Notes on Agents' Behavioral Rules Under Adaptive Learning and Recent Studies of Monetary Policy

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Abstract

These notes try to clarify some recent discussions on the formulation of individual intertemporal behavior under adaptive learning in representative agent models. First, we discuss two suggested approaches and related issues in the context of a simple consumption-saving model. Second, we show that the recent analysis of learning and monetary policy based on "Euler equations" provides a consistent and valid approach, contrary to some claims.

1 Introduction

In the literature on adaptive learning in infinite horizon representative agent settings it is often assumed that agents base their behavior on an Euler equation that is derived under subjective expectations.[1] This formulation has recently been criticized in that it does not require that the intertemporal budget constraint be satisfied for the agent since the constraint is not explicitly used when deriving the behavioral rule of the agent.[2]

Another point of criticism has been that the formulation is not natural since it postulates that agents are making forecasts of their future consumption, which is their own choice variable. Recently, (Preston 2002) has proposed an interesting reformulation of (linearized) intertemporal behavior under learning in which agents are assumed to incorporate a "subjective version" of their intertemporal budget constraint in their behavior under learning.[3] In an earlier version he also asserts that the temporary equilibrium equations, described in part in terms of Euler equations with subjective expectations, in (Bullard and Mitra 2002) and (Evans and Honkapohja 2002a) is subject to inconsistency when subjective expectations are used in equilibrium equations that have normally been derived under rational expectations.

In these notes we first clarify the relationship between two formulations of intertemporal behavior under adaptive learning and show that the intertemporal accounting consistency holds in an *ex post* sense along the sequence of temporary equilibria under "Euler equation" learning. This is done in the simple context of a consumption-saving model. Second, we consider the (Preston 2002) model of monetary policy under learning and show that, under plausible assumptions, the usual system based on Euler equations with subjective expectations can be obtained from Preston's approach. Thus the claims about inconsistency of the Euler equation approach are not correct.

2 A Permanent Income Model

Consider a model in which income follows an exogenous process and there is a representative consumer who makes consumption-saving decisions.[4] The

[1] This is done e.g. in the early paper (Marcet and Sargent 1989), Example e and in some models of Chapter 10 of (Evans and Honkapohja 2001). See also the discussion in (Sargent 1993).

[2] This point has been made in the specific context of New Keynesian models of monetary policy. The approach based on Euler equations is used e.g. in (Bullard and Mitra 2002) and (Evans and Honkapohja 2002a).

[3] The earlier version is dated February 2002.

[4] The results remain unchanged if it is assumed instead that there is finite (or infinite) number of consumers with identical characteristics, including their forecasts and learning

consumer has a standard intertemporal utility function

$$\hat{E}_t \sum_{s=t}^{\infty} \beta^{s-t} U(C_s) \tag{1}$$

and the accounting identity for net assets W_s is

$$W_{s+1} = R_s W_s - C_s + Y_s. \tag{2}$$

For the initial period of the economy net assets are taken to be zero, i.e. $W_t = 0$.[5] R_s is the one-period real gross rate of return factor for a safe one-period loan, assumed known at s. Because we are in a general equilibrium framework we do not take it to be fixed and its value will be determined by market clearing. Output Y_s follows an exogenous process

$$Y_s = M Y_{s-1}^{\rho} V_s \tag{3}$$

or

$$\log Y_s = \mu + \rho \log Y_{s-1} + v_s,$$

where $|\rho| < 1$ and v_s is white noise. Expectations are not necessarily rational, which is indicated by ˆ in the expectations operator. There is also an intertemporal budget constraint of the form

$$C_t + \sum_{s=t+1}^{\infty} \mathcal{R}_{t+1,s} C_s = Y_t + \sum_{s=t+1}^{\infty} \mathcal{R}_{t+1,s} Y_s, \tag{4}$$

where $\mathcal{R}_{t+1,s} = (R_{t+1} \dots R_s)^{-1}$ is the market discount factor.

Maximizing (1) subject to (4) yields the Euler equation as a necessary condition. It has the familiar form

$$U'(C_t) = \beta R_t \hat{E}_t U'(C_{t+1}) \tag{5}$$

and in equilibrium $C_t = Y_t$, as output is assumed to be perishable. In this temporary equilibrium framework, agents' demand for consumption goods C_t depends on their forecast $\hat{E}_t U'(C_{t+1})$ and on the interest rate factor R_t,

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rules.

[5] Note that this is a very simple general equilibrium model of a closed economy. Thus there cannot be any net paper assets (like bonds) before the economy starts.

in accordance with (5). Imposing the market clearing condition $C_t = Y_t$ we see that (5) determines the interest rate according to

$$R_t^{-1} = \beta(\hat{E}_t U'(C_{t+1}))/U'(Y_t).$$

This gives us the temporary equilibrium at t.

We now log-linearize (5) at a non-stochastic steady state. Standard computations yield

$$c_t = \hat{E}_t c_{t+1} - \sigma r_t, \tag{6}$$

where $c_t = \log(C_t/\bar{C})$, r_t is the net return, based on the approximation $r_t \approx \log(R_t/\bar{R})$ and $\sigma = -\frac{U'(\bar{C})}{U''(\bar{C})\bar{C}}$ is the coefficient of intertemporal substitution (or risk aversion). (6) is the consumer's demand schedule giving current consumption demand as a function of the interest rate r_t and forecasts about the next period.

The log-linearization of the output process gives

$$y_t = \rho y_{t-1} + v_t, \tag{7}$$

where $y_t = \log(Y_t/\bar{Y})$. (Bars over the variables denote the non-stochastic steady state.) The rational expectations equilibrium (REE) of the linearized model is given by

$$r_t = -(1-\rho)\sigma^{-1} y_t$$

and for rational forecasts we have

$$E_t c_{t+1} = \rho y_t. \tag{8}$$

2.1 Learning Based on Euler Equations

To formulate learning in terms of the linearized Euler equation (6), which we will call EE approach subsequently, we suppose that agents are learning, using a PLM corresponding to the REE:

$$\hat{E}_t c_{t+1} = m_t + n_t y_t, \tag{9}$$

where (m_t, n_t) are obtained using a regression of c_s on y_{s-1} using data $s = 1, \ldots, t-1$. The data are then used to update parameter estimates to (m_{t+1}, n_{t+1}) and we proceed to period $t+1$.

Note that the rational forecast function (8) is a particular case of (9) and the basic question is whether $(m_t, n_t) \to (0, \rho)$ over time. This can easily be verified, for example using E-stability arguments.[6] Suppose we have (9) where the time subscripts are dropped from the parameters, i.e. $\hat{E}_t c_{t+1} = m + n y_t$. Temporary equilibrium, given forecasts $\hat{E}_t c_{t+1}$, in the linearized model is

$$r_t = -\sigma^{-1}(y_t - \hat{E}_t c_{t+1}) = -\sigma^{-1}[y_t(1-n) - m]$$

and the ALM is

$$T(m, n) = (0, \rho).$$

The E-stability differential equations are thus

$$\frac{d(m, n)}{d\tau} = (0, \rho) - (m, n),$$

which yields convergence of adaptive learning in this model.

Is this a plausible formulation? One of the necessary conditions for individual optimization is on the margin between today's consumption and tomorrow's consumption, and implementation of this FOC requires a forecast of that agent's own C_{t+1}. It might seem odd to have an agent forecasting his own behavior, but it is actually very natural. In the REE future consumption is related to the key exogenous state variable (e.g. income in the model of consumption). In a temporary equilibrium with learning agents are just trying to infer this relationship from past data and in forecasting they use the estimated relationship. The agent needs to plan what level of consumption he will choose in the following period and he also considers the perceived relation of consumption to the key exogenous variable. His best guess, given the AR(1) income process, is plausibly a linear function of current income. Thinking a single step ahead, in this way, appears to us to be one plausible and natural form of bounded rationality.

Note that, at first sight, this formulation of agent's behavior rule does not seem to require explicitly the intertemporal life-time budget constraint (4) or transversality condition. Yet it is not inconsistent with such a constraint as the agent can be thought to solve the intertemporal problem under subjective expectations. When the behavior rule of the agent is based on the Euler equation, only the one-step forward margin, the flow budget constraint and

[6] For the connection between least squares learning and E-stability see (Evans and Honkapohja 2001).

one-step forecasts are explicitly used.[7]

A boundedly rational agent making use only of the current Euler equation and an appropriate forecast function will converge to the household optimum under least squares learning. It can, moreover, be shown that, along the sequence of temporary equilibria during the convergent learning, *ex post* consistency in the accounting over the infinite horizon is fulfilled. To see this we note that, iterating the flow accounting identity, we have

$$C_t + \sum_{s=t+1}^{T} \mathcal{R}_{t+1,s} C_s = Y_t + \sum_{s=t+1}^{T} \mathcal{R}_{t+1,s} Y_s + \mathcal{R}_{t+1,T} W_{T+1}.$$

In the sequence of temporary equilibria $C_s = Y_s$ for all s, which implies that $\mathcal{R}_{t+1,T} W_{T+1} = 0$ and so the *ex post* transversality condition must hold. If learning is convergent, then intertemporal consistency is achieved. Once the EE learning has reached the REE, the agent has the correct forecast function (8) and his behavior based on the Euler equation generates the REE sequence (c_s^*, r_s^*) of consumptions and interest rates. This type of behavior by the agent is then consistent with full intertemporal optimization since if he is faced with the sequence of interest rates r_s^* he would choose the consumption sequence c_s^* which does satisfy the transversality condition.

In other economic models, learning based on Euler equations may fail to be stable. In cases of instability one could argue that if the economy diverges along an explosive path, the household would begin to think through the implications of its lifetime budget constraint and/or transversality condition and eventually alter its behavior. Of course, in the divergent case the log-linearization is also invalid since the economy will not stay near the steady state.

2.2 Learning With Perceptions Over an Infinite Horizon

A different form of learning behavior is suggested in a recent paper by (Preston 2002) in the context of a New Keynesian model of monetary policy. His interesting ideas can also be simply presented in the current context.

[7] Note also that, in many derivations of the REE, the intertemporal budget constraint is checked only at REE prices. Indeed, there could be problems with existence of solutions to household optimum at arbitrary prices sequences.

The starting point is to log-linearize the intertemporal budget constraint (4) at the non-stochastic steady state, which yields

$$\hat{E}_t \bar{C} c_t + \sum_{s=t+1}^{\infty} \bar{\mathcal{R}}_{t+1,s} \bar{C} \hat{E}_t c_s = \bar{Y} y_t + \sum_{s=t+1}^{\infty} \bar{\mathcal{R}}_{t+1,s} \bar{Y} \hat{E}_t y_s, \tag{10}$$

where in fact $\bar{\mathcal{R}}_{t+1,s} = (1/\bar{R})^{s-t} = \beta^{s-t}$ and $\bar{C} = \bar{Y}$ at the steady state. Next, we iterate the linearized Euler equation (6) backwards for $s \geq t+1$, giving

$$\hat{E}_t c_s = c_t + \sigma \sum_{j=t}^{s-1} \hat{E}_t r_j. \tag{11}$$

Substituting (11) into (10) leads to

$$c_t + \sum_{s=t+1}^{\infty} \beta^{s-t} [c_t + \sigma \sum_{j=t}^{s-1} \hat{E}_t r_j] = y_t + \sum_{s=t+1}^{\infty} \beta^{s-t} \hat{E}_t y_s.$$

Rearranging the summation and manipulation give a linearized consumption function in the form

$$c_t = \sum_{s=t}^{\infty} \beta^{s-t} [(1-\beta) \hat{E}_t y_s - \sigma \beta \hat{E}_t r_s]. \tag{12}$$

We will call this the infinite horizon (IH) approach to modeling adaptive learning by the agent.

There are several important comments about this formulation.

First, note that if (12) is the behavioral rule of the learning agent, then the agent must make forecasts about future income/output and rates of return into the infinite future. The agent is thus assumed to be extremely far-sighted even though he is boundedly rational.

Second, it can be asked whether the EE approach is consistent with (12). This is naturally the case, since the derivation of (12) relies in part on (6). Moreover, advancing (12) and multiplying by β one period gives

$$\beta c_{t+1} = \sum_{s=t+1}^{\infty} \beta^{s-t} [(1-\beta) \hat{E}_{t+1} y_s - \sigma \beta \hat{E}_{t+1} r_s],$$

to which one can apply the subjective expectations $\hat{E}_t(.)$. Once this has been done, it is seen that

$$c_t = (1-\beta) y_t - \sigma \beta r_t + \beta \hat{E}_t c_{t+1},$$

so that by using market clearing $c_t = y_t$ the Euler equation (6) also obtains.

This derivation presumes that the law of iterated expectations holds for the subjective expectations of the agent. For standard formulations of adaptive learning this is usually assumed. For example, suppose that agents do not know the relationship between y_t and r_t and assume that the return r_t is a linear function of the key state variable y_t, so that at time t they have the PLM

$$r_t = d_t + f_t y_t. \tag{13}$$

For simplicity, we assume that they know the true process of y_t, (7). The agents forecasts are assumed to behave as follows

$$\hat{E}_t \hat{E}_{t+1} r_s = \hat{E}_t(d_{t+1} + f_{t+1}\hat{E}_{t+1} y_s) = d_t + f_t \hat{E}_t y_s,$$

which says that in iterating expectations back to an earlier period the point estimates of the PLM parameters are shifted back to the earlier values.[8] This is the standard formulation in the adaptive learning literature, and can be viewed as an axiom of the approach.

Third, it is of interest to consider whether learning using the forecasts based on (13) converges. We again study this using E-stability, so that the PLM is $r_t = d + f y_t$. Then (12) can be written as

$$\begin{aligned} c_t &= \sum_{s=t}^{\infty} \beta^{s-t}[(1-\beta)\hat{E}_t y_s - \sigma\beta \hat{E}_t r_s] \\ &= \sum_{s=t}^{\infty} \beta^{s-t}\{[(1-\beta) - \sigma\beta f]\hat{E}_t y_s - \sigma\beta d\}. \end{aligned}$$

We have

$$\sum_{s=t}^{\infty} \beta^{s-t}\hat{E}_t y_s = \sum_{s=t}^{\infty} \beta^{s-t}\rho^{s-t} y_t = \frac{1}{1-\beta\rho} y_t$$

and we get

$$c_t = \frac{1-\beta-\sigma\beta f}{1-\beta\rho} y_t - \frac{\sigma\beta d}{1-\beta}.$$

[8] More generally, one could have the agents also learn the parameters of the process for y_t. Then they would also have a PLM of the form $y_t = a_t + b_t y_{t-1} + v_t$. In this case the iterated expectations would take the form $\hat{E}_t(a_{t+1} + b_{t+1}\hat{E}_{t+1} y_{s-1}) = a_t + b_t \hat{E}_t y_{s-1}$.

The temporary equilibrium value of the rate of return is determined from the Euler equation (6), so that

$$r_t = -\sigma^{-1}(y_t - \hat{E}_t c_{t+1}) = -\sigma^{-1}(y_t - \frac{1-\beta-\sigma\beta f}{1-\beta\rho}\rho y_t + \frac{\sigma\beta d}{1-\beta}).$$

The T-mapping is thus

$$\begin{aligned} d &\rightarrow -\frac{\beta d}{1-\beta} \\ f &\rightarrow -\sigma^{-1}(1-\frac{1-\beta-\sigma\beta f}{1-\beta\rho}\rho). \end{aligned}$$

The differential equation defining E-stability consists of two independent linear odes with negative coefficients on the variables d and f, respectively and so we have E-stability.

2.3 Further Discussion

Comparing the two approaches to agent's behavior under learning we see that the EE approach has the agents making forecasts only one period ahead. It is thus assumed that the agent is relatively short sighted. In contrast, in the IH approach the agent has to make forecasts over the entire infinite future. Thus the agent is very far-sighted. These two approaches represent different ways of modeling agent's behavior under adaptive, boundedly rational learning.

It should be noted that, quite naturally, the agent forecasts different quantities in the EE and IH approaches. Thus the natural PLM have different parameters and the respective mappings from the PLM to the ALM are also different.

We have seen that the two approaches are not inconsistent in the sense that it is possible to derive the EE formulation from the IH approach under certain plausible conditions. We have convergence of learning for both approaches in this model. It is not obvious which approach is preferable. In terms of the degree of farsightedness the two approaches represent extreme cases. In the EE approach the boundedly rational agents look ahead only for one period while in the IH approach they look ahead into the infinite future.

In judging the approaches one must also take note of the empirical observation that in reality public and private forecasting institutions have only a limited time horizon, often at most two years, for detailed business cycle forecasting. Very long term projections are also made by forecasting institutions but these projections are very broad as they usually show only long term trends of relatively few variables. Perhaps the “right” approach is in between these two extremes.

3 Learning and Monetary Policy

(Preston 2002) has recently analyzed a model of monetary policy with the IH approach for the micro-foundations of agents' behavior under learning. He shows that if the central bank uses various types of Taylor type rules, then the learning dynamics are E-stable if and only if the Taylor principle is satisfied. These results are exactly the same as found in (Bullard and Mitra 2002). However, (Preston 2002) argues (especially in the earlier version) that this result is not necessarily to be expected since the analysis in (Bullard and Mitra 2002) relies “in part on the assumption of rational expectations” (p.5 of the February 2002 version) whereas his analysis does not do so.[9] We now demonstrate that the analysis of (Preston 2002) is consistent with that of (Bullard and Mitra 2002) and (Evans and Honkapohja 2002a). As a consequence, the analysis in the latter papers is not inconsistent and they are a valid way of studying stability under learning.

3.1 Framework

We start with the framework presented in Section 2 of Preston which essentially uses a dynamic stochastic equilibrium model of (Woodford 1999) and (Woodford 1996). (Preston 2002) derives an optimal consumption rule for a representative household and an optimal pricing rule for a representative firm.[10] These two equations are, respectively,

$$C_t^i = \hat{E}_t^i \left\{ \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta)Y_T - \beta\sigma(i_T - \pi_{T+1}) + \beta(g_T - g_{T+1})] \right\}, \quad (14)$$

$$p_t^i = \hat{E}_t^i \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} \left[\frac{(1-\alpha\beta)(\omega+\sigma^{-1})}{(1+\omega\theta)} x_T + \alpha\beta\pi_{T+1} \right] \right\}, \quad (15)$$

[9] The same claim applies to the (Evans and Honkapohja 2002a) study of optimal discretionary policy.

[10] We refer the reader to (Preston 2002) for the details of these derivations. Our pricing equation (15) corrects a typographical error.

where g_t is an appropriate taste shock. (In some variations of the model g_t represents a government spending shock).

Under our representative agent assumption agents have identical expectations and thus consumption and price setting (for firms able to set prices) is the same across agents, i.e. for all relevant variables z we have $\hat{E}_t^i z = \hat{E}_t z$ and thus $C_t^i = \int_j C_t^j dj \equiv C_t$ and $p_t^i = \int_j p_t^j dj \equiv p_t$. Given expectations, the temporary equilibrium values of output Y_t and the inflation rate π_t are determined by the market clearing condition $Y_t = C_t$ and by the relationship between the aggregate price level and prices currently being set, given by $\pi_t = \alpha^{-1}(1-\alpha)p_t$. The equation for Y_t is often reexpressed in terms of the output gap $x_t = Y_t - Y_t^n$, where Y_t^n is the natural rate of output.

Integrating (14)-(15) over i and using these relationships gives

$$x_t = \hat{E}_t \left\{ \sum_{T=t}^{\infty} \beta^{T-t} [(1-\beta)x_{T+1} - \sigma(i_T - \pi_{T+1}) + r_T^n] \right\} \tag{16}$$

where $r_T^n = g_T - g_{T+1} + Y_{t+1}^n - Y_t^n$, and

$$\pi_t = \hat{E}_t \left\{ \sum_{T=t}^{\infty} (\alpha\beta)^{T-t} [\kappa x_T + (1-\alpha)\beta\pi_{T+1})] \right\} \tag{17}$$

where

$$\kappa = (1-\alpha)\alpha^{-1}(1-\alpha\beta)(\omega+\sigma^{-1})(1+\omega\theta)^{-1}.$$

(Preston 2002) then conducts the analysis using equations (16) and (17) as the behavioral rule for households and firms.

The analysis in (Bullard and Mitra 2002) and (Evans and Honkapohja 2002a), on the other hand, is based on the EE approach and thus starts from the following two equations

$$x_t = \hat{E}_t x_{t+1} - \sigma \left(i_t - \hat{E}_t \pi_{t+1} \right) + r_t^n, \tag{18}$$

$$\pi_t = \kappa x_t + \beta \hat{E}_t \pi_{t+1}. \tag{19}$$

In the earlier version (Preston 2002) claims that equations (16) and (17) reduce to (18) and (19), respectively, only under the assumption of RE with the subjective expectations operator, $\hat{E}_t$, being replaced by the actual objective expectations under RE, that is, E_t in (18) and (19).[11] Consequently, it is

[11] Section 2.3 of the August 2002 version of (Preston 2002) no longer claims this explicitly and points out that, under certain assumptions, one could instead use the Euler equation as a starting point.

suggested that it is inconsistent to start from (18) and (19) to analyze learning since it is claimed to be only valid under RE. According to the earlier version of (Preston 2002), the key assumption which makes it impossible to reduce (16) and (17) to (18) and (19) are that agents are agnostic about the beliefs of others so that the law of iterated expectations fails to hold for the average expectations operator $\hat{E}_t$ *even* if it holds for the individual expectations operator $\hat{E}_t^i$. However, it should be noted that agents are assumed to have homogenous expectations in the framework of (Preston 2002), so that $\hat{E}_t^i = \hat{E}_t^j$ for all i and j and thus, *for analytical purposes, the law of iterated expectations in fact holds for the average expectations.* Naturally, the agents cannot in general make this inference about average expectations, but given the PLMs below the agents do not need this inference!

We now show how to derive (18) and (19) from (14) and (15). This implies that (18) and (19) are a valid framework for studying learning.

3.2 Derivation of Aggregate Euler Equations

The key assumption that will allow us to derive (18) and (19) from (14) and (15) is that the subjective expectations of individual agents obey the law of iterated expectations, i.e. for any variable z

$$\hat{E}_t^i \hat{E}_{t+s}^i z = \hat{E}_t^i z \text{ for } s = 0, 1, 2, \ldots.$$

As indicated above, this is a standard assumption for agents making forecasts from linear laws of motion estimated by Least Squares.

For example, in (Bullard and Mitra 2002), agent i has a perceived law of motion (PLM) of the form[12]

$$\begin{aligned} x_t &= a_{x,t}^i + b_{x,t}^i r_t^n + \epsilon_{xt}, \\ \pi_t &= a_{\pi,t}^i + b_{\pi,t}^i r_t^n + \epsilon_{\pi t} \end{aligned}$$

which can be used to form future forecasts for any $T > t$,

$$\hat{E}_t^i x_T = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i r_T^n, \tag{20}$$

$$\hat{E}_t^i \pi_T = a_{\pi,t}^i + b_{\pi,t}^i \hat{E}_t^i r_T^n. \tag{21}$$

[12] (Evans and Honkapohja 2002a) allow for an exogenous random shock to the inflation equation (19) and consequently they examine a PLM that depends on this shock. Our central points do not depend on the specific PLM, and hold also if the PLM includes lagged endogenous variables, as in (Evans and Honkapohja 2002b).

Note that if each agent i has identical parameter estimates (and knows the persistence parameter ρ in the process of r_t^n, a simplifying assumption without any loss of generality), then the forecasts of each agent are the same, that is, $\hat{E}_t^i = \hat{E}_t^j$ for all i and j. This, of course, implies that $\hat{E}_t^i = \hat{E}_t$ for all i in the analysis. We re-emphasize that there is no need for any single agent to make this inference when forming the forecasts needed in his decision making. In other words, every agent i forms his own forecast independently of the other agents in the economy and uses this forecast in his optimal consumption or pricing rule. It follows that the optimal consumption and pricing rules of each agent given by (14) and (15) are the same, that is, $C_t^i = C_t$ and $p_t^i = p_t$ for all i. (In principle the rules given by (14) and (15) could vary across households/firms if the future forecasts are different across them but homogenous forecasts force them to be the same.)

As discussed before, (20) implies for $j \geq 1$ that

$$\hat{E}_{t+j}^i x_T = a_{x,t+j}^i + b_{x,t+j}^i \hat{E}_{t+j}^i r_T^n$$

and when we take expectations of the above expression at time t we obtain

$$\hat{E}_t^i \hat{E}_{t+j}^i x_T = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i \hat{E}_{t+j}^i r_T^n = a_{x,t}^i + b_{x,t}^i \hat{E}_t^i r_T^n = \hat{E}_t^i x_T \qquad (22)$$

In other words, it is assumed that the law of iterated expectations holds at the individual level.[13] With assumption (22) and identical expectations across agents, one can show that, for analytical purposes, it is possible to obtain (18) from equation (14). Although there are several ways to obtain the desired results, we give a derivation that focuses on the individual Euler equation. This will reinforce points made earlier in these notes and emphasize the details of individual decision making.

We begin by taking quasi-differences of (14). Advancing (14) by one time unit, taking expectations $\hat{E}_t^i$ of both sides, and using the law of iterated expectations, we obtain

$$C_t^i - \beta \hat{E}_t^i C_{t+1}^i = \hat{E}_t^i [(1-\beta) Y_t - \beta\sigma(i_t - \pi_{t+1}) + \beta(g_t - g_{t+1})], \text{ or}$$

$$C_t^i = \beta \hat{E}_t^i C_{t+1}^i + (1-\beta)(x_t + Y_t^n) - \beta\sigma(i_t - \hat{E}_t^i \pi_{t+1}) + \beta(g_t - g_{t+1}), \quad (23)$$

where for simplicity we assume that g_t, g_{t+1}, Y_t^n and Y_{t+1}^n are known at t.

[13] We have kept on purpose the superscript i for individuals, though the analysis assumes identical expectations.

To implement (23) each agent must forecast their consumption next period. Market clearing and the representative agent assumption imply that $C_t^i = Y_t$ for all i, t, i.e. consumption of each agent is in fact equal to mean/aggregate output in each period. We assume that each agent observes this equality from historical data, and thus forecasts its consumption next period by its forecast of aggregate output.[14] Using also $Y_t = x_t + Y_t^n$, for all t, we obtain

$$\hat{E}_t^i C_{t+1}^i = \hat{E}_t^i x_{t+1} + Y_{t+1}^n.$$

Here we are following the literature in assuming that Y_{t+1}^n is observable at t, in which case it is natural to assume that $\hat{E}_t^i C_{t+1}^i$ would incorporate this information and use least squares to forecast the unknown component x_{t+1}.[15] Hence

$$C_t^i = \beta \hat{E}_t^i x_{t+1} + (1-\beta)x_t + Y_t^n - \beta\sigma(i_t - \hat{E}_t^i \pi_{t+1}) + \beta r_t^n, \qquad (24)$$

where $r_t^n = g_t - g_{t+1} + Y_{t+1}^n - Y_t^n$.

Equation (24) is our behavioral equation giving consumption demand as a function of interest rates, current income and one-step ahead forecasts of income and inflation. As discussed earlier, although (24) does not explicitly impose the lifetime budget constraint, it is a consistent and plausible way of implementing bounded rationality, which in stable systems will indeed lead to satisfaction of the intertemporal budget constraint. Finally, from market-clearing $C_t^i = Y_t = x_t + Y_t^n$ and using $\hat{E}_t^i x_{t+1} = \hat{E}_t x_{t+1}$ and $\hat{E}_t^i \pi_{t+1} = \hat{E}_t \pi_{t+1}$ we arrive at the aggregate Euler equation (18).

The derivation of (19) from (15) is analogous. Taking quasi-differences of (15) and using the law of iterated expectations at the individual level leads to the individual agent Euler equation

$$p_t^i = \alpha\beta \hat{E}_t^i p_{t+1}^i + (1-\alpha\beta)(\omega + \sigma^{-1})(1+\omega\theta)^{-1} x_t + \alpha\beta \hat{E}_t^i \pi_{t+1}.$$

Note that in this Euler equation agent i's expectations of future values of x_T and π_{T+1} are appropriated condensed into $\hat{E}_t^i p_{t+1}^i$, the price the firm expects

[14] Note that we do not need to make any *a priori* assumption that agents know that all agents are identical, and we do not need to assume that agents make deductions based upon this.

[15] However, nothing hinges on this point. In more general representative agent set-ups, each agent would forecast its consumption at $t+1$ by a least squares regression on all relevant information variables.

to set next period if it is again a price setter. Finally, we make use of

$$p_t^i = p_t \text{ and } \pi_t = \alpha^{-1}(1-\alpha)p_t \text{ all } t,$$

which implies that[16]

$$\hat{E}_t^i p_{t+1}^i = \alpha(1-\alpha)^{-1}\hat{E}_t^i \pi_{t+1}.$$

It follows that

$$p_t^i = \alpha\beta(1-\alpha)^{-1}\hat{E}_t^i \pi_{t+1} + (1-\alpha\beta)(\omega+\sigma^{-1})(1+\omega\theta)^{-1}x_t. \qquad (25)$$

Equation (25) is our behavioral equation giving individual price setting as a function of the current output gap and the one-step ahead forecasts of inflation. Integrating over households and using $\pi_t = \alpha^{-1}(1-\alpha)p_t$ we arrive at the aggregate Euler equation (19).

Recently, (Honkapohja and Mitra 2002) have considered cases in which the central bank uses its own forecasts of inflation and output (rather than private sector forecasts) in its interest rate rule. This poses no additional complication for the above derivation of the system (18) and (19) from (14) and (15), given the assumption (which we have maintained throughout) that the consumption schedule is conditioned on current interest rates, so that x_t, π_t and i_t are simultaneously determined in the usual way by market clearing.

3.3 Some Final Remarks

The EE and IH approaches to modeling agent's behavior rule are not identical and lead to different detailed learning dynamics. Thus there is in general no guarantee that the convergence conditions for the two dynamics are identical, though this happens to be the case in the permanent income model of Section 2 and is apparently also the outcome for some interest rate rules in the model of monetary policy, see (Preston 2002). It is an open question whether there exist models in which the convergence conditions differ for the EE and IH approaches or whether a general equivalence of the convergence conditions can be established.

[16] Because there is an exact linear relation between these variables, if agents form expectations using least squares learning, the expectations $\hat{E}_t^i p_{t+1}^i$ and $\hat{E}_t^i \pi_{t+1}$ will exactly satisfy the stated relationship provided the explanatory variables and sample period are the same for both variables, as we of course assume.

References

BULLARD, J., AND K. MITRA (2002): "Learning About Monetary Policy Rules," *Journal of Monetary Economics*, 49, 1105–1129.

EVANS, G. W., AND S. HONKAPOHJA (2001): *Learning and Expectations in Macroeconomics*. Princeton University Press, Princeton, New Jersey.

——— (2002a): "Expectations and the Stability Problem for Optimal Monetary Policies," *Review of Economic Studies, forthcoming.*

——— (2002b): "Monetary Policy, Expectations and Commitment," mimeo, www.valt.helsinki.fi/RAKA/seppo.hmtl and http://darkwing.uoregon.edu/ gevans/.

HONKAPOHJA, S., AND K. MITRA (2002): "Performance of Monetary Policy with Internal Central Bank Forecasting," manuscript, www.valt.helsinki.fi/RAKA/seppo.hmtl.

MARCET, A., AND T. J. SARGENT (1989): "Convergence of Least-Squares Learning Mechanisms in Self-Referential Linear Stochastic Models," *Journal of Economic Theory*, 48, 337–368.

PRESTON, B. (2002): "Learning about Monetary Policy Rules when Long-Horizon Forecasts Matter," mimeo.

SARGENT, T. J. (1993): *Bounded Rationality in Macroeconomics*. Oxford University Press, Oxford.

WOODFORD, M. (1996): "Control of Public Debt: A Requirement for Price Stability?," Working paper, NBER No. 5684.

——— (1999): "Price Level Determination Under Interest-Rate Rules," mimeo, Princeton University.