G_{\alpha_1}(\mathbf{1}_{B_{k_0}^c}\beta)\right\|_\infty \sup_{n \geqslant 1} \mathcal{E}_{\alpha_1}(u_n, u_n) < \varepsilon.$$

The above implies that

$$\begin{aligned}
&\limsup_{n \to \infty} \int_{\mathbb{R}^d} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx \\
&\quad= \limsup_{n \to \infty} \left(\int_{B_{k_0}} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx + \int_{B_{k_0}^c} \left|u_n(x) - u(x)\right|^2 \beta(x)\,dx \right) \\
&\quad\leqslant \varepsilon c_{k_0} \sup_{n \geqslant 1} \mathcal{E}_1(u_n, u_n) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have $\lim_{n \to \infty} \int_{\mathbb{R}^d} |u_n(x) - u(x)|^2 \beta(x)\,dx = 0$. This establishes the theorem. $\square$

Remark 2.7. Examples of symmetric Lévy processes on $\mathbb{R}^d$ with characteristic exponent Ψ satisfying condition (2.17) are the following.

(i) Isotropically symmetric α-stable process on $\mathbb{R}^d$, where $\Psi(\eta) = c|\eta|^\alpha$ for $c > 0$ and $\alpha \in (0, 2]$. Note that when $\alpha = 2$, it is just Brownian motion on $\mathbb{R}^d$.
(ii) Relativistic α-stable process on $\mathbb{R}^d$, where $\Psi(\eta) = (c|\eta|^2 + m^2)^{\alpha/2} - m^\alpha$ with $c > 0$, $\alpha \in (0, 2)$ and $m > 0$. Recall that for $0 < p < 1$, we have the reversed triangle inequality: $(a + b)^p \geqslant a^p + b^p$ for $a, b \geqslant 0$. So $\Psi(\eta) \geqslant c^{\alpha/2}|\eta|^\alpha$.
(iii) Sum of an isotropically symmetric α-stable process on $\mathbb{R}^d$ with any independent symmetric Lévy process on $\mathbb{R}^d$, where $\alpha \in (0, 2]$.
(iv) Sum of a relativistic α-stable process on $\mathbb{R}^d$ with any independent symmetric Lévy process on $\mathbb{R}^d$, where $\alpha \in (0, 2)$.

Now we turn to the case that the motion ξ is a diffusion process.

Theorem 2.8. *Let ξ be a symmetric diffusion whose infinitesimal generator $\mathcal{A}$ is of the form* (1.2) *with a global Lipschitz ρ that is bounded between two positive constants. In this case, $(E, m) = (\mathbb{R}^d, \rho(x)\,dx)$. Then Assumption* 2.1 *holds for ξ and the condition* (2.4) *holds for every $\beta \in \mathbf{K}_\infty(\xi)$ with non-trivial β^+.*

Proof. Let ξ be a symmetric diffusion whose infinitesimal generator $\mathcal{A}$ is of the form (1.2) with a global Lipschitz ρ that is bounded between two positive constants. Clearly this includes Brownian motion as a special case. Note that we can rewrite $\mathcal{A}$ as

$$\mathcal{A} = \sum_{i,j=1}^d \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial}{\partial x_j}\right) + \sum_{j=1}^d \left(\sum_{i=1}^d a_{ij}(x)\frac{\partial \log \rho}{\partial x_i}\right)\frac{\partial}{\partial x_j}.$$

By Aronson [2, Theorem 10], the diffusion process ξ has a jointly continuous transition density function $p(t, x, y)$ such that for every $T > 0$, there is a constant $c_T \geqslant 1$ so that

$$c_T^{-1} t^{-d/2} \exp\left(-\frac{c_T\,|x-y|^2}{t}\right) \leqslant p(t, x, y) \leqslant c_T t^{-d/2} \exp\left(-\frac{|x-y|^2}{c_T t}\right) \tag{2.21}$$

for every $(t, x, y) \in (0, T] \times \mathbb{R}^d \times \mathbb{R}^d$. Therefore the symmetric diffusion process ξ satisfies the Assumption 2.1. When $\rho \equiv 1$ on $\mathbb{R}^d$, it is known that (see, e.g., [20, (I.0.10)]) that the positive constant c_T in (2.21) can be chosen to be independent of $T > 0$.

We claim that when ξ is a symmetric diffusion whose infinitesimal generator $\mathcal{A}$ is of the form (1.2), condition (2.4) holds for every measure $\beta \in \mathbf{K}_\infty(\xi)$. Note that for (2.4), we do not need to assume the function ρ in (1.2) is Lipschitz continuous. The diffusion process ξ is symmetric with respect to the symmetrizing measure $m(dx) := \rho(x)\,dx$ on $\mathbb{R}^d$ and its Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathbb{R}^d, \rho(x)\,dx)$ is given by

$$\mathcal{F} = W^{1,2}(\mathbb{R}^d) := \left\{u \in L^2\left(\mathbb{R}^d, dx\right):\ \nabla u \in L^2\left(\mathbb{R}^d, dx\right)\right\}, \tag{2.22}$$

$$\mathcal{E}(u, v) = \int_{\mathbb{R}^d} \sum_{i,j=1}^d \rho(x) a_{ij}(x) \frac{\partial u(x)}{\partial x_i}\frac{\partial v(x)}{\partial x_j}\,dx \quad \text{for } u, v \in \mathcal{F}. \tag{2.23}$$

For $\alpha > 0$, define

$$\mathcal{E}_\alpha(u,u) := \mathcal{E}(u,u) + \alpha \int_{\mathbb{R}^d} u(x)^2\, m(dx) \quad \text{for } u \in \mathcal{F}.$$

Clearly there is a constant $c \geqslant 1$ so that for every $\alpha \geqslant 0$ and $u \in W^{1,2}(\mathbb{R}^d)$,

$$c^{-1} \int_{\mathbb{R}^d} \left(\left|\nabla u(x)\right|^2 + \alpha u(x)^2 \right) dx \leqslant \mathcal{E}_\alpha(u,u) \leqslant c \int_{\mathbb{R}^d} \left(\left|\nabla u(x)\right|^2 + \alpha\, u(x)^2 \right) dx.$$

Hence the classical Rellich–Kondrachov compact embedding theorem for Sobolev space $W^{1,2}(\mathbb{R}^d)$ (or Theorem 2.4) tells us that the restriction map $u \mapsto \mathbf{1}_{B_k} u$ is a compact embedding from $(\mathcal{F}, \mathcal{E}_1)$ into $L^2(B_k, dx)$ for every ball B_k. The rest of the proof is the same as that for Theorem 2.6. □

Remark 2.9. Kato classes $\mathbf{K}(\xi)$ and $\mathbf{K}_\infty(\xi)$ can be defined for signed measures, see [5]. Theorems 2.6 and 2.8 in fact hold for any signed measure $\beta \in \mathbf{K}_\infty(\xi)$ with non-trivial β^+, with exactly the same proof. These two theorems provide more examples for which the main results in [6] apply.

2.2. *h*-Transform

Let h be the normalized positive L^2-eigenfunction of $\mathcal{A} + \beta$ corresponding to $\lambda_1 := \lambda_1(\beta)$ with $\int_E h^2(x)\, m(dx) = 1$. Then

$$h(x) = e^{\lambda_1 t} P_t^\beta h(x), \quad x \in E. \tag{2.24}$$

The function h is called the ground state for Schrödinger operator $\mathcal{A} + \beta$. It follows from [6, p. 379] that h is bounded, continuous and strictly positive.

We do h-transform for Schrödinger operator $\mathcal{A} + \beta$. For $t > 0$ and $x \in E$, define

$$P_t^h f(x) := \frac{e^{\lambda_1 t}}{h(x)} \Pi_x\left[e_\beta(t) h(\xi_t) f(\xi_t)\right], \quad f \in \mathcal{B}^+(E).$$

It is easy to check that $\{P_t^h, t \geqslant 0\}$ forms a strongly continuous Markovian semigroup in $L^2(E, h^2 m)$. In fact, for $x \in E$, $t \geqslant 0$ and $f \in \mathcal{B}^+(E)$,

$$\Pi_x^h\left[f(\xi_t)\right] := P_t^h f(x)$$

defines a family of probability measures $\{\Pi_x^h, x \in E\}$ on $(\Omega^0, \mathcal{G}_\infty^0, \{\mathcal{G}_t^0, t \geqslant 0\})$. For emphasis, the Hunt process ξ under these new probability measures $\{\Pi_x^h, x \in E\}$ will be denoted as ξ^h. It is shown [9] that ξ^h is $h^2 m$-symmetric and irreducible. Moreover, if we denote by $(\mathcal{E}^h, \mathcal{F}^h)$ the symmetric Dirichlet form of ξ^h in $L^2(E; h^2 m)$, then we have by [9] that $f \in \mathcal{F}^h$ if and only if $fh \in \mathcal{F}$ and that for $f \in \mathcal{F}^h$,

$$\mathcal{E}^h(f,f)=\mathcal{E}(fh,fh)-\int_E f(x)^2h(x)^2\big(\lambda_1+\beta(x)\big)\,m(dx)$$
$$=\mathcal{E}^\beta(fh,fh)-\lambda_1\int_E f(x)^2h(x)^2\,m(dx).$$

The constant function 1 belongs to $\mathcal{F}^h$ and $\mathcal{E}^h(1,1)=0$. Consequently, ξ^h is recurrent. Hence the bottom of the spectrum of $\sigma(\mathcal{E}^h)$ is 0, that is

$$\lambda_1^h(\beta):=\inf\left\{\mathcal{E}^h(u,u)\colon u\in\mathcal{F}^h,\ \int_E u^2(x)h^2(x)\,m(dx)=1\right\}=0.$$

Let $\lambda_2^h:=\lambda_2^h(\beta)$ be the spectral gap of the self-adjoint operator associated with $(\mathcal{E}^h,\mathcal{F}^h)$:

$$\lambda_2^h(\beta):=\inf\left\{\mathcal{E}^h(u,u)\colon u\in\mathcal{F}^h,\ \int_E u^2(x)h^2(x)\,m(dx)=1,\ \int_E u(x)h^2(x)\,m(dx)=0\right\}.$$

It follows that

$$\lambda_2^h=\lambda_2(\beta)-\lambda_1(\beta)>0, \tag{2.25}$$

which implies the following Poincaré inequality:

$$\big\|P_t^h\phi\big\|_{L^2(E;h^2m)}\leqslant e^{-\lambda_2^h t}\|\varphi\|_{L^2(E;h^2m)} \tag{2.26}$$

for any $\varphi\in L^2(E;h^2m)$ with $\int_E\varphi(x)h^2(x)\,m(dx)=0$. It is proved in [6] that there exists a constant $C>0$ such that

$$\big|h(x)P_t^h\varphi(x)\big|\leqslant Ce^{-\lambda_2^h t}\|\varphi\|_{L^2(E;h^2m)}\quad\text{for } t\geqslant 1, \tag{2.27}$$

for every $\varphi\in L^2(E,h^2m)$ with $\int_E\varphi(x)h^2(x)\,m(dx)=0$. Taking $\varphi(x)=g(x)/h(x)$ with g being a function in $L^2(E,m)$ and satisfying $\int_E g(x)m(dx)=0$, we get

$$e^{\lambda_1 t}\Pi_x\big[e_\beta(t)g(\xi_t)\big]\leqslant Ce^{-\lambda_2^h t}\|g\|_{L^2(E;m)}\quad\text{for } t\geqslant 1.$$

Therefore by (2.2) for any function g in $L^2(E,m)\cap L^\infty(E,m)$ satisfying $\int_E g(x)\,m(dx)=0$, we have

$$e^{\lambda_1 t}\Pi_x\big[e_\beta(t)g(\xi_t)\big]\leqslant Ce^{-\lambda_2^h t}\|g\|_{L^2(E;m)}\quad\text{for every } t\geqslant 0. \tag{2.28}$$

The following result was proved in [6].

Lemma 2.10. *Suppose that conditions* (2.4) *and* (2.5) *hold. For any $f\in L^2(E,m)$, we have*

$$\lim_{t\to\infty}e^{\lambda_1 t}\Pi_x\big[e_\beta(t)f(\xi_t)\big]=h(x)\int_E f(y)h(y)\,m(dy),\quad x\in E. \tag{2.29}$$

2.3. Dawson–Watanabe superprocesses

In this section, assume that ξ is a symmetric Feller process on E; that is, for every $f \in C_\infty(E)$, $P_t f \in C_\infty(E)$ for every $t > 0$ and $\lim_{t\to 0} P_t f = f$ uniformly on E. Here $\{P_t, t \geqslant 0\}$ is the transition semigroup of ξ. The infinitesimal generator of ξ in the Banach space $(C_\infty(E), \|\cdot\|_\infty)$ is called the Feller generator, which will (also) be denoted as $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$. Let $M_{\mathrm{F}}(E)$ denote the set of finite measures on E, and assume that $\beta \in \mathbf{K}_\infty(\xi)$ and $0 \leqslant \kappa \in \mathbf{K}_\infty(\xi)$. Suppose $X = \{X_t, \geqslant 0, \mathbb{P}_\mu, \mu \in M_{\mathrm{F}}(E)\}$ is a time-homogeneous $(\mathcal{A}, \beta, \kappa)$-super-Markov process corresponding to the operator $\mathcal{A}u + \beta u - \kappa u^2$. More precisely, X is a strong Markov process with $X_t \in M_{\mathrm{F}}(E)$, $t \geqslant 0$, and the Laplace functional

$$\mathbb{P}_\mu\big[\exp\big(\langle -f, X_t\rangle\big)\big] = \exp\big(\big\langle -v(t,\cdot), \mu\big\rangle\big) \tag{2.30}$$

with $\mu \in M_{\mathrm{F}}(E)$, $f \in \mathcal{B}_{\mathrm{b}}^+(E)$, where v is the unique solution of the integral equation

$$v(t,x) + \varPi_x\left[\int_0^t \kappa(\xi_s) e_\beta(s) v(t-s,\xi_s)^2\, ds\right] = \varPi_x\big[e_\beta(t) f(\xi_t)\big].$$

The minimal augmented filtration of X satisfying the standard condition will be denoted as $\{\mathcal{G}_t, t \geqslant 0\}$. The first two moments for X_t are given as follows: for every $f \in \mathcal{B}_{\mathrm{b}}^+(E)$ and $t \geqslant 0$,

$$\mathbb{P}_\mu\big[\langle f, X_t\rangle\big] = \varPi_\mu\big[e_\beta(t) f(\xi_t)\big], \tag{2.31}$$

$$\mathrm{Var}_\mu\big(\langle f, X_t\rangle\big) = 2\varPi_\mu\left[\int_0^t \kappa(\xi_r) e_\beta(r)\big(\varPi_{\xi_r}\big[e_\beta(t-r) f(\xi_{t-r})\big]\big)^2\, dr\right]. \tag{2.32}$$

Suppose that g is a non-negative, bounded, Borel function on E. Let $v(t,x)$ be the unique solution of the integral equation: for $t \geqslant 0$ and $x \in E$,

$$v(t,x) + \varPi_x\left[\int_0^t e_\beta(r)\kappa(\xi_r) v(t-r,\xi_r)^2 dr\right] = \varPi_x\left[\int_0^t e_\beta(r) g(\xi_r)\, dr\right]. \tag{2.33}$$

Then

$$\mathbb{P}_\mu\left[\exp\left(-\int_0^t \langle g, X_r\rangle\, dr\right)\right] = \exp\big(\big\langle -v(t,\cdot), \mu\big\rangle\big) \quad \text{for } t \geqslant 0,\ \mu \in M_{\mathrm{F}}(E). \tag{2.34}$$

Replacing the above g by θg and then differentiating (2.33) and (2.34) at $\theta = 0$ yields that for $t \geqslant 0$ and $\mu \in M_{\mathrm{F}}(E)$,

$$\mathbb{P}_\mu\left[\int_0^t \langle g, X_r\rangle\, dr\right] = \varPi_\mu\left[\int_0^t e_\beta(r) g(\xi_r)\, dr\right] \tag{2.35}$$

and

$$\mathrm{Var}_\mu\left(\int_0^t \langle g, X_r\rangle\,dr\right) = 2\Pi_\mu\left[\int_0^t e_\beta(r_1)\kappa(\xi_{r_1})\left(\Pi_{\xi_{r_1}}\left[\int_0^{t-r_1} e_\beta(r_2)g(\xi_{r_2})\,dr_2\right]\right)^2 dr_1\right]. \tag{2.36}$$

In order to adapt the Perkins' extension of the time dependent martingale problem, let us describe the martingale problem for X. X is the unique solution of the following martingale problem:

$$\mathbb{P}_\mu(X_0 = \mu) = 1,$$

and for every $\phi \in \mathcal{D}(\mathcal{A})$,

$$M_t(\phi) = \langle\phi, X_t\rangle - \langle\phi, X_0\rangle - \int_0^t \langle(\mathcal{A}+\beta)\phi, X_s\rangle\,ds \tag{2.37}$$

is a $\{\mathcal{G}_t\}$-local martingale such that

$$\langle M(\phi)\rangle_t = \int_0^t \langle\kappa\phi^2, X_s\rangle\,ds.$$

Perkins [16] extends the above result to time dependent functions. For the convenience of readers, we now recall some definitions and results from [16]. Let $\mathcal{P}$ be the σ-field of $\{\mathcal{G}_t\}$-predictable sets in $\mathbb{R}_+ \times \Omega$, and define

$$\mathcal{L}^2 = \Bigg\{g : \mathbb{R}_+ \times \Omega \times E \to \mathbb{R}\colon\ g \text{ is } \mathcal{P}\times\mathcal{B}(E)\text{-measurable}, \\ \mathbb{P}_\mu\left(\int_0^t \langle\kappa g_s^2, X_s\rangle\,ds\right) < \infty \quad \forall t > 0\Bigg\}.$$

It is shown in [16, Proposition II.5.4] that there is a martingale measure $M(t,x)$ so that for every $\phi \in \mathcal{D}(\mathcal{A})$, the $M_t(\phi)$ in (2.37) can be expressed as $\int_0^t \int \phi(x)\,dM(s,x)$. Moreover, for $g \in \mathcal{L}^2$, stochastic integral $M_t(g) := \int_0^t \int g(s,x)\,dM(s,x)$ is well defined and is a square integrable $\{\mathcal{G}_t\}$-martingale with

$$\langle M(g)\rangle_t = \int_0^t \langle\kappa g_s^2, X_s\rangle\,ds \quad \text{for every } t \geqslant 0 \text{ a.s.} \tag{2.38}$$

Definition 2.11. Let $T > 0$. A function $g : [0,T] \times E \to \mathbb{R}$ is said in $\mathcal{D}(\vec{\mathcal{A}})_T$ if and only if:

(i) For any x in E, $t \to g(t,x)$ is absolutely continuous and there is a jointly Borel measurable version of its Radon–Nikodym derivative $\frac{\partial g}{\partial t}(t,x)$ which is bounded on $[0,T] \times E$ and continuous in x for each $t \in [0,T]$.
(ii) For any $t \in [0,T]$, $g(t,\cdot)$ is in the domain of the Feller generator of $\mathcal{A}$ and that $\mathcal{A}g_t$ is bounded on $[0,T]$.

The following result is taken from [16, Proposition II.5.7].

Proposition 2.12. *If $g \in \mathcal{D}(\vec{\mathcal{A}})_T$, then a.s. for every $t \in [0,T]$,*

$$\langle g_t, X_t\rangle = \langle g_0, X_0\rangle + \int_0^t \int g(s,x)\,dM(s,x) + \int_0^t \left\langle \left(\frac{\partial}{\partial s} + \mathcal{A} + \beta\right) g(s,x), X_s \right\rangle ds.$$

3. Limit theorems

Throughout this section, we assume $\beta \in \mathbf{K}_\infty(\xi)$ and $\kappa \in \mathbf{K}_\infty(\xi)^+$, where ξ is a symmetric Hunt process on state space E with infinitesimal generator $\mathcal{A}$. Let $(X, \mathbb{P}_\mu, \mu \in M_{\mathrm{F}}(E))$ be the Dawson–Watanabe superprocess corresponding to operator $\mathcal{A}u + \beta u - \kappa u^2$.

Lemma 3.1. *Assume condition* (2.4) *holds and that $\lambda_1 := \lambda_1(\beta) < 0$. Then for any $f \in L^2(E,m)$,*

$$\lim_{t\to\infty} e^{\lambda_1 t}\mathbb{P}_{\delta_x}\big[\langle f, X_t\rangle\big] = h(x)\int_E f(y)h(y)\,m(dy) \quad \textit{for every } x \in E. \tag{3.1}$$

Proof. By (2.31),

$$e^{\lambda_1 t}\mathbb{P}_{\delta_x}\big[\langle f, X_t\rangle\big] = e^{\lambda_1 t}\Pi_x\big[e_\beta(t) f(\xi_t)\big].$$

So (3.1) follows immediately from (2.29). □

Let

$$M_t^h := e^{\lambda_1 t}\langle h, X_t\rangle, \quad t \geqslant 0 \tag{3.2}$$

Lemma 3.2. *For every $x \in E$, M_t^h is a positive martingale with respect to $\mathbb{P}_{\delta_x}$, and therefore there exists a limit $M_\infty^h \in [0,\infty)$, $\mathbb{P}_{\delta_x}$-a.s.*

Proof. By the Markov property of X, (2.31) and (2.24),

$$\mathbb{P}_{\delta_x}\big[M_{t+s}^h \big| \mathcal{F}_t\big] = e^{\lambda_1 t}\mathbb{P}_{X_t}\big[e^{\lambda_1 s}\langle h, X_s\rangle\big] = e^{\lambda_1 t}\big\langle e^{\lambda_1 s}\Pi_\cdot\big[e_\beta(s)h(\xi_s)\big], X_t\big\rangle = e^{\lambda_1 t}\langle h, X_t\rangle = M_t^h.$$

This proves that M^h is a positive martingale and so it has an almost sure limit M_∞^h as $t \to \infty$. □

Proposition 3.3. *Suppose that condition* (2.4) *holds and that* $\lambda_1 := \lambda_1(\beta) < 0$.

(i) *For every* $f \in L^2(E;m) \cap \mathcal{B}_b(E)$ *and* $x \in E$,

$$\lim_{t\to\infty} e^{\lambda_1 t}\langle f, X_t\rangle = M_\infty^h \int_E f(x)h(x)\,m(dx) \quad \textit{in probability with respect to } \mathbb{P}_{\delta_x}. \tag{3.3}$$

(ii) *Let* $\{t_n\}$ *be any sequence such that* $\sum_{n=1}^\infty e^{-\varepsilon t_n} < \infty$ *for some* $\varepsilon \in (0, (-\lambda - 1) \wedge (2\lambda_2^h))$. *Then for every* $f \in L^2(E;m) \cap \mathcal{B}_b(E)$ *and* $x \in E$,

$$\lim_{n\to\infty} e^{\lambda_1 t_n}\langle f, X_{t_n}\rangle = M_\infty^h \int_E f(x)h(x)\,m(dx) \quad \mathbb{P}_{\delta_x}\textit{-a.s.} \tag{3.4}$$

Proof. The main idea in the following proof is as follows. For any $f \in L^2(E;m) \cap \mathcal{B}_b(E)$, we decompose it orthogonally into $ch + g$ and show that $e^{\lambda_1 t}\langle g, X_t\rangle$ vanishes as $t \to \infty$.

(i) Let $f \in L^2(E;m) \cap \mathcal{B}(E)$ and $g(x) = f(x) - h(x)\int_E f(x)h(x)\,m(dx)$. Then

$$e^{\lambda_1 t}\langle f, X_t\rangle = M_t^h \int_E f(x)h(x)\,m(dx) + e^{\lambda_1 t}\langle g, X_t\rangle.$$

By (2.31) and (2.32),

$$\mathbb{P}_{\delta_x}\left[\left(e^{\lambda_1 t}\langle g, X_t\rangle\right)^2\right] = e^{2\lambda_1 t}\left(\mathbb{P}_{\delta_x}\left[\langle g, X_t\rangle\right]\right)^2 + \mathrm{Var}_{\delta_x}\left(\langle g, X_t\rangle\right) = I + II, \tag{3.5}$$

where

$$I := e^{2\lambda_1 t}\left(\Pi_x\left[e_\beta(t)g(\xi_t)\right]\right)^2 = \left(h(x)P_t^h(g/h)(x)\right)^2,$$

and

$$II := 2e^{2\lambda_1 t}\Pi_x\left[\int_0^t \kappa(\xi_r)e_\beta(r)\left(\Pi_{\xi_r}\left[e_\beta(t-r)g(\xi_{t-r})\right]\right)^2 dr\right].$$

Note that since g and h are orthogonal in $L^2(E,m)$, $\varphi := g/h \in L^2(E, h^2 m)$ with

$$\int_E \varphi(x)h^2(x)\,m(dx) = \int_E g(x)h(x)\,m(dx) = 0.$$

Therefore by (2.27),

$$I \leqslant c_1 e^{-2\lambda_2^h t}\|g/h\|^2_{L^2(E;h^2m)} = c_1 e^{-2\lambda_2^h t}\|g\|^2_{L^2(E;m)}, \quad t \geqslant 1. \tag{3.6}$$

We rewrite II as

$$II = 2\Pi_x\left[\int_0^t \kappa(\xi_r)e_{\beta+2\lambda_1}(r)\left(e^{\lambda_1(t-r)}\Pi_{\xi_r}\left[e_\beta(t-r)g(\xi_{t-r})\right]\right)^2 dr\right].$$

Let $\varepsilon \in (0,\ (-\lambda_1) \wedge (2\lambda_2^h))$. We have by (3.6)

$$\begin{aligned}II &\leqslant 2c_1\Pi_x\left[\int_0^t \kappa(\xi_r)e_{\beta+2\lambda_1}(r)e^{-2\lambda_2^h(t-r)}\,dr\right]\|g\|^2_{L^2(E;m)}\\ &\leqslant c_2e^{-\varepsilon t}\Pi_x\left[\int_0^\zeta \kappa(\xi_r)e_{\beta+2\lambda_1+\varepsilon}(r)\,dr\right]\|g\|^2_{L^2(E;m)}.\end{aligned}$$

By definition of $\lambda_1 = \lambda_1(\beta)$,

$$\begin{aligned}&\inf\left\{\mathcal{E}(u,u) - (2\lambda_1+\varepsilon)\int_E u^2\,m(dx) - \int_E u^2\beta\,m(dx):\ u \in \mathcal{F},\ \int_E u^2(x)\,m(dx) = 1\right\}\\ &\quad = -\lambda_1 - \varepsilon > 0.\end{aligned}$$

This implies by [21, Lemma 3.5] that

$$\inf\left\{\mathcal{E}(u,u) - (2\lambda_1+\varepsilon)\int_E u^2\,m(dx) + \int_E u^2\beta^-\,m(dx):\ u \in \mathcal{F},\ \int_E u^2(x)\beta^+\,m(dx) = 1\right\} > 1.$$

Let $\alpha := -(2\lambda_1+\varepsilon)$ and ξ^α be the subprocess of ξ killed at rate α. Clearly $\mathbf{K}_\infty(\xi) \subset \mathbf{K}_\infty(\xi^\alpha)$. As the Dirichlet form of ξ^α is $(\mathcal{E}_\alpha, \mathcal{F})$, applying Theorem 2.3(iv) to process ξ^α, we have

$$\sup_{x\in E}\Pi_x\left[\int_0^\zeta \kappa(\xi_r)e_{\beta+2\lambda_1+\varepsilon}(r)\,dr\right] < \infty,$$

and so

$$II \leqslant c_3e^{-\varepsilon t}\|g\|^2_{L^2(E;m)}.$$

We conclude then

$$\mathbb{P}_{\delta_x}\left[\left(e^{\lambda_1 t}\langle g, X_t\rangle\right)^2\right] \leqslant \left(c_1e^{-\lambda_2^h t} + c_3e^{-\varepsilon t}\right)\|g\|^2_{L^2(E;m)}. \tag{3.7}$$

This implies that $e^{\lambda_1 t}\langle f, X_t\rangle - M_t^h\int_E f(x)h(x)\,m(dx)$ converges to 0 in $L^2(\mathbb{P}_{\delta_x})$ as $t \to \infty$ and so (3.3) follows as M_t^h converges to M_∞^h a.s. as $t \to \infty$.

(ii) Let f and g be the same as above, respectively. By (3.7),

$$\sum_{n=1}^\infty \mathbb{P}_{\delta_x}\left[\left(e^{\lambda_1 t_n}\langle g, X_t\rangle\right)^2\right] \leqslant C\sum_{n=1}^\infty e^{-\varepsilon t_n} < \infty.$$

By Borel–Cantelli's lemma, we have $\lim_{n\to\infty} e^{\lambda_1 t_n}\langle g, X_{t_n}\rangle = 0$ a.s. and so (3.4) holds as $\lim_{t\to\infty} M_t^h = M_\infty^h$ a.s. $\square$

We remark that Proposition 3.3 remains true if condition $\kappa \in \mathbf{K}_\infty(\xi)$ is replaced by $\kappa \in \mathcal{B}_\mathrm{b}^+(\mathbb{R}^d)$. This is because in this case, $\sup_{x\in E} \Pi_x[\int_0^\zeta \kappa(\xi_r)e_{\beta+2\lambda_1+\varepsilon}(r)\,dr]$ in above proof can be bounded by

$$\|\kappa\|_\infty \sup_{x\in E} \Pi_x\left[\int_0^\zeta e^{-\delta r} e_{\beta+2\lambda_1+\varepsilon+\delta}(r)\,dr\right] \leqslant c\int_0^\infty e^{-\delta r}\,dr < \infty,$$

where $\delta > 0$ is a sufficiently small constant so that $-\lambda_1 - \varepsilon - \delta > 0$. Here in the first inequality, we used Theorem 2.3(iii) applied to the subprocess ξ^α with $\alpha = -2\lambda_1 - \varepsilon - \delta$.

We now turn our attention to almost sure scaling limit theorem for superprocesses. Throughout the remainder of this section, we assume that either

(i) $(E, m) = (\mathbb{R}^d, dx)$ and ξ is a symmetric Lévy process in $\mathbb{R}^d$ with characteristic exponent Ψ satisfying condition (2.17); or
(ii) $(E, m) = (\mathbb{R}^d, \rho(x)\,dx)$ and ξ is a symmetric diffusion on $\mathbb{R}^d$ with infinitesimal generator

$$\mathcal{A} = \rho(x)^{-1}\nabla\cdot(\rho A\nabla) = \nabla\cdot(A\nabla) + A\nabla(\log\rho)\cdot\nabla, \tag{3.8}$$

where $A(x) = (a_{ij}(x))_{1\leqslant i,j\leqslant d}$ is bounded and uniformly elliptic on $\mathbb{R}^d$ with $a_{ij} \in C_\mathrm{b}^1(\mathbb{R}^d)$ and $\rho \in C_\mathrm{b}^1(\mathbb{R}^d)$ is bounded between two positive constants.

We further assume that $\beta \in \mathbf{K}_\infty(\xi) \cap C_\mathrm{b}(\mathbb{R}^d)$ and $\kappa \in \mathcal{B}_\mathrm{b}^+(\mathbb{R}^d)$.

Note that a diffusion process ξ with infinitesimal generator $\mathcal{A}$ given by (3.8) is symmetric with respect to the measure $m(dx) := \rho(x)\,dx$ and its Dirichlet form $(\mathcal{E}, \mathcal{F})$ in $L^2(\mathbb{R}^d, m)$ is given by (2.22), (2.23). It is known that in both cases, the process ξ has double Feller property; that is, it has strong Feller property as well as the Feller property (i.e. $P_t(C_\infty(\mathbb{R}^d)) \subset C_\infty(\mathbb{R}^d)$ for every $t > 0$ and $\lim_{t\to\infty}\|P_t f - f\|_\infty = 0$ for every $f \in C_\infty(\mathbb{R}^d)$). Moreover, $C_\mathrm{c}^2(\mathbb{R}^d)$ is in the domain of Feller generator $\mathcal{A}$ of ξ.

Here is our main result on almost sure scaling limit theorem for superprocesses.

Theorem 3.4. *Under the above assumption, suppose that $\lambda_1 := \lambda_1(\beta) < 0$. Then there exists $\Omega_0 \subset \Omega$ of probability one (that is, $\mathbb{P}_{\delta_x}(\Omega_0) = 1$ for every $x \in \mathbb{R}^d$) such that, for every $\omega \in \Omega_0$ and for every bounded Borel measurable function f on $\mathbb{R}^d$ with compact support whose set of discontinuous points has zero m-measure, we have*

$$\lim_{t\to\infty} e^{\lambda_1 t}\langle f, X_t\rangle = M_\infty^h \int_{\mathbb{R}^d} f(x)h(x)\,m(dx). \tag{3.9}$$

Note that by Theorems 2.6 and 2.8, under the condition of Theorem 3.4 for process ξ, Assumption 2.1 is satisfied and the condition (2.4) holds for every $\beta \in \mathbf{K}_\infty(\xi)$ with non-trivial β^+. So Proposition 3.3 applies. To prove Theorem 3.4 we need some lemmas. Let U be a bounded open set in $\mathbb{R}^d$. For every $\varepsilon > 0$, choose $\phi_\varepsilon \in C_\mathrm{c}^2(\mathbb{R}^d)$ such that $0 \leqslant \phi_\varepsilon \leqslant \mathbf{1}_U$ and $\phi_\varepsilon = 1$ on

$U^\varepsilon = \{x \in U, d(x, \partial U) > \varepsilon\}$, where $d(x, \partial U)$ denotes the distance between x and ∂U. Clearly we have

$$\langle \mathbf{1}_U h, X_t\rangle \geqslant \langle \phi_\varepsilon h, X_t\rangle, \quad t \geqslant 0. \tag{3.10}$$

Lemma 3.5. *Assume that either*:

(i) $(E, m) = (\mathbb{R}^d, \rho(x)\,dx)$ *and* ξ *a symmetric diffusion with infinitesimal generator* $\mathcal{A}$ *given by* (3.8); *or*
(ii) $(E, m) = (\mathbb{R}^d, dx)$ *and* ξ *is a symmetric Lévy process in* $\mathbb{R}^d$ *with characteristic exponent* Ψ *satisfying condition* (2.17).

Assume that $\beta \in \mathbf{K}_\infty(\xi) \cap C_b(\mathbb{R}^d)$. *Then both* h *and* $\phi_\varepsilon h$ *are in the domain of the Feller generator* $\mathcal{A}$ *of* ξ *with*

$$M := \sup_{x\in\mathbb{R}^d} \bigl|\mathcal{A}(\phi_\varepsilon h)(x)\bigr| < \infty.$$

Moreover, $\mathcal{A}(\phi_\varepsilon h)(x) \in L^1(\mathbb{R}^d, m)$.

Proof. Recall that (see Section 2.2) the function h is bounded continuous and strictly positive, and

$$h(x) = e^{\lambda_1 t}\Pi_x\bigl[e_\beta(t)h(\xi_t)\bigr] = \Pi_x\bigl[e_{\beta+\lambda_1}(t)h(\xi_t)\bigr] \quad \text{for every } x \in \mathbb{R}^d \text{ and } t > 0.$$

It follows by the Markov property of ξ and the double Feller property of ξ,

$$\begin{aligned}\Pi_x\bigl[h(\xi_t)\bigr] - h(x) &= -\Pi_x\left[\int_0^t \bigl(\beta(\xi_s) + \lambda_1\bigr)e^{\int_s^t(\beta(\xi_r)+\lambda_1)\,dr}h(\xi_t)\,ds\right]\\ &= -\Pi_x\left[\int_0^t \bigl(\beta(\xi_s) + \lambda_1\bigr)h(\xi_s)\,ds\right].\end{aligned}$$

This implies that $\frac{1}{t}\Pi_x[h(\xi_t)] - h(x)$ converges uniformly on $\mathbb{R}^d$ to $-(\beta + \lambda_1)h$ as $t \downarrow 0$; that is h is in the domain of the Feller generator $\mathcal{A}$ of ξ and $\mathcal{A}h = -(\beta + \lambda_1)h$.

When ξ is a symmetric diffusion whose infinitesimal generator $\mathcal{A}$ is given by (3.8), as $\nabla(A\nabla h) = -(\beta + \lambda_1)\rho h$, it is known (cf. [14]) that $h \in C^1(\mathbb{R}^d)$. Therefore

$$\begin{aligned}&\frac{1}{t}\Pi_x\left[\left(\phi_\varepsilon(\xi_t) - \phi_\varepsilon(\xi_0) - \int_0^t \mathcal{A}\phi_\varepsilon(\xi_s)\,ds\right)\left(h(\xi_t) - h(\xi_0) - \int_0^t \mathcal{A}h(\xi_s)\,ds\right)\right]\\ &\quad= \frac{1}{t}\Pi_x\left[\int_0^t \nabla\phi_\varepsilon(\xi_s)\cdot A(\xi_s)\nabla h(\xi_s)\,ds\right]\end{aligned}$$

converges uniformly on $\mathbb{R}^d$ to $\nabla\phi_\varepsilon(x)\cdot A(x)\nabla h(x)$ as $t \downarrow 0$. We denote the latter by $\Gamma(\phi_\varepsilon, h)(x)$.

When ξ is a symmetric Lévy process in $\mathbb{R}^d$ with characteristic exponent Ψ satisfying condition (2.17), it is known in this case that $C_c^2(\mathbb{R}^d) \subset \mathcal{D}(\mathcal{A})$. Moreover, since for every $k \geqslant 0$ and $p \geqslant 1$,

$$|\eta|^k e^{-t\Psi(\eta)} \in L^p(\mathbb{R}^d) \quad \text{and} \quad \int_0^t |\eta|^k e^{-s\Psi(\eta)}\, ds \in L^p\big(\mathbb{R}^d\big) \quad \text{for every } t > 0,$$

we have for every $t > 0$ and $f \in L^2(\mathbb{R}^d)$, $P_t f \in C^1(\mathbb{R}^d)$ and $\int_0^t P_s f\, ds \in C^1(\mathbb{R}^d)$. (Furthermore, the transition density function $p(t, x, y)$ of ξ, which is given by (2.6), is C^∞-smooth in x and in y.) As $\mathcal{A}h = -(\beta + \lambda_1)h$ and $h \in L^2(\mathbb{R}^d) \cap \mathcal{B}_b(\mathbb{R}^d)$, we deduce from the identity

$$h = P_t h - \int_0^t P_s \mathcal{A}h\, ds = P_t h + \int_0^t P_s\big((\beta + \lambda_1)h\big)\, ds$$

that $h \in C^1(\mathbb{R}^d)$. Recall that the characteristic exponent of ξ has decomposition (2.7). The function

$$\varphi(x) := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial\phi_\varepsilon(x)}{\partial x_i}\frac{\partial h(x)}{\partial x_j} + \int_{\mathbb{R}^d}\big(\phi_\varepsilon(x+y) - \phi_\varepsilon(x)\big)\big(h(x+y) - h(x)\big)\, J(dy),$$

is bounded on $\mathbb{R}^d$, as by mean-value theorem, the last term above can be bounded as

$$\begin{aligned}
&\left|\Bigg(\int_{\{|y|\leqslant 1\}} + \int_{\{|y|>1\}}\Bigg)\big(\phi_\varepsilon(x+y) - \phi_\varepsilon(x)\big)\big(h(x+y) - h(x)\big)\, J(dy)\right| \\
&\quad \leqslant \left|\int_{\{|y-x|\leqslant 1\}}\big(\nabla\phi_\varepsilon(z_{x,y})\cdot y\big)\big(\nabla h(z_{x,y})\cdot y\big)\, J(dy)\right| + 4\|\phi_\varepsilon\|_\infty \|h\|_\infty J\big(\big\{|y| > 1\big\}\big) \\
&\quad \leqslant c\|\nabla\phi_\varepsilon\|_\infty\Big(\sup_{z\in\overline{U}}\big|\nabla h(z)\big|\Big)\int_{0<|y|\leqslant 1}|y|^2\, J(dy) + 4\|\phi_\varepsilon\|_\infty\|h\|_\infty J\big(\big\{|y| > 1\big\}\big) \\
&\quad < \infty,
\end{aligned}$$

where $z_{x,y}$ in the second inequality is a point in the line segment that connects x and $x + y$, and (2.8) is used in the last inequality. Moreover, a similar calculation shows that $\varphi(x)$ can be approximated uniformly on $\mathbb{R}^d$ by

$$\varphi_k(x) := \frac{1}{2}\sum_{i,j=1}^d a_{ij}\frac{\partial\phi_\varepsilon(x)}{\partial x_i}\frac{\partial h(x)}{\partial x_j} + \int_{\{|y|>1/k\}}\big(\phi_\varepsilon(x+y) - \phi_\varepsilon(x)\big)\big(h(x+y) - h(x)\big)\, J(dy)$$

and so φ is bounded and continuous on $\mathbb{R}^d$. Now using the Lévy system of ξ, we see that

$$\frac{1}{t}\Pi_x\left[\left(\phi_\varepsilon(\xi_t)-\phi_\varepsilon(\xi_0)-\int_0^t \mathcal{A}\phi_\varepsilon(\xi_s)\,ds\right)\left(h(\xi_t)-h(\xi_0)-\int_0^t \mathcal{A}h(\xi_s)\,ds\right)\right]$$
$$=\frac{c_{d,\alpha}}{t}\Pi_x\left[\int_0^t \varphi(\xi_s)\,ds\right]$$

converges uniformly on $\mathbb{R}^d$ to $c_{d,\alpha}\varphi(x)$ as $t\downarrow 0$. We denote the latter as $\Gamma(\phi_\varepsilon,h)(x)$.

Thus in both cases of ξ, we have

$$\begin{aligned}
&\lim_{t\downarrow 0}\frac{1}{t}\big(T_t(\phi_\varepsilon h)(x)-\phi_\varepsilon(x)h(x)\big)\\
&\quad=\lim_{t\downarrow 0}\frac{1}{t}\Pi_x\big[(\phi_\varepsilon h)(\xi_t)-(\phi_\varepsilon h)(\xi_0)\big]\\
&\quad=\lim_{t\downarrow 0}\frac{1}{t}h(x)\Pi_x\big[\phi_\varepsilon(\xi_t)-\phi_\varepsilon(\xi_0)\big]+\lim_{t\downarrow 0}\frac{1}{t}\phi_\varepsilon(x)\Pi_x\big[h(\xi_t)-h(\xi_0)\big]\\
&\qquad+\lim_{t\downarrow 0}\frac{1}{t}\Pi_x\big[\big(\phi_\varepsilon(\xi_t)-\phi_\varepsilon(\xi_0)\big)\big(h(\xi_t)-h(\xi_0)\big)\big]
\end{aligned}$$

converges uniformly on $\mathbb{R}^d$ to

$$h(x)\mathcal{A}\phi_\varepsilon(x)+\phi_\varepsilon(x)\mathcal{A}h(x)+\Gamma(\phi_\varepsilon,h)(x).$$

This proves that $\phi_\varepsilon h$ is in the domain of the Feller generator $\mathcal{D}(\mathcal{A})$ of ξ. In particular, $\|\mathcal{A}(\phi_\varepsilon h)\|_\infty<\infty$. That $\mathcal{A}(\phi_\varepsilon h)(x)\in L^1(\mathbb{R}^d,m)$ follows from the Cauchy–Schwarz inequality and the fact that $\mathcal{E}(\phi_\varepsilon,\phi_\varepsilon)+\mathcal{E}(h,h)<\infty$. $\square$

Remark 3.6. The assumption that ξ is either a symmetric diffusion having infinitesimal generator $\mathcal{A}$ of type (3.8) or ξ is a symmetric Lévy process on $\mathbb{R}^d$ with characteristic exponent Ψ satisfying condition (2.17) is only used in this paper to show that $\phi_\varepsilon h$ is in the domain of the Feller generator of ξ and that it satisfies the Assumption 2.1 and condition (2.4). The proof above for h being in the domain of Feller generator of ξ requires only that ξ has double Feller property. Lemma 3.5 is used below in order to apply Ito type formula (Proposition 2.12) for superprocess X.

By Lemma 3.5, we can apply Proposition 2.12 to $g(t,x)=e^{\lambda_1 t}\phi_\varepsilon(x)h(x)$ to get that for every $t\in[n\delta,(n+1)\delta)$,

$$\begin{aligned}
\big\langle e^{\lambda_1 t}\phi_\varepsilon h, X_t\big\rangle &= e^{\lambda_1 n\delta}\langle\phi_\varepsilon h, X_{n\delta}\rangle+\int_{n\delta}^t\int e^{\lambda_1 s}\phi_\varepsilon(x)h(x)\,dM(s,x)\\
&\quad+\int_{n\delta}^t\left\langle\left(\frac{\partial}{\partial s}+\mathcal{A}+\beta\right)e^{\lambda_1 s}\phi_\varepsilon h, X_s\right\rangle ds.
\end{aligned}$$

Note that

$$\left(\frac{\partial}{\partial s}+\mathcal{A}+\beta\right)e^{\lambda_1 s}\phi_\varepsilon(x)h(x)=e^{\lambda_1 s}\big(\mathcal{A}(\phi_\varepsilon h)(x)+(\beta+\lambda_1)\phi_\varepsilon(x)h(x)\big).$$

Put $g(x):=|\mathcal{A}(\phi_\varepsilon h)(x)+(\beta+\lambda_1)\phi_\varepsilon(x)h(x)|$, which is bounded and $L^2(\mathbb{R}^d,m)$-integrable by Lemma 3.5. Since $\beta(x)$ is bounded in U, we have from the above and (3.10) that for $t\in[n\delta,(n+1)\delta)$,

$$e^{\lambda_1 t}\langle \mathbf{1}_U h, X_t\rangle \geqslant e^{\lambda_1 t}\langle \phi_\varepsilon h, X_t\rangle \geqslant e^{\lambda_1 n\delta}\langle \phi_\varepsilon h, X_{n\delta}\rangle - \left|D_t^\varepsilon\right| - S_n^{\delta,\varepsilon}, \tag{3.11}$$

where

$$D_t^\varepsilon=\int_{n\delta}^{t}\int e^{\lambda_1 s}\phi_\varepsilon(x)h(x)\,dM(s,x)\quad \text{for } t\in\big[n\delta,(n+1)\delta\big),$$

and

$$S_n^{\delta,\varepsilon}=e^{\lambda_1 n\delta}\int_{n\delta}^{(n+1)\delta}\langle g, X_s\rangle\,ds.$$

Lemma 3.7. *Under the conditions of Theorem* 3.4*, for every probability measure μ on $\mathbb{R}^d$ and each fixed $\varepsilon>0$,*

$$\lim_{t\to\infty} D_t^\varepsilon = 0\quad \mathbb{P}_\mu\text{-a.s.} \tag{3.12}$$

Proof. By the Borel–Cantelli lemma, it suffices to show that for every $\varepsilon'>0$,

$$\sum_{k\geqslant 1}\mathbb{P}_\mu\Big[\sup_{t\in[k\delta,(k+1)\delta]}\left|D_t^\varepsilon\right|\geqslant\varepsilon'\Big]<\infty. \tag{3.13}$$

Since D_t^ε is a martingale, we have by Doob's maximal inequality that for each $k\geqslant 1$,

$$\mathbb{P}_\mu\Big[\sup_{t\in[k\delta,(k+1)\delta]}\left|D_t^\varepsilon\right|\geqslant\varepsilon'\Big]\leqslant\frac{\mathbb{P}_\mu[(D^\varepsilon_{(k+1)\delta})^2]}{\varepsilon'^2}.$$

Recall that for $f\in\mathcal{B}_{\mathrm{b}}(\mathbb{R}^d)$,

$$P_t^\beta f(x)=\Pi_x\big[e_\beta(t)f(\xi_t)\big],\quad x\in\mathbb{R}^d,\ t\geqslant 0.$$

By (2.24), (2.31), (2.38), and the Markov property of X, for every $\delta\in(0,1)$,

$$\begin{aligned}
\mathbb{P}_\mu\big[\big(D^\varepsilon_{(k+1)\delta}\big)^2\big] &= \mathbb{P}_\mu\Bigg[\Bigg(\int_{k\delta}^{(k+1)\delta}\int e^{\lambda_1 s}\phi_\varepsilon(x)h(x)\,dM(s,x)\Bigg)^2\Bigg] \\
&\leqslant e^{2\lambda_1 k\delta}\mathbb{P}_\mu\Bigg[\int_{k\delta}^{(k+1)\delta}\big\langle\kappa(\phi_\varepsilon h)^2, X_s\big\rangle\,ds\Bigg] \\
&= e^{2\lambda_1 k\delta}\int_{k\delta}^{(k+1)\delta}\mathbb{P}_\mu\big[\big\langle P^\beta_{s-k\delta}\big(\kappa h^2\big), X_{k\delta}\big\rangle\big]\,ds \\
&\leqslant \|\kappa h\|_\infty e^{(2k-1)\lambda_1\delta}\int_{k\delta}^{(k+1)\delta}\mathbb{P}_\mu\big[\big\langle e^{\lambda_1(s-k\delta)}P^\beta_{s-k\delta}h, X_{k\delta}\big\rangle\big]\,ds \\
&= \|\kappa h\|_\infty e^{(2k-1)\lambda_1\delta}\int_{k\delta}^{(k+1)\delta}\mathbb{P}_\mu\big[\langle h, X_{k\delta}\rangle\big]\,ds \\
&= \delta\|\kappa h\|_\infty e^{(k-1)\lambda_1\delta}\big\langle e^{\lambda_1 k\delta}P^\beta_{k\delta}h, \mu\big\rangle \\
&\leqslant \|\kappa h\|_\infty e^{(k-1)\lambda_1\delta}\big\langle h, \mu\big\rangle.
\end{aligned}$$

Thus we have

$$\sum_{k=1}^\infty e^{2\lambda_1 k\delta}\mathbb{P}_\mu\big[\big(D^\varepsilon_{(k+1)\delta}\big)^2\big] < \infty,$$

which implies (3.13). $\square$

Lemma 3.8. *Under the conditions of Theorem* 3.4*, for every probability measure* μ *on* $\mathbb{R}^d$*,*

$$\lim_{n\to\infty}\big(S_n^{\delta,\varepsilon} - \mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}\big|\mathcal{G}_{n\delta}\big]\big) = 0 \quad \mathbb{P}_\mu\text{-a.s.}$$

Proof. Note that

$$\mathbb{P}_\mu\big[\big(S_n^{\delta,\varepsilon} - \mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}\big|\mathcal{G}_{n\delta}\big]\big)^2\big] = \mathbb{P}_\mu\big[\mathbb{P}_\mu\big[\big(S_n^{\delta,\varepsilon}\big)^2\big|\mathcal{G}_{n\delta}\big] - \big(\mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}\big|\mathcal{G}_{n\delta}\big]\big)^2\big].$$

By the Markov property of X we have

$$\mathbb{P}_\mu\big[\big(S_n^{\delta,\varepsilon}\big)^2\big|\mathcal{G}_{n\delta}\big] = e^{2\lambda_1 n\delta}\mathbb{P}_{X_{n\delta}}\Bigg(\int_0^\delta\langle g, X_s\rangle\,ds\Bigg)^2,$$

and

$$\mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}\big|\mathcal{G}_{n\delta}\big] = e^{\lambda_1 n\delta}\mathbb{P}_{X_{n\delta}}\int_0^\delta \langle g, X_s\rangle\,ds.$$

Then we have

$$\mathbb{P}_\mu\big[\big(S_n^{\delta,\varepsilon}\big)^2\big|\mathcal{G}_{n\delta}\big] - \mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}|\mathcal{G}_{n\delta}\big]^2 = e^{2\lambda_1 n\delta}\,\mathrm{Var}_{X_{n\delta}}\left(\int_0^\delta \langle g, X_s\rangle\,ds\right).$$

By the variation formula (2.36), we have for $\delta \in (0,1)$,

$$\begin{aligned}
&\mathrm{Var}_{X_{n\delta}}\left(\int_0^\delta \langle g, X_s\rangle\,ds\right)\\
&\quad = 2\Pi_{X_{n\delta}}\left[\int_0^\delta e_\beta(s)\kappa(\xi_s)\left(\Pi_{\xi_s}\left[\int_0^{\delta-s} e_\beta(r)g(\xi_r)\,dr\right]\right)^2 ds\right]\\
&\quad \leqslant 2\|\kappa\|_\infty\|g\|_\infty^2\Pi_{X_{n\delta}}\left[\int_0^\delta e_\beta(s)\left(\Pi_{\xi_s}\left[\int_0^{\delta-s} e_\beta(r)\,dr\right]\right)^2 ds\right]\\
&\quad \leqslant c_1\|\kappa\|_\infty\Pi_{X_{n\delta}}\left[\int_0^\delta e_\beta(s)\,ds\right]\\
&\quad \leqslant c_2\|\kappa\|_\infty\,\langle 1, X_{n\delta}\rangle,
\end{aligned}$$

where in the last two inequalities, we used (2.2) with $t = 1$. Therefore together with (2.31),

$$\begin{aligned}
\mathbb{P}_\mu\big[\big(S_n^{\delta,\varepsilon} - \mathbb{P}_\mu\big[S_n^{\delta,\varepsilon}\big|\mathcal{G}_{n\delta}\big]\big)^2\big] &\leqslant c_2\,\|\kappa\|_\infty e^{2\lambda_1 n\delta}\mathbb{P}_\mu\langle 1, X_{n\delta}\rangle\\
&= c_2\|\kappa\|_\infty e^{2\lambda_1 n\delta}\big\langle \Pi_\cdot\big[e_\beta(n\delta)\big], \mu\big\rangle\\
&= c_2\|\kappa\|_\infty\, e^{\lambda_1 n\delta/2}\big\langle \Pi_\cdot\big[e_{\frac{3}{2}\lambda_1+\beta}(n\delta)\big], \mu\big\rangle.
\end{aligned}$$

By definition of $\lambda_1 = \lambda_1(\beta)$,

$$\inf\left\{\mathcal{E}(u,u) - \frac{3}{2}\lambda_1\int_{\mathbb{R}^d} u^2\,m(dx) - \int_{\mathbb{R}^d} u^2\beta\,m(dx)\colon\, u\in\mathcal{F}, \int_{\mathbb{R}^d} u^2(x)\,m(dx) = 1\right\} = -\frac{1}{2}\lambda_1 > 0.$$

This implies by [21, Lemma 3.5] that

$$\inf\left\{\mathcal{E}(u,u) - \frac{3}{2}\lambda_1\int_{\mathbb{R}^d} u^2\,m(dx) + \int_{\mathbb{R}^d} u^2\beta^-\,m(dx)\colon\, u\in\mathcal{F}, \int_{\mathbb{R}^d} u^2(x)\beta^+\,m(dx) = 1\right\} > 1.$$

Let $\alpha := -3\lambda_1/2$ and denote by ξ^α the subprocess of ξ killed at rate α. Clearly the resolvent of ξ^α is dominated by that of ξ and so $\mathbf{K}_\infty(\xi) \subset \mathbf{K}_\infty(\xi^\alpha)$. As the Dirichlet form of ξ^α is $(\mathcal{E}_\alpha, \mathcal{F})$, by applying Theorem 2.3(iii) to ξ^α, there is a constant $C > 0$ such that

$$\sup_{x\in\mathbb{R}^d, t\geqslant 0} \Pi_x\left[e_{\frac{3}{2}\lambda_1+\beta}(t)\right] \leqslant C.$$

Therefore we have

$$\mathbb{P}_\mu\left[\left(S_n^{\delta,\varepsilon} - \mathbb{P}_\mu\left[S_n^{\delta,\varepsilon} \middle| \mathcal{G}_{n\delta}\right]\right)^2\right] \leqslant C\, e^{\frac{1}{2}\lambda_1 n\delta}$$

and consequently

$$\sum_{n=0}^{\infty} \mathbb{P}_\mu\left[\left(S_n^{\delta,\varepsilon} - \mathbb{P}_{\delta_x}\left[S_n^{\delta,\varepsilon} \middle| \mathcal{G}_{n\delta}\right]\right)^2\right] < \infty.$$

The lemma follows by an application of Borel–Cantelli's lemma. $\square$

Lemma 3.9. *Under the conditions of Theorem* 3.4*, we have for every probability measure* μ *on* $\mathbb{R}^d$*,*

$$\liminf_{t\to\infty} e^{\lambda_1 t} \langle \mathbf{1}_U h, X_t\rangle \geqslant M_\infty^h \int_U h(x)^2\, m(dx) \quad \mathbb{P}_\mu\text{-a.s.} \tag{3.14}$$

Proof. Recall inequality (3.11). By Theorems 2.6, 2.8 and Proposition 3.3,

$$\lim_{n\to\infty} e^{\lambda_1 n\delta} \langle \phi_\varepsilon h, X_{n\delta}\rangle = M_\infty^h \int_{\mathbb{R}^d} \phi_\varepsilon(x) h(x)^2\, m(dx).$$

Note that

$$\mathbb{P}_\mu\left[S_n^{\delta,\varepsilon} \middle| \mathcal{G}_{n\delta}\right] = e^{\lambda_1 n\delta} \int_0^\delta \left\langle \Pi_\cdot\left[e_\beta(s) g(\xi_s)\right], X_{n\delta}\right\rangle ds = e^{\lambda_1 n\delta} \left\langle \int_0^\delta P_s^\beta g\, ds, X_{n\delta}\right\rangle.$$

It follows from (3.11), Proposition 3.3(ii), Lemmas 3.7 and 3.8 that for every $\varepsilon > 0$,

$$\liminf_{t\to\infty} e^{\lambda_1 t} \langle \mathbf{1}_U h, X_t\rangle \geqslant M_\infty^h \int_{U^\varepsilon} h(x)^2\, m(dx) - M_\infty^h \int_{\mathbb{R}^d} \left(\int_0^\delta P_s^\beta g(x)\, ds\right) h(x)\, m(dx).$$

First letting $\delta \to 0$ and then $\varepsilon \to 0$, we obtain the desired result (3.14). $\square$

Proof of Theorem 3.4. Let $\mathcal{U} = \{U_k, k \geqslant 1\}$ be a countable base of open sets on $\mathbb{R}^d$ that is closed under finite union. Define

$$\Omega_0 := \left\{\omega \in \Omega\colon \lim_{t\to\infty} e^{\lambda_1 t} \left\langle \mathbf{1}_{U_k} h, X_t(\omega)\right\rangle \geqslant M_\infty^h \int_{U_k} h(x)^2\, m(dx)\right\}.$$

By Lemma 3.9, $\mathbb{P}_{\delta_x}(\Omega_0) = 1$ for every $x \in \mathbb{R}^d$. For any open set U, there exists a sequence of increasing open sets $\{U_{n_k}, k \geqslant 1\}$ in $\mathcal{U}$ so that $\bigcup_{k=1}^{\infty} U_{n_k} = U$. We have for every $\omega \in \Omega_0$,

$$\liminf_{t\to\infty} e^{\lambda_1 t}\langle \mathbf{1}_U h, X_t(\omega)\rangle \geqslant \liminf_{t\to\infty} e^{\lambda_1 t}\langle \mathbf{1}_{U_{n_k}} h, X_t(\omega)\rangle = M_\infty^h(\omega) \int_{U_{n_k}} h(x)^2\, m(dx)$$

for every $k \geqslant 1$. Letting $k \to \infty$ yields that

$$\liminf_{t\to\infty} e^{\lambda_1 t}\langle \mathbf{1}_U h, X_t(\omega)\rangle \geqslant M_\infty^h(\omega) \int_U h(x)^2\, m(dx).$$

The remaining part of the proof is similar to that of [6, Theorem 3.7]. We omit the details. □

Combining Theorem 3.4 with Lemma 3.1, we obtain

Corollary 3.10 *(Strong law of large numbers). Suppose that the conditions of Theorem* 3.4 *hold. Let Ω_0 be the same as in Theorem* 3.4. *Then*

$$\lim_{t\to\infty} \frac{X_t(B)(\omega)}{\mathbb{P}_{\delta_x}[X_t(B)]} = \frac{M_\infty^h(\omega)}{h(x)}$$

for every $\omega \in \Omega_0$, and for every relatively compact Borel subset B in $\mathbb{R}^d$ having $m(B) > 0$ and $m(\partial B) = 0$.

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