# Tensor products and restrictions in type $A$

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**Abstract.** The goal of this article is to give an exposition of some recent results on tensor products and restrictions for rational representations of the general linear group in positive characteristic. The exposition is based on our papers [11, 12, 13]. We also outline the relations with the LLT algorithm and the ideal structure of the group algebra of the finitary symmetric group.

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## Introduction

We begin with some motivating discussion about *translation functors*, following Jantzen [39]. Let $G$ be a reductive algebraic group over an algebraically closed field $\mathbb{F}$ of characteristic $p > 0$, and let $\mathcal{C}$ denote the category of all rational $\mathbb{F}G$-modules. Writing $X$ (resp. $X^+$) for the set of integral weights (resp. dominant integral weights) corresponding to the root system of $G$, we have for each $\lambda \in X^+$ the modules $L(\lambda), \Delta(\lambda), \nabla(\lambda)$ and $T(\lambda)$ which are the irreducible, standard (Weyl), costandard (induced) and indecomposable tilting modules of highest weight $\lambda$ respectively. We would of course like to describe the inverse decomposition numbers $[L(\lambda) : \Delta(\mu)]$, allowing us to compute the formal characters of irreducibles as linear combinations of Weyl characters, and also the $\nabla$-filtration multiplicities $[T(\lambda) : \nabla(\mu)]_\nabla$. Jantzen's translation functors have played a key role in attacking (and in the analogous quantum problems, solving) these questions.

We recall briefly the definition of the translation functor $T_\lambda^\mu$, for $\lambda, \mu \in X$ lying in the closure of the same alcove. Let $\mathcal{C}(\mu)$ denote the *linkage class* corresponding to $\mu$, that is, the full subcategory of $\mathcal{C}$ consisting of all modules with composition factors of the form $\{L(w \cdot \mu) \mid w \in W_p\}$ where $W_p$ denotes the affine Weyl group, acting on $X$ by the usual dot action. There is an exact projection functor

$$pr_\mu : \mathcal{C} \to \mathcal{C}(\mu)$$

given on objects by taking the largest submodule belonging to $\mathcal{C}(\mu)$. Then, the translation functor

$$T_\lambda^\mu : \mathcal{C}(\lambda) \to \mathcal{C}(\mu)$$

is the functor $pr_\mu \circ (? \otimes \nabla(\nu))$ where $\nu$ is the unique dominant weight conjugate under the Weyl group to $(\mu - \lambda)$. When $\lambda$ and $\mu$ belong to the interior of the same alcove (or more generally, the same facet) the functor $T_\lambda^\mu$ is an equivalence of categories.

The key situation to consider is when $\lambda \in X^+$ lies in some alcove $A$ and $\mu$ lies on the interior of a wall in the upper closure of $A$. Let $\lambda s$ denote the reflection of $\lambda$ in the wall containing $\mu$ ($s$ can viewed as a simple reflection in $W_p$). Then, $T_\lambda^\mu L(\lambda) \cong L(\mu)$ but the module $T_\mu^\lambda L(\mu)$ is far more complicated. It is known that $T_\mu^\lambda L(\mu)$ has simple head and socle isomorphic to $L(\lambda)$, and also that $T_\mu^\lambda L(\mu)$ contains $L(\lambda s)$ as a composition factor with multiplicity 1. In particular, this implies that $T_\mu^\lambda L(\mu)$ has Loewy length at least 3. Moreover, the Lusztig conjecture is equivalent to the statement that $T_\mu^\lambda L(\mu)$ has length exactly 3 for $\lambda$ within a certain region (this equivalence was proved by Andersen [2, 2.16]). So understanding the structure of $T_\mu^\lambda L(\mu)$ even for such special configurations of $\lambda$ and $\mu$ is a fundamental problem.

**Question.** What does $T_\mu^\lambda L(\mu)$ look like in general, for $\mu, \lambda \in X^+$ lying in the closure of the same alcove but neither lying in the closure of the facet containing the other? For example, when is it non-zero? When is it irreducible or indecomposable? Can one give a lower bound on its Loewy length?

In this article, we will descibe some recent results [12] which answer these questions in special cases for $G = GL_n(\mathbb{F})$. For instance, we will see *in our special cases* that the Loewy length of $T_\mu^\lambda L(\mu)$ has a natural lower bound equal to

$$2 \dim \operatorname{End}_G(T_\mu^\lambda L(\mu)) - 1.$$

This lower bound can take any odd value, for suitable choices of $\lambda, \mu$ and sufficiently large $n$. The length 3 case mentioned above is then a special case of our results.

Actually, we will not work with the functor $T_\mu^\lambda$ in type $A$, but instead introduce functors $F_\alpha$ and $E_\alpha$ for $\alpha \in \mathbb{Z}/p\mathbb{Z}$ which roughly speaking are given by tensoring with the natural $GL_n(\mathbb{F})$-module $V$ or its dual $V^*$, then projecting onto certain linkage classes determined by $\alpha$. Our results are in keeping with the philosophy behind the algorithm of Lascoux, Leclerc and Thibon [4, 52, 53, 66]: in type $A$, one should always expect to be able to generalize results involving the affine Weyl group $W_p$, with associated alcove combinatorics, to results involving the Fock space of the affine Kac-Moody algebra $\hat{\mathfrak{gl}}_p$, with associated Young diagram combinatorics. Just as at the level of representation theory, the 'wall crossing functor' $\Theta_s = T_\mu^\lambda \circ T_\lambda^\mu$ plays the role of the simple reflection $s \in W_p$, our functors $E_\alpha$ and $F_\alpha$ play the role (in a way we make precise later) of the simple root generators $e_\alpha, f_\alpha \in \hat{\mathfrak{gl}}_p$.

The remainder of the article is organized as follows. We begin in section 1 with some quite general results from [13] about the structure of tensor products of the form $M \otimes \nabla(\nu)$ in characteristic $p$, remembering that for a $G$-module $M$, $T_\lambda^\mu M$ is by definition a certain linkage class of $M \otimes \nabla(\nu)$ for suitable $\nu$. Then for the remainder of the paper, we specialize to $G = GL(n)$, when there are very close connections between the tensor product $M \otimes \nabla(\nu)$ for special $\nu$ and the restriction of $M$ to the subgroup $GL(n-1)$. In section 2, we state in detail the definitions of the functors $E_\alpha$ and $F_\alpha$ and our main results from [12]. The proof of these depended on first reformulating the results in terms of the branching problems from $GL(n)$ to $GL(n-1)$ studied in our earlier work [44, 45, 46, 48, 49, 8, 9, 15]. In section 3, we explain the connection between the functors $F_\alpha$ and the LLT algorithm [53, 52], basing our account on the recent results of Varagnolo and Vasserot [66]. In section 4 we discuss some of the other connections between tensor products and restrictions in $GL(n)$ obtained in [11], in particular, the relationship with tilting modules and the work of Mathieu and Papadopoulou [55]. As applications of these techniques, section 5 contains the corollaries of our main results for the symmetric group, while the relations between tensor ideals [3, 29, 30] and ideals of group algebras of the finitary symmetric group [6] are outlined in section 6.

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## Notation

*General conventions:* If $G$ and $H$ are two groups, $L$ is an $\mathbb{F}G$-module and $M$ is an $\mathbb{F}H$-module we write $L \boxtimes M$ for the *outer* tensor product of $L$ and $M$ (which is a module over $G \times H$). If $N$ is another $\mathbb{F}G$-module we write $L \otimes N$ for the *inner* tensor product of $L$ and $N$ (which is a $G$-module). If $L$ is irreducible, $I$ is indecomposable, and $M$ is an arbitrary $\mathbb{F}G$-module, then $[M : L]$ stands for the multiplicity of $L$ as a composition factor of $M$, and $(M : I)$ stands for the multiplicity of $I$ as an indecomposable summand of $M$. If $G$ is an algebraic group, a $G$-module will always mean a rational $\mathbb{F}G$-module, unless otherwise stated.

*Notation in arbitrary type:* If $G$ is an arbitrary reductive algebraic group over $\mathbb{F}$, we will follow Jantzen [39] for notation. In particular, $R$ denotes the root system of $G$ with respect to a fixed maximal torus $T$, $R^+ \subset R$ denotes the set of positive roots determined by a choice of Borel subgroup $B^+$ containing $T$, and $\{\alpha_1, \dots, \alpha_\ell\} \subset R^+$ is the corresponding base for $R$. We write $X(T)$ for the character group $\mathrm{Hom}(T, \mathbb{F}^\times)$, $Y(T)$ for the cocharacter group $\mathrm{Hom}(\mathbb{F}^\times, T)$ and let $\langle \cdot, \cdot \rangle$ be the natural pairing $X(T) \times Y(T) \to \mathbb{Z}$. For $\alpha \in R$, $\alpha^\vee$ denotes the corresponding coroot in $Y(T)$, and $X^+(T)$ denotes the set $\{\lambda \in X(T) \mid \langle \lambda, \alpha_i^\vee \rangle \geq 0, i = 1, \dots, \ell\}$ of dominant weights. Given a weight $\nu \in X(T)$ and a $T$-module $M$, $M_\nu$ will denote the $\nu$-weight space of $M$, and the formal character of $M$ is $\mathrm{ch}\, M$.

*Notation in type A:* In addition, if $G = GL(n) = GL_n(\mathbb{F})$, we will make the following choices. We always take $T$ to be all diagonal matrices in $GL(n)$ and $B^+$ to be all upper triangular matrices. We identify the weight lattice $X(T)$ with the set $X(n)$ of all $n$-tuples $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ of integers, $\lambda$ corresponding to the character $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_1^{\lambda_1} \dots t_n^{\lambda_n}$, and $X^+(T)$ with the set $X^+(n) = \{\lambda \in X(n) \mid \lambda_1 \geq \cdots \geq \lambda_n\}$. We also write $\epsilon_i$ for the weight $(0, \dots, 0, 1, 0, \dots, 0)$ with 1 in the $i$th position. The natural $n$-dimensional $GL(n)$-module with highest weight $\epsilon_1$ will be denoted by $V$. Its dual $V^*$ has highest weight $-\epsilon_n$. Whenever we need to clarify the group $GL(n)$ that we are referring to, we will add a subscript $n$ to our notation, giving us $GL(n)$-modules $V_n, L_n(\lambda)$, $\Delta_n(\lambda)$, $\nabla_n(\lambda)$ and $T_n(\lambda)$.

# 1. General results

Let $G$ denote an arbitrary reductive algebraic group over $\mathbb{F}$. Throughout the article, we consider two types of problems. Firstly, we are interested in tensor products of irreducible (standard, costandard, tilting) $G$-modules. Secondly, we study the restrictions of irreducible (standard, costandard, tilting) modules from $G$ to its Levi subgroups. We also want to reveal various connections between the two types of problems.

Some of the problems have 'characteristic-free' answers but we try to consider those which do depend on the characteristic. For example, assume for a moment that the ground field $\mathbb{F}$ has characteristic 0. Let $c^\lambda_{\mu\nu}$ be the multiplicity of $L(\lambda)$ in $L(\mu) \otimes L(\nu)$. If $G = GL(n)$, the constant $c^\lambda_{\mu\nu}$ is a Littlewood-Richardson coefficient. Then, in *any* characteristic, the tensor product $\nabla(\mu) \otimes \nabla(\nu)$ has a $\nabla$-filtration, with $\nabla(\lambda)$ appearing $c^\lambda_{\mu\nu}$ times. Indeed, the fact that a $\nabla$-filtration exists follows from the fundamental Donkin-Mathieu theorem on good filtrations (see [65, 19, 54]), and the multiplicities do not depend on the characteristic as the formal characters of costandard modules do not (they are given by Weyl's character formula).

Moreover, by a theorem of Cline, Parshall, Scott and van der Kallen [18], [39, 4.13], $\mathrm{Ext}^1_G(\Delta(\lambda), \nabla(\gamma)) = 0$ for any $\lambda$ and $\gamma$. Also, it is well known (see for example [39, 4.13]) that $\dim \mathrm{Hom}_G(\Delta(\lambda), \nabla(\gamma)) = \delta_{\lambda\gamma}$ where $\delta_{\lambda\gamma}$ is the Kronecker delta. It follows that

$$\dim \mathrm{Hom}_G(\Delta(\lambda), \nabla(\mu) \otimes \nabla(\nu)) = c^\lambda_{\mu\nu}. \tag{1}$$

We say a vector $v \in M$ is *primitive* if it is invariant with respect to the unipotent radical $U^+$ of $B^+$. By the universal property of standard modules [39, 2.13 b)], we see from (1) that the space of the primitive vectors of weight $\lambda$ in $\nabla(\mu) \otimes \nabla(\nu)$ has dimension $c^\lambda_{\mu\nu}$, just like in characteristic 0.

However, we do get a 'modular problem' if we want to understand which of these primitive vectors generate simple modules. Equivalently, we are interested in the *socle* of $\nabla(\mu)\otimes\nabla(\nu)$ or in the dimension of the spaces

$$\mathrm{Hom}_G(L(\lambda),\nabla(\mu)\otimes\nabla(\nu)) \tag{2}$$

for all triples $(\lambda,\mu,\nu)$. These dimensions can be thought of as *modular Littlewood-Richardson coefficients* (in type $A$). By dualizing, the space (2) is isomorphic to $\mathrm{Hom}_G(\Delta(-w_0\mu),L(-w_0\lambda)\otimes\nabla(\nu))$, where $w_0$ is the longest element of the Weyl group $W=N_G(T)/T$. So, as $\lambda,\mu,\nu$ are arbitrary, our problem is equivalent to describing the spaces

$$\mathrm{Hom}_G(\Delta(\lambda),L(\mu)\otimes\nabla(\nu)), \tag{3}$$

for arbitrary triples $(\lambda,\mu,\nu)$. Equivalently, we want to understand the primitive vectors in tensor products of the form $L(\mu)\otimes\nabla(\nu)$.

Our first result on the space (3) generalizes a well known fact in characteristic 0. Let $\mathrm{Dist}(G)$ be the algebra of distributions of $G$ as in [39, I.7], which is generated by $\mathrm{Dist}(T)$ and the 'divided power' root generators $X_\alpha^{(n)},Y_\alpha^{(n)}$ for $\alpha\in R^+, n\geq 1$. Write $X_i^{(n)}=X_{\alpha_i}^{(n)}, Y_i=Y_{\alpha_i}^{(n)}$ for $i=1,\ldots,\ell$. If $G$ is semisimple and simply connected, $\mathrm{Dist}(G)$ coincides with the hyperalgebra of $G$ arising from the Chevalley construction. We note that any $G$-module is a $\mathrm{Dist}(G)$-module in a natural way; see [39, I.7.11, II.1.20]. For any $G$-module $M$, a dominant weight $\nu\in X(T)^+$ and any weight $\gamma\in X(T)$ we define

$$M^\nu := \{v\in M \mid X_i^{(b_i)}v=0 \text{ for all } b_i>\langle\nu,\alpha_i^\vee\rangle \text{ and } i=1,2,\ldots,\ell\}$$

and let $M^\nu_\gamma := M^\nu\cap M_\gamma$ denote its $\gamma$-weight space.

**Theorem 1.1.** *Let $\lambda,\nu\in X^+(T)$, and $M$ be any (rational) $G$-module. Then*

$$\dim\mathrm{Hom}_G(\Delta(\lambda),M\otimes\nabla(\nu))=\dim M^\nu_{\lambda-\nu}.$$

To explain our interest in the theorem, suppose that $M=L(\mu)$ is an irreducible module for some fixed $\mu\in X^+(T)$. Then, for $\nu$ large relative to $\mu$, we have $M^\nu_{\lambda-\nu}=M_{\lambda-\nu}$, and by the theorem, $\dim L(\mu)_{\lambda-\nu}=\dim\mathrm{Hom}_G(\Delta(\lambda),L(\mu)\otimes\nabla(\nu))$. So to compute the formal character of $L(\mu)$ it suffices to describe the Hom-space in 1.1 for $M=L(\nu)$ and $\lambda,\nu$ large. In view of the universality of standard modules, this is equivalent to describing the primitive vectors of weight $\lambda$ in $L(\mu)\otimes\nabla(\nu)$.

A complete proof of 1.1 can be found in [13, Theorem A]. The main idea is to use the well-known presentation of standard modules by generators and relations: for $\mu\in X^+(T)$, we have that

$$\Delta(\mu)\cong U/I(\mu),$$

where $I(\mu)$ is the left ideal of $\mathrm{Dist}(G)$ generated by $\{X_\alpha^{(b_\alpha)}\mid\alpha\in R^+,b_\alpha\geq 1\}\cup\{H-\mu(H)\mid H\in\mathrm{Dist}(T)\}\cup\{Y_i^{(b_i)}\mid 1\leq i\leq\ell, b_i>\langle\mu,\alpha_i^\vee\rangle\}$.

Now suppose that $G=GL(n)$ and embed $GL(n-1)$ into the top left hand corner of $GL(n)$. Observe that if we take $\mu=-\ell\epsilon_n$ for $\ell\geq 0$, then the space $M^\mu_{\lambda-\mu}$ appearing in 1.1 is precisely the space of vectors in $M_{\lambda-\mu}$ which are primitive with respect to the subgroup $GL(n-1)$, and satisfying in addition $X_{n-1}^{(b)}v=0$ for any

$b > \ell$. Recalling that $\nabla_n(-\ell\epsilon_n)$ is precisely the $\ell$th symmetric power $S^\ell(V_n^*)$, one obtains a connection between $GL(n)$-primitive vectors in the tensor product $M \otimes S^\ell(V_n^*)$ and $GL(n-1)$-primitive vectors in the restriction $M\downarrow_{GL(n-1)}$. We obtained the following extension in one important special case [13, Theorem C]:

**Theorem 1.2.** *Fix $\lambda, \mu \in X^+(n)$ with $\lambda_n = \mu_n$ and set $\ell = \sum_{i=1}^n (\lambda_i - \mu_i)$. Then, for any submodule $M$ of $\nabla_n(\lambda)$,*

$$\mathrm{Hom}_{GL(n)}(L_n(\mu), M \otimes S^\ell(V_n^*)) \cong \mathrm{Hom}_{GL(n-1)}(L_{n-1}(\bar{\mu}), M\downarrow_{GL(n-1)})$$

*where $\bar{\mu} = (\mu_1, \dots, \mu_{n-1})$ denotes the restriction of $\mu$ to $T \cap GL(n-1)$.*

We believe it is an important problem to find the socle of $L_n(\lambda)\downarrow_{GL(n-1)}$ for any $\lambda \in X^+(n)$. This socle is described by the right hand side of the equation in 1.2, taking $M = L_n(\lambda)$. We refer to this problem as the *modular branching problem for the general linear group.* A complete answer only exists in the special case that $L_n(\lambda)\downarrow_{GL(n-1)}$ is completely reducible; see [15].

If we take $\ell = 1$ in 1.2, we see that the problems of computing the socle of $L_n(\lambda) \otimes V_n^*$ and the socle of part of the restriction $L_n(\lambda)\downarrow_{GL(n-1)}$, known as the *first level*, are equivalent. The first level of such restrictions has been studied extensively in our earlier work, especially [46, 8]. This connection was exploited in [12]; we will discuss the results in detail in the next section.

Finally, we describe one other general result from [13] about the structure of tensor products of the form $M \otimes \nabla(\nu)$, again valid in arbitrary type. A dominant weight $\lambda$ is called *$p^r$-restricted* if $\langle \lambda, \alpha_i^\vee \rangle < p^r$ for all $i = 1, 2, \dots, \ell$. A semisimple module is called $p^r$-restricted if all of its composition factors have $p^r$-restricted highest weights. Going back to the space (2), we would like to understand when the socle of $\nabla(\mu) \otimes \nabla(\nu)$ is $p$-restricted. A necessary condition is that $\mu$ and $\nu$ are both $p$-restricted. Even though the converse is not quite true (see [13, Remark 2.10] for a counterexample), we have the following result from [13, Theorem B]:

**Theorem 1.3.** *Let $\mu, \nu \in X^+(T)$ and $\alpha_0 \in R$ be the highest root. Suppose that $\mu$ is $p^r$-restricted and $\langle \nu, \alpha_0^\vee \rangle < p^r$. Then, the socle of $\nabla(\mu) \otimes \nabla(\nu)$ is $p^r$-restricted. In particular, the socle of $L(\mu) \otimes L(\nu)$ is $p^r$-restricted.*

We can of course take $\nu$ to be a miniscule weight in 1.3 so:

**Corollary 1.4.** *Let $\mu$ be a dominant $p^r$-restricted weight, and $\nu$ be any miniscule weight. If $M$ is any submodule of $\nabla(\mu)$ then the socle of $M \otimes \nabla(\nu)$ is $p^r$-restricted. In particular, the socle of $L(\mu) \otimes \nabla(\nu)$ is $p^r$-restricted.*

**Remark 1.5.** (I) We note that even under the assumptions of 1.3, the tensor product $\nabla(\mu) \otimes \nabla(\nu)$ can in general have many non-$p^r$-restricted composition factors. The fact that they never appear in the socle is somewhat miraculous.

(II) The result 1.3 can also be interpreted in terms of translation functors as in the introduction. Suppose that $\lambda, \mu \in X(T)$ lie in the closure of the same alcove and that $-p < \langle \lambda - \mu, \alpha^\vee \rangle < p$ for all roots $\alpha$. Then for any $w \in W_p$ such that $w \cdot \mu \in X^+(T)$ is $p^r$-restricted, the socle of $T_\mu^\lambda \nabla(w \cdot \mu)$ is also $p^r$-restricted.

# 2. Translation functors in type $A$

For the remainder of the article, we specialize to the case $G = GL(n)$. Given $(a,b) \in \mathbb{Z} \times \mathbb{Z}$, define the corresponding *p-residue* $\mathrm{res}(a,b)$ to be $(b-a)$ regarded as an element of the ring $\mathbb{Z}/p\mathbb{Z}$. For $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and $\lambda \in X(n)$, define the *$\alpha$-content* of $\lambda$ to be the integer:

$$\mathrm{cont}_\alpha(\lambda) = \left| \left\{ (a,b) \;\middle|\; \begin{array}{c} 1 \le a \le n, 0 < b \le \lambda_a, \\ \mathrm{res}(a,b) = \alpha \end{array} \right\} \right| - \left| \left\{ (a,b) \;\middle|\; \begin{array}{c} 1 \le a \le n, \lambda_a \le b < 0, \\ \mathrm{res}(a,b) = \alpha \end{array} \right\} \right|.$$

Say $\lambda, \mu \in X(n)$ are *linked*, written $\lambda \sim \mu$, if $\mathrm{cont}_\alpha(\lambda) = \mathrm{cont}_\alpha(\mu)$ for all $\alpha \in \mathbb{Z}/p\mathbb{Z}$. The *linkage principle* proved in [17] implies that if $\mathrm{Ext}^1_{GL(n)}(L(\lambda), L(\mu)) \neq 0$, for $\lambda, \mu \in X^+(n)$, then $\lambda \sim \mu$.

Let $\mathcal{C}$ denote the category of all rational $GL(n)$-modules. For any $\lambda \in X(n)$, let $\mathcal{C}(\lambda)$ denote the full subcategory of $\mathcal{C}$ consisting of all $M \in \mathcal{C}$ such that all composition factors of $M$ are of the form $L(\mu)$ for $\mu \sim \lambda$. By the linkage principle, any module $M \in \mathcal{C}$ can be written uniquely as

$$M \cong \bigoplus_{\lambda} pr_\lambda M$$

where $\lambda$ runs over a set of $\sim$-equivalence class represnetatives in $X(n)$, and $pr_\lambda M$ denotes the largest submodule of $M$ belonging to $\mathcal{C}(\lambda)$.

Fix a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. We can now define the functors

$$E_\alpha : \mathcal{C} \to \mathcal{C} \qquad \text{and} \qquad F_\alpha : \mathcal{C} \to \mathcal{C}.$$

We will first define their restrictions to $\mathcal{C}(\lambda)$ for any $\lambda \in X(n)$, and then extend additively to obtain the functors on the whole category $\mathcal{C}$. Given $M \in \mathcal{C}(\lambda)$, we let $F_\alpha M$ (resp. $E_\alpha M$) denote the largest submodule of $M \otimes V$ (resp. $M \otimes V^*$) all of whose composition factors are of the form $L(\mu)$ with

$$\mathrm{cont}_\alpha(\mu) = \mathrm{cont}_\alpha(\lambda) + 1 \quad (\text{resp. } \mathrm{cont}_\alpha(\mu) = \mathrm{cont}_\alpha(\lambda) - 1),$$

and $\mathrm{cont}_\beta(\mu) = \mathrm{cont}_\beta(\lambda)$ for all $\alpha \neq \beta \in \mathbb{Z}/p\mathbb{Z}$. Given a morphism $\theta : M \to N$, $F_\alpha\theta$ is just the restriction to $F_\alpha M$ of the natural map $\theta \otimes 1 : M \otimes V \to N \otimes V$, and similarly for $E_\alpha$. We have

$$M \otimes V \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} F_\alpha M \quad \text{and} \quad M \otimes V^* \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} E_\alpha M$$

for any $M \in \mathcal{C}$.

On any fixed linkage class $\mathcal{C}(\lambda)$, the functor $F_\alpha$ (resp. $E_\alpha$), for a suitable choice of $\alpha$, coincides with the translation functor $T_\lambda^\mu$ defined in [39, II.7.6], for any weight $\mu \in X(n)$ such that the dominant conjugate of $(\mu - \lambda)$ is equal to the highest weight of $V$ (resp. $V^*$). We note initially that the argument of [39, II.7.6] shows easily that the functors $F_\alpha$ and $E_\alpha$ are (left and right) adjoint to one another, and both are exact.

In the next combinatorial definitions, the notions of normal and good first appeared in [46], while the dual notions of conormal and cogood were introduced in [12]. (The reader may be more familiar with normal and good *nodes* – we reserve this terminology for the symmetric group setting when definitions are 'transposed'). In the definitions, we call a map $\psi$ from a set $M \subseteq \mathbb{Z}$ to a set $N \subseteq \mathbb{Z}$ *increasing* (resp. *decreasing*) if $\psi(m) > m$ (resp. $\psi(m) < m$) for all $m \in M$.

Fix $\lambda \in X^+(n)$ and $1 \leq i \leq n$. We say $i$ is *$\lambda$-removable* if *either* $i = n$ *or* $1 \leq i < n$ and $\lambda_i > \lambda_{i+1}$; equivalently, $i$ is $\lambda$-removable if $\lambda - \epsilon_i \in X^+(n)$. We say $i$ is *$\lambda$-addable* if *either* $i = 1$ *or* $1 < i \leq n$ and $\lambda_i < \lambda_{i-1}$; equivalently, $i$ is $\lambda$-addable if $\lambda + \epsilon_i \in X^+(n)$.

Say $i$ is *normal for* $\lambda$ if $i$ is $\lambda$-removable and there is a decreasing injection from the set of

$$\lambda\text{-addable } j \text{ with } i < j \leq n \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(j, \lambda_j + 1)$$

into the set of

$$\lambda\text{-removable } j' \text{ with } i < j' < n \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(j', \lambda_{j'}).$$

Say $i$ is *good for* $\lambda$ if $i$ is normal for $\lambda$ and there is no $j$ that is normal for $\lambda$ with $1 \leq j < i$ and $\mathrm{res}(j, \lambda_j) = \mathrm{res}(i, \lambda_i)$.

Say $i$ is *conormal for* $\lambda$ if $i$ is $\lambda$-addable and there is an increasing injection from the set of

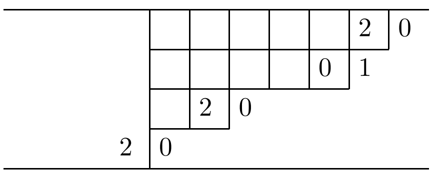
$$\lambda\text{-removable } j \text{ with } 1 \leq j < i \text{ and } \mathrm{res}(j, \lambda_j) = \mathrm{res}(i, \lambda_i + 1)$$

into the set of

$$\lambda\text{-addable } j' \text{ with } 1 < j' < i \text{ and } \mathrm{res}(j', \lambda_{j'} + 1) = \mathrm{res}(i, \lambda_i + 1).$$

Say $i$ is *cogood for* $\lambda$ if $i$ is conormal for $\lambda$ and there is no $j$ that is conormal for $\lambda$ with $i < j \leq n$ and $\mathrm{res}(j, \lambda_j + 1) = \mathrm{res}(i, \lambda_i + 1)$.

**Example 2.1.** We pause to give an example illustrating the definitions. Consider $n = 4, \lambda = (6, 5, 2, 0)$ and $p = 3$. The 3-residues of the addable and removable nodes are:




For $i = 1, 2, 3$ or $4$, $i$ is normal for $\lambda$ if $i = 1, 3, 4$ (when $\mathrm{res}(i, \lambda_i) = 2$) but not if $i = 2$. So 1 is good for $\lambda$. Similarly, for $i = 1, 2, 3$ or $4$, $i$ is conormal for $\lambda$ if $i = 1, 4$ (when $\mathrm{res}(i, \lambda_i + 1) = 0$) or if $i = 2$ (when $\mathrm{res}(i, \lambda_i + 1) = 1$), but not if $i = 3$. So 2 and 4 are cogood for $\lambda$.

In general, for a fixed $\alpha \in \mathbb{Z}/p\mathbb{Z}$, there is at most one good $i$ for $\lambda$ such that $\operatorname{res}(i, \lambda_i) = \alpha$. Moreover there is exactly one such $i$ if and only if there is at least one normal $j$ for $\lambda$ with $\operatorname{res}(j, \lambda_j) = \alpha$. A similar result is true for conormal and cogood.

Our first result [12, Theorem A] describes the effect of $F_\alpha$ on standard modules (the analogous result for costandard modules follows easily since $F_\alpha$ commutes with contravariant duality):

**Theorem 2.2.** *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $F_\alpha \Delta(\lambda)$ is zero unless there is at least one $\lambda$-addable $i$ with $1 \leq i \leq n$ and $\operatorname{res}(i, \lambda_i + 1) = \alpha$. In that case,*

(i) *$F_\alpha \Delta(\lambda)$ has a filtration with factors $\Delta(\lambda + \epsilon_j)$ for all $\lambda$-addable $j$ with $1 \leq j \leq n$ and $\operatorname{res}(j, \lambda_j + 1) = \alpha$, each appearing with multiplicity one;*

(ii) *the head of $F_\alpha \Delta(\lambda)$ is $\bigoplus_j L(\lambda + \epsilon_j)$ where the sum is over all $j$ with $1 \leq j \leq n$ such that $j$ is normal for $\lambda + \epsilon_j$ and $\operatorname{res}(j, \lambda_j + 1) = \alpha$;*

(iii) *if $\lambda$ is $p^r$-restricted, every element of the head of $F_\alpha \Delta(\lambda)$ is also $p^r$-restricted.*

By contravariant duality, (iii) is a special case of 1.4.

To illustrate the theorem, take $\lambda$ as in 2.1. Then, $F_0 \Delta(\lambda)$ has a $\Delta$-filtration with factors (from the bottom up) $\Delta(\lambda + \epsilon_1), \Delta(\lambda + \epsilon_3)$ and $\Delta(\lambda + \epsilon_4)$. Moreover, $F_0 \Delta(\lambda)$ has simple head $L(\lambda + \epsilon_4)$ so is certainly indecomposable.

Next we consider $F_\alpha$ applied to an irreducible module. This is [12, Theorem B]:

**Theorem 2.3.** *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $F_\alpha L(\lambda)$ is zero unless there is at least one $j$ conormal for $\lambda$ and such that $\operatorname{res}(j, \lambda_j + 1) = \alpha$. In that case,*

(i) *$F_\alpha L(\lambda)$ is an indecomposable, contravariantly self-dual module, with simple socle and head isomorphic to $L(\lambda + \epsilon_i)$ where $i$ is the cogood with $\operatorname{res}(i, \lambda_i + 1) = \alpha$;*

(ii) *for any $\mu \in X^+(n)$,*

$$\operatorname{Hom}_{GL(n)}(\Delta(\mu), F_\alpha L(\lambda)) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda + \epsilon_j \text{ for some } j \text{ conormal for } \lambda, \\ & \text{such that } \operatorname{res}(j, \lambda_j + 1) = \alpha; \\ 0 & \text{otherwise;} \end{cases}$$

(iii) *for any $\lambda$-addable $j$ with $1 \leq j \leq n$,*

$$[F_\alpha L(\lambda) : L(\lambda + \epsilon_j)] = \begin{cases} b_j & \text{if } j \text{ is conormal for } \lambda \text{ and } \operatorname{res}(j, \lambda_j + 1) = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

*where $b_j$ denotes the number of $k$ with $1 \leq k \leq j$ such that $k$ is conormal for $\lambda$ and $\operatorname{res}(k, \lambda_k + 1) = \alpha$;*

(iv) *the endomorphism ring $\operatorname{End}_{GL(n)}(F_\alpha L(\lambda))$ is isomorphic to the truncated polynomial ring $\mathbb{F}[T]/(T^b)$, of dimension $b$, where $b$ is the number of $j$ with $1 \leq j \leq n$ such that $j$ is conormal for $\lambda$ and $\operatorname{res}(j, \lambda_j + 1) = \alpha$.*

Again, we illustrate the theorem using the setup of 2.1. Then, $F_0 L(\lambda)$ has simple head and socle $L(\lambda + \epsilon_4)$, and $L(\lambda + \epsilon_1)$ appears in $F_0 L(\lambda)$ with multiplicity

1. So we see at once that the Loewy length of $F_0 L(\lambda)$ is at least 3. In fact, this example is similar to the length 3 case of translation functors mentioned in the introduction: $\lambda$ lies in the upper closure of the facet containing $\lambda + \epsilon_4$ and $\lambda + \epsilon_1$ is the reflection of $\lambda + \epsilon_4$ across a wall containing $\lambda$.

Now suppose quite generally that $N$ is a $G$-module (for any reductive group $G$) such that

(1) $N$ is a submodule of a *finite dimensional* $G$-module with a $\nabla$-filtation;

(2) $N \cong N^\tau$ where $N^\tau$ denotes the contravariant dual of $N$ as in [39, p.205].

Then, according to [12, Proposition 4.7]:

**Lemma 2.4.** *The $G$-module $N$ is completely reducible if and only if*

$$\mathrm{Hom}_G(L(\lambda), N) \cong \mathrm{Hom}_G(\Delta(\lambda), N)$$

*for all $\lambda \in X^+(T)$.*

We apply 2.4 to the module $N = F_\alpha L(\lambda)$, observing that this certainly satisfies (1) and (2). The results in 2.3 then allow us to obtain the following *necessary and sufficient* condition for $F_\alpha L(\lambda)$ to be irreducible:

**Corollary 2.5.** *With the notation of 2.3, $F_\alpha L(\lambda)$ is irreducible if and only if there is a unique $j$ such that $j$ is conormal for $\lambda$ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$. Hence, $L(\lambda) \otimes V$ is completely irreducible if and only if every conormal $i$ is cogood.*

In fact it is well-known that $T_\mu^\lambda L(\mu)$ is irreducible whenever $\lambda$ lies in the upper closure of the facet containing $\mu$ (see [39, Proposition 7.15]), as is the case in our example for $F_1 L(\lambda) \cong L(\lambda + \epsilon_2)$. However, the corollary shows that there are many other more general circumstances when $F_\alpha L(\lambda)$ is irreducible. The significance of this is the following, which follows immediately from 2.3 by exactness of $F_\alpha$:

**Corollary 2.6.** *Fix $\lambda \in X^+(n)$ and $\alpha \in \mathbb{Z}/p\mathbb{Z}$ such that $\lambda$ has a unique conormal $i$ with $\mathrm{res}(i, \lambda_i + 1) = \alpha$. Then, the inverse decomposition number $[L(\lambda + \epsilon_i) : \Delta(\mu)]$ is equal to $\sum_j [L(\lambda) : \Delta(\mu - \epsilon_j)]$ summed over all $\mu$-removable $j$.*

The final result [12, Theorem C] gives further information about the structure of $F_\alpha L(\lambda)$:

**Theorem 2.7.** *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let $1 = s_1 < s_2 < \cdots < s_b$ denote the set of all $j$ with $1 \leq j \leq n$ such that $j$ is conormal for $\lambda$ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$. Then, $N := F_\alpha L(\lambda)$ has a filtration $0 = N_0 < N_1 < \cdots < N_b = N$ such that:*

(i) *for $1 \leq i \leq b$, $N_i / N_{i-1}$ is a non-zero quotient of $\Delta(\lambda + \epsilon_{s_i})$;*

(ii) *for $1 \leq i \leq j \leq b$,*

$$\dim \mathrm{Hom}_{GL(n)}(N_j / N_{j-1}, N_i / N_{i-1}) = [N_i / N_{i-1} : L(\lambda + \epsilon_{s_j})] = 1;$$

(iii) *for $1 \leq i < b$, the extension $0 \to N_i / N_{i-1} \to N_{i+1} / N_{i-1} \to N_{i+1} / N_i \to 0$ does not split;*

(iv) *the Loewy length of* $N_i/N_{i-1}$ *is at least* $b-i+1$*;*
(v) *the Loewy length of* $N$ *is at least* $2b-1$*.*

It is easy to construct examples $\lambda$ so that the number $b$ from 2.3(iv), 2.7 is arbitrarily large. Thus, we see by 2.7(v) that the Loewy length of $F_\alpha L(\lambda)$ can be arbitrarily large.

We have not mentioned the functor $E_\alpha$ adjoint to $F_\alpha$ yet. In fact, there are entirely analogous statements to 2.2–2.7 in this case, all of which follow directly from the above and some combinatorial arguments. Roughly, one needs to swap 'addable' and 'removable', 'normal' and 'conormal', 'good' and 'cogood' in the statements, but there are some other differences too; for precise statements we refer the reader to [12, Theorems A′, B′, C′] (where the functor $E_\alpha$ is denoted $\mathrm{Tr}_\alpha$).

# 3. Connections with the LLT algorithm

We need to switch to working with *polynomial representations.* Let $\Lambda(n) \subset X(n)$ denote all $n$-tuples $(\lambda_1,\dots,\lambda_n)$ satisfying $\lambda_i \geq 0$ for $i=1,\dots,n$, and $\Lambda^+(n) := \Lambda(n) \cap X^+(n)$. Let $\Lambda(n,r) \subset \Lambda(n)$ denote all $n$-tuples $(\lambda_1,\dots,\lambda_n)$ satisfying $|\lambda| := \lambda_1+\dots+\lambda_n = r$, and $\Lambda^+(n,r) := \Lambda(n,r)\cap X^+(n)$. We call elements of $\Lambda(n,r)$ *compositions* of $r$ (with at most $n$ non-zero parts), and elements of $\Lambda^+(n,r)$ *partitions* of $r$ (with at most $n$ non-zero parts).

If $\lambda$ is any partition we denote by $\lambda^t$ the *transpose* partition, i.e. the partition whose Young diagram is the transpose of the Young diagram of $\lambda$. The following result of Donkin from [21] will be important:

**Theorem 3.1.** *For* $\lambda,\mu \in \Lambda^+(n,r)$*,* $[\Delta(\lambda):L(\mu)] = [T(\mu^t):\nabla(\lambda^t)]_\nabla$.

We note that $\lambda \in \Lambda^+(n,r)$ does *not* necessarily imply that $\lambda^t \in \Lambda^+(n,r)$, but we can always find some $m > n$ so that $\lambda^t \in \Lambda^+(m,r)$. The expression $[T(\mu^t):\nabla(\lambda^t)]_\nabla$ in the theorem needs to be interpreted possibly as a $\nabla$-filtration multiplicity inside of the group $GL(m)$ for this larger $m$. Since such multiplicities remain stable as $m$ increases, there is no ambiguity.

Let $q$ be an indeterminate and $U = U_q(\hat{\mathfrak{sl}}_p)$ denote Lusztig's integral form for the quantized enveloping algebra of the affine Lie algebra $\hat{\mathfrak{sl}}_p$. The negative part $U^-$ of this algebra is generated by certain elements $\{f_\alpha \mid \alpha \in \mathbb{Z}/p\mathbb{Z}\}$ together with all of their *quantized divided powers*, subject to various well-known relations. The *Fock space* $\mathcal{F}$ is a certain integrable representation of $U$ with basis given by *all* partitions. A beautiful account of this representation (or rather its unquantized analogue) is given in the book [41]. For its extension to the quantum case, we refer the reader to [57, 42, 53].

Fix $n$ and let $\mathcal{F}_n$ denote the free $\mathbb{Z}[q,q^{-1}]$-module with basis $\{\Delta_\lambda \mid \lambda \in \Lambda^+(n)\}$. This can be regarded as a quotient of $\mathcal{F}$ as a $U^-$-module in an obvious way, so that

[42, 53] give us a well-defined action of $U^-$ on $\mathcal{F}_n$ satisfying for each $\alpha \in \mathbb{Z}/p\mathbb{Z}$:

$$f_\alpha \Delta_\lambda = \sum_i q^{a_{<i} - r_{<i}} \Delta_{\lambda + \epsilon_i}$$

where the sum is over all $\lambda$-addable $i$ such that $\mathrm{res}(i, \lambda_i + 1) = \alpha$, and $a_{<i}$ denotes the number of $\lambda$-addable $j < i$ with $\mathrm{res}(j, \lambda_j + 1) = \alpha$ and $r_{<i}$ denotes the number of $\lambda$-removable $k < i$ with $\mathrm{res}(k, \lambda_k) = \alpha$.

Now assume for simplicity that $p \geq n$. Everything we are saying generalizes to the case $p < n$ too, but the definitions become considerably more technical. Then it is quite easy to see (because $p \geq n$) that $\mathcal{F}_n$ is generated by a $U^-$-module by the vectors $\{\Delta_{i\delta}\}_{i \geq 0}$ where $i\delta$ denotes the weight $(i, i, \dots, i) \in \Lambda^+(n)$ corresponding to the $i$th power of determinant. Moreover, as follows directly from [53] or [52] on passing to the quotient $\mathcal{F}_n$ of the space $\mathcal{F}$ there, there is a unique ring homomorphism

$$- : \mathcal{F}_n \to \mathcal{F}_n$$

such that $\bar{q} = q^{-1}$, $\bar{\Delta}_{i\delta} = \Delta_{i\delta}$ for all $i \geq 0$ and commuting with the action of the $f_\alpha$ for $\alpha \in \mathbb{Z}/p\mathbb{Z}$.

**Theorem 3.2.** *There exist unique bases $\{L_\lambda \mid \lambda \in \Lambda^+(n)\}$ and $\{T_\lambda \mid \lambda \in \Lambda^+(n)\}$ for $\mathcal{F}_n$ which are $-$-invariant and satisfy*

$$L_\lambda = \Delta_\lambda + \sum_{\mu < \lambda} e_{\lambda,\mu}(q) \Delta_\mu, \qquad T_\lambda = \Delta_\lambda + \sum_{\mu < \lambda} d_{\lambda,\mu}(q) \Delta_\mu$$

*for polynomials $e_{\lambda,\mu} \in q^{-1}\mathbb{Z}[q^{-1}]$ and $d_{\lambda,\mu} \in q\mathbb{Z}[q]$.*

The basis $\{T_\lambda\}_{\lambda \in \Lambda^+(n)}$ is called the *lower global crystal basis* or *canonical basis* of $\mathcal{F}_n$, and $\{L_\lambda\}_{\lambda \in \Lambda^+(n)}$ is the *upper global crystal basis*. The (purely combinatorial) construction of these bases follows directly from the more general [52, Theorem 4.1], on passing to the quotient $\mathcal{F}_n$ of $\mathcal{F}$. In fact, the Leclerc-Thibon construction gives similar bases of $\mathcal{F}_n$ even for $n < p$. The strategy in the general case is to work not just with $U_q(\hat{\mathfrak{sl}}_p)$ but with the algebra $U_q(\hat{\mathfrak{gl}}_p)$, which is a sum of $U_q(\hat{\mathfrak{sl}}_p)$ and a Heisenberg algebra, and its action on $\mathcal{F}$ constructed in [42]. There is also a more geometric construction of the canonical basis of $\mathcal{F}_n$ in terms of the Hall algebra associated to the cyclic quiver of type $\hat{A}_p$ originating in [31] and exploited by Varagnolo and Vasserot [66].

To state the main result of [66], we let $\xi$ be a primitive $p$th root of unity in $\mathbb{C}$. For $\lambda \in \Lambda^+(n)$, write $L_\xi(\lambda)$, $\Delta_\xi(\lambda)$, $\nabla_\xi(\lambda)$ and $T_\xi(\lambda)$ for the irreducible, standard, costandard and indecomposable tilting modules for "quantum $GL_n$" over $\mathbb{C}$ at the root of unity $\xi$. Let $\mathcal{G}_\xi$ denote the Grothendieck group of the category of polynomial representations of quantum $GL_n$ at root of unity $\xi$. Then, there are three natural bases for $\mathcal{G}_\xi$, namely, $\{[\Delta_\xi(\lambda)] = [\nabla_\xi(\lambda)]\}_{\lambda \in \Lambda^+(n)}$, $\{[L_\xi(\lambda)]\}_{\lambda \in \Lambda^+(n)}$ and $\{[T_\xi(\lambda)]\}_{\lambda \in \Lambda^+(n)}$ corresponding to the standard, irreducible and tilting modules respectively. Then, the main results of [66] can be stated as:

**Theorem 3.3.** *Suppose that $p \geq n$. Identify the specialization $\mathcal{F}_n \otimes_{\mathbb{Z}[q,q^{-1}]} \mathbb{Z}$ at $q = 1$ with the Grothendieck group $\mathcal{G}_n$ so that*

$$\Delta_\lambda \otimes 1 = [\Delta_\xi(\lambda)]$$

*for all $\lambda \in \Lambda^+(n)$. Then, for all $\lambda \in \Lambda^+(n)$,*

$$L_\lambda \otimes 1 = [L_\xi(\lambda)],$$
$$T_\lambda \otimes 1 = [T_\xi(\lambda)].$$

In particular, this implies that the inverse decomposition numbers $[L_\xi(\lambda) : \Delta_\xi(\mu)]$ and the $\nabla$-filtration multiplicities $[T_\xi(\lambda) : \nabla_\xi(\mu)]_\nabla$ can be computed from knowledge of the lower and upper global crystal bases of $\mathcal{F}_n$:

**Corollary 3.4.** *For $\lambda, \mu \in \Lambda^+(n)$,*

$$[L_\xi(\mu) : \Delta_\xi(\mu)] = e_{\lambda,\mu}(1),$$
$$[T_\xi(\lambda) : \nabla_\xi(\mu)]_\nabla = d_{\lambda,\mu}(1).$$

**Remark 3.5.** (I) We have stated 3.3 somewhat differently from [66]. The result concerning the basis $\{T_\lambda\}_{\lambda\in\Lambda^+(n)}$ stated here follows from [66, Corollary 11.2] by the definition of the decomposition matrix of the $q$-Schur algebra adopted in [66, 52], together with the quantum analogue of 3.1 (proved e.g. in [22]). The result concerning $L_\lambda$ stated here is precisely [66, Theorem 12].

(II) In fact, for $p \geq n$, the result 3.3 in the case of the basis $\{T_\lambda\}_{\lambda\in\Lambda^+(n)}$ can also be deduced directly from Ariki's earlier theorem proving the original LLT conjecture [4, 53]. However, it is *not* immediately clear how to deduce the result about the basis $\{L_\lambda\}_{\lambda\in\Lambda^+(n)}$ from this using tricks like 3.1, the problem being that the necessary restriction $p \geq n$ does not allow one to transpose partitions.

(III) In [66, Proposition 9.3(j)], it is shown that the coefficients $e_{\lambda,\mu}(q)$ coincide with the signed sums of Kazhdhan-Lusztig polynomials appearing in the statement of the (quantum) Lusztig conjecture (for $\lambda, \mu$ lying in interiors of alcoves). Thus, [66] gives an alternative proof of the Lusztig conjecture for quantum groups of type $A$.

(IV) In [66, Proposition 9.3(k)], it is shown that the coefficients $d_{\lambda,\mu}(q)$ coincide with certain Kazhdan-Lusztig polynomials for the affine Hecke algebra of type $A$. In particular, this shows that the coefficients of the polynomials $d_{\lambda,\mu}(q)$ are all non-negative integers, as conjectured in [52]. There is a seemingly quite different approach due to Soergel [63, 64] to computing the multiplicities $[T_\xi(\lambda) : \nabla_\xi(\mu)]_\nabla$ (in arbitrary type!) for $\lambda, \mu$ lying in interiors of alcoves, again involving affine Kazhdan-Lusztig polynomials. The Kazhdan-Lusztig polynomials appearing in both approaches are the same. This has also been verified combinatorially by Goodman and Wenzl [32].

(V) The bases $\{L_\lambda\}_{\lambda\in\Lambda^+(n)}$ and $\{T_\lambda\}_{\lambda\in\Lambda^+(n)}$ were constructed more generally for $p < n$ in [52], as we have said. The analogue of 3.3 for $p < n$ was also conjectured by Leclerc and Thibon. This was not quite proved by Vasserot and Varagnolo: the problem left was to show that the canonical basis constructed there geometrically using the Hall algebra coincided with the combinatorial construction of Leclerc and Thibon. This identification, completing the proof of the Leclerc-Thibon conjecture, has recently been made by Schiffmann [62].

Now we wish to discuss how the results in section 2 relate to 3.3 in the quantum case. We will assume that *all* the earlier results described in section 2 in the classical setting have analogues for quantum groups. We have no doubt that this is true, even in the quantum mixed case, with precisely the same statements: there are no modifications necessary resulting from the different Steinberg tensor product theorems in the two settings. However, to date, full proofs in the quantum case have only been given in roughly half of the results, see [8]; the remaining quantizations will be carried out in [14]. We remark that it is not obvious how to deduce all the results from section 2 in the quantum case directly from 3.3: the latter at present only gives information about multiplicities, not about submodule structure.

First, we observe that at $q = 1$, $f_\alpha \Delta_\lambda = \sum_i \Delta_{\lambda+\epsilon_i}$, summed over all $\lambda$-addable $i$ with $\mathrm{res}(i, \lambda_i + 1) = \alpha$, which is precisely the same as the effect of the (quantum analogue of the) functor $F_\alpha$ on the basis $[\Delta_\xi(\lambda)]$ of $\mathcal{G}_\xi$. Consequently, we can identify the operator $f_\alpha$ and the functor $F_\alpha$ in their actions on the Grothendieck group. So we can calculate the composition multiplicities (resp. the tilting module multiplicities) in $F_\alpha L_\xi(\lambda)$ (resp. $F_\alpha T_\xi(\lambda)$) algorithmically by first computing $f_\alpha L_\lambda$ (resp. $f_\alpha T_\lambda$) using the known action of $f_\alpha$ on the $\Delta_\lambda$'s, then rewriting the resulting expression in terms of the $L_\lambda$'s (resp. the $T_\lambda$'s). This observation was first made in [26] in a slightly different setting.

Let us give a very simple example. Suppose that $n = 4, p = 3$ and $\lambda = (4, 2, 0)$. This is the highest weight of the Steinberg module, so

$$L_{(4,2,0)} = \Delta_{(4,2,0)}.$$

Now apply $f_1$ to deduce that

$$f_1 L_{(4,2,0)} = \Delta_{(5,2,0)} + q\Delta_{(4,3,0)} + q^2\Delta_{(4,2,1)}$$

which on rewriting in terms of the $L$ basis gives

$$f_1 L_{(4,2,0)} = L_{(5,2,0)} + (q + q^{-1})L_{(4,3,0)} + (q^2 + 1 + q^{-2})L_{(4,2,1)}.$$

Now, 2.3(iii) tells us that $F_1 L_\xi((4, 2, 0))$ has Loewy length at least 5 and contains the composition factors $L_\xi((5, 2, 0))$, $L_\xi((4, 3, 0))$ and $L_\xi((4, 2, 1))$ with multiplicities $1, 2$ and $3$ respectively.

In fact, in general, it is easy to see using the positivity 3.5(IV) that quite generally, one can always write

$$f_\alpha L_\lambda = \sum_\mu b_{\lambda,\mu}(q) L_\mu$$

for Laurent polynomials $b_{\lambda,\mu}(q)$ which are always positive linear combinations of quantum integers. Evaluating at $q = 1$, $b_{\lambda,\mu}(1)$ computes the composition multiplicity $[F_\alpha L_\xi(\lambda) : L_\xi(\mu)]$. It seems reasonable to expect that in the quantum case, the Loewy length of $F_\alpha L_\xi(\lambda)$ is exactly as predicted in 2.7(v), and moreover that the powers of $q$ in the polynomial $b_{\lambda,\mu}(q)$ indicate the Loewy levels that the irreducible $L_\xi(\mu)$ appears in $F_\alpha L_\xi(\lambda)$.

One consequence of these remarks is that there is an alternative approach to proving the quantum analogue of 2.5, very similar to the result of [26]. In the language of crystal bases, 2.5 is equivalent to describing precisely when the operator $f_\alpha$ sends an upper global crystal basis element to a single upper global crystal basis element. It would be interesting to determine in a similar way when $f_\alpha$ sends a lower global crystal basis element to a single lower global crystal basis element, that is, when $f_\alpha T_\lambda = T_\mu$ for some $\mu$. In terms of representation theory, this is:

**Question.** For $\lambda \in \Lambda^+(n)$, when is $F_\alpha T(\lambda)$ an indecomposable tilting module?

# 4. Relating tensor products and restrictions

We have already mentioned in section 1 one source of connections between tensor products and restrictions to Levi subgroups. We will now describe two more ways such connections arise. The results in this section were all obtained in [11].

Fix $a \geq 1$ and $\nu = (n_1, \dots, n_a) \in \Lambda(a, n)$ (a composition of $n$ with $a$ non-zero parts). Let $GL(\nu) = GL(n_1) \times \cdots \times GL(n_a)$ denote the *standard Levi subgroup* of $GL(n)$ consisting of all invertible block diagonal matrices with block sizes $n_1, \dots, n_a$. Of course, if $\nu = (n)$ then $GL(\nu) = GL(n)$ while, at the other extreme, if $\nu = (1, \dots, 1)$ then $GL(\nu)$ is the maximal torus $T < GL(n)$. The following theorem is [11, Theorem 2.8].

**Theorem 4.1.** *Let $\nu \in \Lambda(a, n)$ and $\mu^{(1)}, \dots, \mu^{(a)} \in \Lambda^+(n)$ be partitions such that $\mu^{(i)}$ has at most $n_i$ non-zero rows for each $i$. Let $\bar\mu^{(i)} = (\mu^{(i)}_1, \dots, \mu^{(i)}_{n_i}) \in \Lambda^+(n_i)$. For any polynomial $GL(n)$-module $M$,*

$$\mathrm{Hom}_{GL(n)}(M, \nabla_n(\mu^{(1)}) \otimes \cdots \otimes \nabla_n(\mu^{(a)})) \cong \mathrm{Hom}_{GL(\nu)}(M{\downarrow_{GL(\nu)}}, \nabla_{n_1}(\bar\mu^{(1)}) \boxtimes \cdots \boxtimes \nabla_{n_a}(\bar\mu^{(a)})).$$

The following corollary of 4.1 (with $a = 2$) should be compared with 1.2:

**Corollary 4.2.** *Fix $\lambda, \mu \in \Lambda^+(n)$ with $\mu_n = 0$. Then,*

$$\mathrm{Hom}_{GL(n)}(L_n(\lambda), \nabla_n(\mu) \otimes S^\ell(V)) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\bar\mu), L_n(\lambda) \downarrow_{GL(n-1)})$$

*where $\ell = |\lambda| - |\mu|$.*

The main tool used in the proof of 4.1 is a *polynomial induction functor* from Levi subgroups. This notion goes back to [20] (see also [24]). However we prove a new property of this functor (see 4.3 below), which is crucial for 4.1.

Let $M_{\mathbb{F}}(n)$ (resp. $M_{\mathbb{F}}(n,r)$) be the category of polynomial $GL(n)$-modules (resp. of degree $r$), and analogously, let $M_{\mathbb{F}}(\nu)$ (resp. $M_{\mathbb{F}}(\nu,r)$) be the category of polynomial $GL(\nu)$-modules (resp. of degree $r$). The restriction of a polynomial module (resp. a polynomial module of degree $r$) from $GL(n)$ to $GL(\nu)$ is again a polynomial module (resp. a polynomial module of degree $r$). So we have the exact restriction functor

$$R^n_\nu : M_{\mathbb{F}}(n) \to M_{\mathbb{F}}(\nu).$$

We now describe how to construct a functor which is right adjoint to $R^n_\nu$. Let $A(n)$ denote the subalgebra of the algebra of regular functions $\mathbb{F}[GL(n)]$ generated by the functions $\{c_{ij} \mid 1 \leq i,j \leq n\}$, where $c_{ij}$ picks out the $ij$-entry of a matrix $g \in GL(n)$. There are two commuting *left* actions of $GL(n)$ on $A(n)$, the *left regular* and *right regular* actions, which we define for $g, g' \in GL(n), f \in A(n)$ by $(g \cdot_l f)(g') = f(g^{-1}g')$ and $(g \cdot_r f)(g') = f(g'g)$ respectively.

For $M \in M_{\mathbb{F}}(\nu)$, we define the $GL(n)$-module

$$(M \otimes A(n))^{GL(\nu)}$$

where the $GL(n)$-action on the induced module comes from the right regular action of $GL(n)$ on $A(n)$ and the trivial action on $M$, and the action of $GL(\nu)$ on $M \otimes A(n)$ under which we are taking fixed points comes from the given action on $M$ and the left regular action on $A(n)$.

Define the *polynomial induction functor*

$$I^n_\nu : M_{\mathbb{F}}(\nu) \to M_{\mathbb{F}}(n)$$

by letting $I^n_\nu M := (M \otimes A(n))^{GL(\nu)}$, with the obvious definition on morphisms. For $M \in M_{\mathbb{F}}(\nu)$, $I^n_\nu M$ can be described alternatively as the largest polynomial submodule of $\mathrm{Ind}^{GL(n)}_{GL(\nu)} M$, where $\mathrm{Ind}^{GL(n)}_{GL(\nu)}$ denotes the usual induction functor in the sense of algebraic groups [39, I.3.3]. One easily checks that that $I^n_\nu$ is right adjoint to $R^n_\nu$, so is left exact and sends injectives in $M_{\mathbb{F}}(\nu)$ to injectives in $M_{\mathbb{F}}(n)$. Moreover, for $M \in M_{\mathbb{F}}(n)$, and any $N \in M_{\mathbb{F}}(\nu)$ with a $\nabla$-filtration,

$$\mathrm{Ext}^i_{GL(\nu)}(R^n_\nu M, N) \cong \mathrm{Ext}^i_{GL(n)}(M, I^n_\nu N)$$

for all $i \geq 0$. As observed in [11], this implies, using the well-known analogous result about restrictions of modules with $\nabla$-filtrations to Levi subgroups, that $I^n_\nu$ sends modules with $\nabla$-filtrations to modules with $\nabla$-filtrations.

Now fix $m \leq n$, and let $\nu = (m, 1, \dots, 1)$, a composition of $n$. Then, $GL(m)$ is a normal subgroup of the Levi subgroup $GL(\nu) \leq GL(n)$, and we have an exact inflation functor $\mathrm{Infl}^\nu_m : M_{\mathbb{F}}(m) \to M_{\mathbb{F}}(\nu)$ which sends each $M_{\mathbb{F}}(m,r)$ into $M_{\mathbb{F}}(\nu, r)$. Let $H(m) \leq GL(\nu)$ be the $(n-m)$-dimensional torus such that $GL(\nu) = GL(m) \times H(m)$. Then, the fixed point functor $M \mapsto M^{H(m)}$ is right adjoint to $\mathrm{Infl}^m_\nu$ (see e.g. [39, I.6.4]). Now define the functors

$$I^n_m : M_{\mathbb{F}}(m) \to M_{\mathbb{F}}(n) \quad \text{and} \quad R^n_m : M_{\mathbb{F}}(n) \to M_{\mathbb{F}}(m)$$

by the compositions

$$I_m^n := I_\nu^n \circ \mathrm{Infl}_m^\nu, \qquad R_m^n := (-)^{H(m)} \circ R_\nu^n.$$

We note that $I_m^n$ is right adjoint to the exact functor $R_m^n$, hence is left exact and sends injectives to injectives.

Our main motivation for introducing the functors $I_m^n$ and $I_\nu^n$ was the following result [11, Theorem 2.7] describing polynomial induction applied to an outer tensor product:

**Theorem 4.3.** *Let $\nu \in \Lambda(a,n)$. Take any modules $M_1, \dots, M_a$ with $M_i \in M_{\mathbb{F}}(n_i)$, so that $M_1 \boxtimes \cdots \boxtimes M_a \in M_{\mathbb{F}}(\nu)$. Then*

$$I_\nu^n(M_1 \boxtimes \cdots \boxtimes M_a) \cong (I_{n_1}^n M_1) \otimes \cdots \otimes (I_{n_a}^n M_a).$$

Finally, we observe that for $\mu \in \Lambda^+(n)$ and $m \leq n$ with $\mu_{m+1} = \cdots = \mu_n = 0$, we showed in [11] that

$$I_m^n \nabla_m(\bar{\mu}) \cong \nabla_n(\mu) \qquad \text{and} \qquad I_m^n L_m(\bar{\mu}) \cong L_n(\mu)$$

where $\bar{\mu} = (\mu_1, \dots, \mu_m) \in \Lambda^+(m)$. Combining this with 4.3, our original result 4.1 follows immediately from 'Frobenius reciprocity'.

**Remark 4.4.** There is also an analogous result to 4.3 for quantum $GL_n$. The details will appear in [10].

Now we discuss our second, quite different relation between tensor products and restrictions to Levi subgroups. This comes from Donkin's characteristic free version of Howe duality (see [11, section 3] for more details). Denote

$$\Lambda^+(n \times m) := \{\lambda = (\lambda_1, \lambda_2, \dots) \in \Lambda^+(n) \mid \lambda_1 \leq m\}.$$

The following theorem is [11, Theorem 3.5]:

**Theorem 4.5.** *Fix $a, n, m \geq 1$, and choose $\nu \in \Lambda(a,n)$. Let $GL(\nu)$ denote the standard Levi subgroup of $GL(n)$. Choose $\lambda \in \Lambda^+(n \times m)$ and $\mu^{(i)} \in \Lambda^+(n_i \times m)$ for $i = 1, \dots, a$. Then,*

(i) $(T_n(\lambda)\downarrow_{GL(\nu)} : T_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\mu^{(a)})) =$

$$[L_m((\mu^{(1)})^t) \otimes \cdots \otimes L_m((\mu^{(a)})^t) : L_m(\lambda^t)].$$

(ii) $[L_n(\lambda)\downarrow_{GL(\nu)} : L_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes L_{n_a}(\mu^{(a)}))] =$

$$(T_m((\mu^{(1)})^t) \otimes \cdots \otimes T_m((\mu^{(a)})^t) : T_m(\lambda^t)).$$

**Remark 4.6.** There is a generalization of 4.5 involving the Andersen and Jantzen filtrations, obtained in [16]. We illustrate this in a special case. For $\lambda \in \Lambda^+(n \times m), \mu \in \Lambda^+(n \times m_1)$ and $\nu \in \Lambda^+(n \times m_2)$ with $m = m_1 + m_2$, the space

$$E_\lambda(T_n(\mu) \otimes T_n(\nu)) = \mathrm{Hom}_{GL(n)}(T_n(\mu) \otimes T_n(\nu), \nabla_n(\lambda))$$

has an *Andersen filtration* $\mathfrak{A}^i E_\lambda(T_n(\mu)\otimes T_n(\nu))$ $(i \geq 0)$ (taking notation from [56]). The space $\Delta_m(\lambda^t)$ has a *Jantzen filtration* $\mathfrak{J}^i\Delta_m(\lambda^t)$ $(i \geq 0)$. We observe in [16] that the dimension of $\mathfrak{A}^i E_\lambda(T_n(\mu)\otimes T_n(\nu))$ is precisely the composition multiplicity $[\mathfrak{J}^i\Delta_m(\lambda^t)\downarrow_{GL(m_1)\times GL(m_2)}: L_{m_1}(\mu^t)\boxtimes L_{m_2}(\nu^t)]$. The result 4.5(i) follows directly since the top factor in the Andersen filtration computes $(T_n(\mu)\otimes T_n(\nu) : T_n(\lambda))$ and the top factor in the Jantzen filtration computes the composition multiplicity $[L_m(\lambda^t)\downarrow_{GL(m_1)\times GL(m_2)}: L_{m_1}(\mu^t)\boxtimes L_{m_2}(\nu^t)]$.

A special case of 4.5(ii) gives (compare with 1.2 and 4.2):

**Corollary 4.7.** *For $\lambda \in \Lambda^+(n\times m)$ and $\mu \in \Lambda^+(n\times(m-1))$ with $|\lambda| \geq |\mu|$ put $\ell = |\lambda| - |\mu|$. Then*

$$(T_n(\mu)\otimes {\textstyle\bigwedge}^\ell(V) : T_n(\lambda)) = [L_m(\lambda^t)\downarrow_{GL(m-1)}: L_{m-1}(\mu^t)].$$

The significance of this corollary is that it explains the connection between the work of Mathieu and Papadopoulou [55] and the paper [15]. Both papers obtained a general character formula for $L_n(\lambda)$ for a special class of highest weights $\lambda$, which we called the *completely splittable weights* in [15]. The approach in [15] depended on first understanding precisely the restriction $L_n(\lambda)\downarrow_{GL(n-1)}$ in the special cases that it is completely reducible, that is, the right hand side of 4.7 for special $\lambda$. On the other hand, Mathieu and Papadopoulou exploited the results of [29, 30] to determine the structure of $T_n(\lambda)\otimes \bigwedge^i(V)$ for special $\lambda$, modulo a certain 'tilting ideal'. We will discuss further relations between branching rules and tilting ideals in section 6.

Finally, we note that taking $\nu = (1, 1, \dots, 1)$ in 4.1 and 4.5, we obtain the following character formulas:

**Corollary 4.8.** *Let $\lambda \in \Lambda^+(n\times m)$, $\mu = (\mu_1, \dots, \mu_n) \in \Lambda(n\times m)$. Let $V$ and $W$ be the natural $GL(n)$- and $GL(m)$-modules respectively. Then,*

(i) $\dim L_n(\lambda)_\mu = (S^{\mu_1}(V)\otimes\cdots\otimes S^{\mu_n}(V) : Q_n(\lambda))$, *where $Q_n(\lambda)$ is the the injective hull of $L_n(\lambda)$ in the category of polynomial $GL(n)$-modules.*

(ii) $\dim L_n(\lambda)_\mu = (\bigwedge^{\mu_1}(W)\otimes\cdots\otimes\bigwedge^{\mu_n}(W) : T_m(\lambda^t))$.

(iii) $\dim T_n(\lambda)_\mu = [\bigwedge^{\mu_1}(W)\otimes\cdots\otimes\bigwedge^{\mu_n}(W) : L_m(\lambda^t)]$.

Of these, (i) is a result of Donkin [21, Lemma 3.4(i)] and (ii) is due to Mathieu and Papadopoulou [55].

# 5. The symmetric group

The results in section 2 on translation functors can be translated into analogous results abouts symmetric groups using the techniques of the Schur functors. In this section we describe these results on the symmetric groups, most but not all of which can be found in [12], and shortly discuss the 'translation' techniques.

Let $\Sigma_r$ be the symmetric group on $r$ letters. If $\lambda$ is a partition of $r$ we write $\lambda \vdash r$. We denote by $S^\lambda$ (resp. $Y^\lambda$, $M^\lambda$) the Specht (resp. Young, permutation) module over $\mathbb{F}\Sigma_r$ corresponding to a partition $\lambda \vdash r$, and by $D^\lambda$ the irreducible $\mathbb{F}\Sigma_r$-module, corresponding to a *p-regular* partition $\lambda \vdash r$. The reader is referred to [36], [37] or [38] for these and other standard notions of the representation theory of symmetric groups. We denote by sgn the 1-dimensional sign representation of $\Sigma_r$.

Fix a partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash r$. We identify $\lambda$ with its *Young diagram*

$$\lambda = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \le \lambda_i\}.$$

The elements of $\mathbb{N} \times \mathbb{N}$ are called *nodes*. A node of the form $(i, \lambda_i)$ is called a *removable node* (of $\lambda$) if $\lambda_i > \lambda_{i+1}$; a node of the form $(i, \lambda_i + 1)$ is called an *addable node* (for $\lambda$) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$. If $A = (i, \lambda_i)$ is a removable node, we denote by

$$\lambda_A = \lambda \setminus \{A\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots)$$

the partition of $r-1$ obtained by removing $A$ from $\lambda$. If $B = (i, \lambda_i + 1)$ is an addable node, we denote by

$$\lambda^B = \lambda \cup \{B\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots)$$

the partition of $r+1$ obtained by adding $B$ to $\lambda$. The $p$-residue of a node $A = (i,j)$ is defined as in section 2: $\operatorname{res} A = (j - i) \pmod p$.

The next definitions make sense for both 'French' and 'English' notation for the Young diagrams. They are the same as the analogous definitions in section 2 except that we have transposed partitions.

A removable node $A$ (of $\lambda$) is called *normal* if for every addable node $B$ to the right of $A$ with $\operatorname{res} B = \operatorname{res} A$ there exists a removable node $C(B)$ strictly between $A$ and $B$ with $\operatorname{res} C(B) = \operatorname{res} A$, and $B \neq B'$ implies $C(B) \neq C(B')$. A removable node is called *good* if it is the leftmost among the normal nodes of a fixed residue.

An addable node $B$ (for $\lambda$) is called *conormal* if for every removable node $A$ to the left of $B$ with $\operatorname{res} A = \operatorname{res} B$ there exists an addable node $C(A)$ strictly between $B$ and $A$ with $\operatorname{res} C(A) = \operatorname{res} B$, and $A \neq A'$ implies $C(A) \neq C(A')$. An addable node is called *cogood* if it is the rightmost among the conormal nodes of a fixed residue.

For $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and a partition $\lambda$, define the *$\alpha$-content* of $\lambda$ to be the integer $\operatorname{cont}_\alpha(\lambda) := |\{A \in \lambda \mid \operatorname{res} A = \alpha\}|$, which is again a special case of the corresponding definition in section 2. For two partitions $\lambda$ and $\mu$ we write $\lambda \sim \mu$, if $\operatorname{cont}_\alpha(\lambda) = \operatorname{cont}_\alpha(\mu)$ for all $\alpha \in \mathbb{Z}/p\mathbb{Z}$. The 'Nakayama Conjecture' (see e.g. [38]) claims that $\mathbb{F}\Sigma_r$-modules $D^\lambda$ and $D^\mu$ belong to the same block if and only if $\lambda \sim \mu$.

Fix a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. We define the functors

$$\operatorname{Ind}^\alpha : \mathbb{F}\Sigma_r\text{-mod} \to \mathbb{F}\Sigma_{r+1}\text{-mod} \qquad \text{and} \qquad \operatorname{Res}_\alpha : \mathbb{F}\Sigma_r\text{-mod} \to \mathbb{F}\Sigma_{r-1}\text{-mod}.$$

by defining them first on a module $M$ in an (arbitrary fixed) block and then extending additively to the whole of $\mathbb{F}\Sigma_r$-mod. Assume $M$ belongs to the block corresponding to the residue contents $c_0, c_1, \dots, c_{p-1}$—this means that for any irreducible module $D^\lambda$ in this block, $\mathrm{cont}_\beta(\lambda) = c_\beta$ for all $\beta \in \mathbb{Z}/p\mathbb{Z}$. We now let $\mathrm{Ind}^\alpha M$ (resp. $\mathrm{Res}_\alpha M$) denote the largest submodule of $M \uparrow^{\Sigma_{r+1}}$ (resp. $M \downarrow_{\Sigma_{r-1}}$) all of whose composition factors are of the form $D^\mu$ with

$$\mathrm{cont}_\alpha(\mu) = c_\alpha + 1 \qquad (\text{resp. } \mathrm{cont}_\alpha(\mu) = c_\alpha - 1),$$

and

$$\mathrm{cont}_\beta(\mu) = c_\beta \quad \text{for all } \alpha \neq \beta \in \mathbb{Z}/p\mathbb{Z}.$$

Given a morphism $\theta : M \to N$, $\mathrm{Ind}^\alpha \theta$ is just the restriction to $\mathrm{Ind}^\alpha M$ of the natural map $\hat{\theta} : M \uparrow^{\Sigma_{r+1}} \to N \uparrow^{\Sigma_{r+1}}$, induced by $\theta$, and similarly for $\mathrm{Res}_\alpha$. We have

$$M \uparrow^{\Sigma_{r+1}} \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Ind}^\alpha M \quad \text{and} \quad M \downarrow_{\Sigma_{r-1}} \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Res}_\alpha M.$$

The functors just defined are called *Robinson's $\alpha$-induction* and *$\alpha$-restriction functors* (cf. [38, 6.3.16]).

We collect the following known results about the effect of $\mathrm{Ind}^\alpha$ applied to Specht modules. Of these, part (i) is very well-known, see for example [36, 17.14], part (ii) is [12, Theorem D] while part (iii) follows easily from [46, Theorem 0.4] by Frobenius reciprocity.

**Theorem 5.1.** *Fix a partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then,* $\mathrm{Ind}^\alpha S^\lambda$ *is zero unless there is at least one addable node $B$ with* $\mathrm{res}\, B = \alpha$*. In that case:*

(i) *There is a filtration $(0) = S_0 \subset S_1 \subset \cdots \subset S_k = \mathrm{Ind}^\alpha S^\lambda$ with $S_i/S_{i-1} \cong S^{\lambda^{B_i}}$, $i = 1, 2, \dots, k$, where $B_1, B_2, \dots, B_k$ are the addable nodes (for $\lambda$) of residue $\alpha$ counted from left to right.*

(ii) *Assume that $\lambda^t$ is $p$-regular, and $\mu \vdash r+1$ is a partition such that $\mu^t$ is $p$-regular. Then*

$$\mathrm{Hom}_{\Sigma_{r+1}}((S^\mu)^*, \mathrm{Ind}^\alpha S^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some addable } B \text{ of residue } \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iii) *Assume that $\lambda$ is $p$-regular. Then the head of* $\mathrm{Ind}^\alpha S^\lambda$ *is $\bigoplus_B D^{\lambda^B}$ where the sum is taken over all addable nodes $B$ of residue $\alpha$ such that $B$ is normal for $\lambda^B$ (and all such $\lambda^B$ are $p$-regular).*

Next we consider $\mathrm{Ind}^\alpha$ applied to an irreducible module.

**Theorem 5.2.** *Fix a $p$-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then,* $\mathrm{Ind}^\alpha D^\lambda$ *is zero unless $\lambda$ has at least one conormal node of residue $\alpha$. In that case,*

(i) $\mathrm{Ind}^\alpha D^\lambda$ *is an indecomposable, self-dual module, with simple socle and head isomorphic to $D^{\lambda^B}$ where $B$ is the cogood node of residue $\alpha$;*

(ii) $\mathrm{Ind}^\alpha D^\lambda$ *is irreducible if and only if there is a unique conormal node of residue $\alpha$;*

(iii) *for any $p$-regular $\mu \vdash r+1$,*

$$\mathrm{Hom}_{\Sigma_{r+1}}(S^\mu, \mathrm{Ind}^\alpha D^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some conormal } B \text{ with } \mathrm{res}\, B = \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any addable node $B$ such that $\lambda^B$ is p-regular,*

$$[\operatorname{Ind}^\alpha D^\lambda : D^{\lambda^B}] = \begin{cases} d_B & \text{if } B \text{ is conormal for } \lambda \text{ and } \operatorname{res} B = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

*where $d_B$ denotes the number of conormal nodes $C$ to the left of $B$ (counting $B$ itself) such that* $\operatorname{res} C = \alpha$*;*

(v) *the endomorphism ring* $\operatorname{End}_{\Sigma_{r+1}}(\operatorname{Ind}^\alpha D^\lambda)$ *is isomorphic to the truncated polynomial ring $\mathbb{F}[T]/(T^d)$, of dimension $d$, where $d$ is the number of conormal nodes $B$ with* $\operatorname{res} B = \alpha$.

As a consequence of 5.2(ii), we have:

**Corollary 5.3.** *For p-regular $\lambda$, the induced module $D^\lambda \uparrow^{\Sigma_{r+1}}$ is completely reducible if and only if all conormal nodes have different residues.*

We note that a criterion of complete reducibility for $D^\lambda \uparrow^{\Sigma_{r+1}}$ different from the one in 5.3 was found in [48]. Bessenrodt and Olsson informed us that they were able to prove directly that the two combinatorial conditions are equivalent.

Our next result gives further information about the structure of $\operatorname{Ind}^\alpha D^\lambda$:

**Theorem 5.4.** *Fix a p-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let $B_1, B_2 \dots B_d$ be all conormal nodes of residue $\alpha$ counted from left to right. Then, $N := \operatorname{Ind}^\alpha D^\lambda$ has a filtration $0 = N_0 < N_1 < \dots < N_d = N$ such that:*

(i) *for $1 \le i \le d$, $N_i/N_{i-1}$ is a non-zero quotient of $S^{\lambda^{B_i}}$;*

(ii) *for $1 \le i \le j \le d$ such that $\lambda^{B_j}$ is p-regular,* $[N_i/N_{i-1} : D^{\lambda^{B_j}}] = 1$;

(iii) *Let $1 < j < d$. The extension*

$$0 \to N_j/N_{j-1} \to N_{j+1}/N_{j-1} \to N_{j+1}/N_j \to 0$$

*does not split;*

(iv) *If all $\lambda^{B_j}$ for $1 \le j \le d$ are p-regular then the Loewy length of $N$ is at least* $2d - 1$.

We state now the dual results to 5.1, 5.2, 5.4, for the functor $\operatorname{Res}_\alpha$. The results of 5.5(i),(ii), 5.6, 5.7were obtained in [45]-[49].

**Theorem 5.5.** *Fix a partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\operatorname{Res}_\alpha S^\lambda$ is zero unless there is at least one removable node $A$ with* $\operatorname{res} A = \alpha$*. In that case:*

(i) *There is a filtration $(0) = S_0 \subset S_1 \subset \dots \subset S_k = \operatorname{Res}_\alpha S^\lambda$ with $S_i/S_{i-1} \cong S^{\lambda_{A_i}}$, $i = 1, 2, \dots, k$, where $A_1, A_2, \dots, A_k$ are the removable nodes (of $\lambda$) of residue $\alpha$ counted from right to left.*

(ii) *Assume that $\lambda$ is p-regular, and $\mu \vdash r - 1$ is another p-regular partition. Then*

$$\operatorname{Hom}_{\Sigma_{r-1}}((S^\mu)^*, \operatorname{Res}_\alpha S^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^A \text{ for some removable } A \text{ of residue } \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iii) *Assume that $\lambda$ is p-regular. Then the head of* $\operatorname{Res}_\alpha S^\lambda$ *is $\bigoplus_A D^{\lambda_A}$ where the sum is taken over all removable nodes $A$ of residue $\alpha$ such that $A$ is conormal for $\lambda_A$ (and every such $\lambda_A$ is p-regular).*

**Theorem 5.6.** *Fix a $p$-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\mathrm{Res}_\alpha D^\lambda$ is zero unless $\lambda$ has at least one normal node of residue $\alpha$. In that case,*

(i) *$\mathrm{Res}_\alpha D^\lambda$ is an indecomposable, self-dual module, with simple socle and head isomorphic to $D_{\lambda_A}$ where $A$ is the good node of residue $\alpha$;*

(ii) *$\mathrm{Res}_\alpha D^\lambda$ is irreducible if and only if there is a unique normal node of residue $\alpha$;*

(iii) *for any $p$-regular $\mu \vdash r-1$,*

$$\mathrm{Hom}_{\Sigma_{r-1}}(S^\mu, \mathrm{Res}_\alpha D^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda_A \text{ for some normal } A \text{ with } \operatorname{res} A = \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any removable node $A$ such that $\lambda_A$ is $p$-regular,*

$$[\mathrm{Res}_\alpha D^\lambda : D^{\lambda_A}] = \begin{cases} d_A & \text{if } A \text{ is normal for } \lambda \text{ and } \operatorname{res} A = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

*where $d_A$ denotes the number of normal nodes $D$ to the right of $A$ (counting $A$ itself) such that $\operatorname{res} D = \alpha$;*

(v) *the endomorphism ring $\mathrm{End}_{\Sigma_{r-1}}(\mathrm{Res}_\alpha D^\lambda)$ is isomorphic to the truncated polynomial ring $\mathbb{F}[T]/(T^d)$, of dimension $d$, where $d$ is the number of normal nodes $A$ with $\operatorname{res} A = \alpha$.*

In particular, 5.6(ii) gives a criterion for complete reducibility of $D^\lambda \downarrow_{\Sigma_{r-1}}$: the restriction $D^\lambda \downarrow_{\Sigma_{r-1}}$ is completely reducible if and only if all normal nodes have different residues.

**Theorem 5.7.** *Fix a $p$-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let $A_1, A_2 \dots A_d$ be all normal nodes of residue $\alpha$ counted from right to left. Then, $N := \mathrm{Res}_\alpha D^\lambda$ has a filtration $0 = N_0 < N_1 < \cdots < N_d = N$ such that:*

(i) *for $1 \leq i \leq d$, $N_i/N_{i-1}$ is a non-zero quotient of $S^{\lambda_{A_i}}$;*

(ii) *for $1 \leq i \leq j \leq d$ with $\lambda_{A_j}$ is $p$-regular, $[N_i/N_{i-1} : D^{\lambda_{A_j}}] = 1$;*

(iii) *Let $1 < j < d$. The extension*

$$0 \to N_j/N_{j-1} \to N_{j+1}/N_{j-1} \to N_{j+1}/N_j \to 0$$

*does not split.*

(iv) *If all $\lambda_{A_j}$ for $1 \leq j \leq d$ are $p$-regular then the Loewy length of $N$ is at least $2d-1$.*

Now we want to say a little more about the translation techniques to go from the results on tensor products in section 2 to the results stated here about $\mathrm{Ind}^\alpha$ (and analogously, how to deduce the results stated here about $\mathrm{Res}_\alpha$ from branching rules from $GL(n)$ to $GL(n-1)$).

Fix now integers $n, r$ and compositions

$$\nu = (n_1, \dots, n_a) \in \Lambda(a, n) \quad \text{and} \quad \rho = (r_1, \dots, r_a) \in \Lambda(a, r)$$

for some $a$ such that $n \geq r$ and $n_i \geq r_i$ for $i = 1, \dots, a$. We denote by $\Sigma_\rho \cong \Sigma_{r_1} \times \cdots \times \Sigma_{r_a} < \Sigma_r$ the standard Young subgroup of $\Sigma_r$ corresponding to the composition $\rho$. Let $GL(\nu)$ be the corresponding Levi subgroup of $GL(n)$, and $\Sigma_\rho$

be the corresponding Young subgroup $\Sigma_{r_1} \times \cdots \times \Sigma_{r_a}$ of $\Sigma_r$. We first consider the effect of Schur functors (which amount to taking certain weight spaces) on restrictions from $GL(n)$ to $GL(\nu)$.

Let $\omega$ be the weight

$$\omega = \Big(\underbrace{\overbrace{1,\dots,1}^{r_1 \text{ entries}},0,\dots,0}_{n_1 \text{ entries}},\underbrace{\overbrace{1,\dots,1}^{r_2 \text{ entries}},0,\dots,0}_{n_2 \text{ entries}},\dots,\underbrace{\overbrace{1,\dots,1}^{r_a \text{ entries}},0,\dots,0}_{n_a \text{ entries}}\Big).$$

Let $\mathrm{Perm}_r$ denote the subgroup of $GL(n)$ consisting of all permutation matrices $g$ with $g_{i,i} = 1$ whenever $\omega_i = 0$. Obviously, $\mathrm{Perm}_r$ is isomorphic to $\Sigma_r$. We *always* identify $\mathrm{Perm}_r$ with the symmetric group $\Sigma_r$ in the obvious way.

Given any $M \in M_{\mathbb{F}}(n,r)$, the $\omega$-weight space $M_\omega$ of $M$ is stable under the action of $\mathrm{Perm}_r$. We now define the *Schur functor*

$$F_{n,r} : M_{\mathbb{F}}(n,r) \to \mathrm{mod}\,\mathbb{F}\Sigma_r$$

on objects by letting $F_{n,r}M := M_\omega$, and by restriction on morphisms.

The next lemma describe a number of useful properties of $F_{n,r}$. Parts (i)-(iii) are proved in [33, §6], and the rest of 5.8 is proved in (or follows easily from) [21, (3.5),(3.6)].

**Lemma 5.8.** *Fix $\lambda \in \Lambda^+(n,r)$ and $\mu \in \Lambda(n,r)$.*

(i) *$F_{n,r}\nabla_n(\lambda) \cong S^\lambda$;*

(ii) *$F_{n,r}\Delta_n(\lambda) \cong (S^\lambda)^* \cong S^{\lambda^t} \otimes \mathrm{sgn}$;*

(iii) *$F_{n,r}L_n(\lambda)$ is zero unless $\lambda$ is $p$-restricted, in which case $F_{n,r}L_n(\lambda) \cong D^{\lambda^t} \otimes \mathrm{sgn}$.*

(iv) *$F_{n,r}(\bigwedge^{\mu_1}(V) \otimes \cdots \otimes \bigwedge^{\mu_n}(V)) \cong M^\mu \otimes \mathrm{sgn}$;*

(v) *$F_{n,r}(S^{\mu_1}(V) \otimes \cdots \otimes S^{\mu_n}(V)) \cong M^\mu$;*

(vi) *$F_{n,r}T_n(\lambda) \cong Y^{\lambda^t} \otimes \mathrm{sgn}$.*

(vii) *$F_{n,r}P_n(\lambda) \cong F_{n,r}Q_n(\lambda) \cong Y^\lambda$, where $P_n(\lambda)$ (resp. $Q_n(\lambda)$) is a projective cover (resp. injective hull) of $L_n(\lambda)$ in $M_{\mathbb{F}}(n,r)$.*

Let $\mathrm{Perm}_\rho := \mathrm{Perm}_r \cap GL(\nu)$. Then $\mathrm{Perm}_\rho$ is isomorphic to the Young subgroup $\Sigma_\rho < \Sigma_r$. We can now define a more general Schur functor

$$F_{\nu,\rho} : M_{\mathbb{F}}(\nu,r) \to \mathrm{mod}\,\mathbb{F}\mathrm{Perm}_\rho$$

given on $M \in M_{\mathbb{F}}(\nu,r)$ by letting $F_{\nu,\rho}M$ denote the weight space $M_\omega$, noting that this is stable under the action of $\mathrm{Perm}_\rho < GL(\nu)$. The next lemma is well known. Part (i) follows from the definitions, see [11, 4.8], and part (ii) can be found in [23, Lemma 2.5], [11, 4.13].

**Lemma 5.9.** (i) *Given a module $M \in M_{\mathbb{F}}(n,r)$,*

$$F_{\nu,\rho}(R^n_\nu M) \cong (F_{n,r}M)\downarrow^{\mathbb{F}\mathrm{Perm}_r}_{\mathbb{F}\mathrm{Perm}_\rho}.$$

(ii) *Given modules $M_i \in M_{\mathbb{F}}(n,r_i)$ for $i = 1,\dots,a$,*

$$F_{n,r}(M_1 \otimes \cdots \otimes M_a) \cong (F_{n,r_1}M_1 \boxtimes \cdots \boxtimes F_{n,r_a}M_a)\uparrow^{\mathbb{F}\Sigma_r}_{\mathbb{F}\Sigma_\rho}.$$

Using these two lemmas, it is quite straightforward to reformulate 4.5 as the following result about the symmetric group (see [11, section 4]):

**Theorem 5.10.** *Fix partitions $\lambda \vdash r$ and $\mu^{(i)} \vdash r_i$ for $i = 1, \dots, a$. Regard $\lambda^t$ as an element of $\Lambda^+(n, r)$ and each $(\mu^{(i)})^t$ as elements of $\Lambda^+(n_i, r_i)$ or $\Lambda^+(n, r_i)$. Then,*

(i) $\big(Y^\lambda{\downarrow_{\Sigma_\rho}}\colon Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}}\big) = \big[L_n(\mu^{(1)}) \otimes \cdots \otimes L_n(\mu^{(a)}) : L_n(\lambda)\big]$;

(ii) $\big((Y^{\mu^{(1)}} \boxtimes \cdots \boxtimes Y^{\mu^{(a)}}){\uparrow^{\Sigma_r}}\colon Y^\lambda\big) = \big[L_n(\lambda){\downarrow_{GL(\nu)}}\colon L_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes L_{n_a}(\mu^{(a)})\big]$.

*Moreover, if all the partitions are $p$-regular, then*

(iii) $\big[D^\lambda{\downarrow_{\Sigma_\rho}}\colon D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}\big] = \big(T_n(\mu^{(1)}) \otimes \cdots \otimes T_n(\mu^{(a)}) : T_n(\lambda)\big)$;

(iv) $\big[(D^{\mu^{(1)}} \boxtimes \cdots \boxtimes D^{\mu^{(a)}}){\uparrow^{\Sigma_r}}\colon D^\lambda\big] = \big(T_n(\lambda){\downarrow_{GL(\nu)}}\colon T_{n_1}(\mu^{(1)}) \boxtimes \cdots \boxtimes T_{n_a}(\mu^{(a)})\big)$.

The deduction of the results on $\mathrm{Ind}^\alpha$ from the results in section 2 is also quite straightforward. Parts (iii) and (iv) of 5.4 and 5.7 are new so we sketch their proof here in more detail. The proof for $\mathrm{Res}^\alpha$ is similar to that for $\mathrm{Ind}^\alpha$ and we concentrate on the latter one.

So fix a $p$-regular partition $\lambda$ of $r$, an integer $n > r$, and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Set

$$I := \mathrm{Ind}^\alpha D^\lambda \qquad \text{and} \qquad N := F_{-\alpha} L_n(\lambda^t).$$

It follows from 5.8(iii) and 5.9(ii) that $I \otimes \mathrm{sgn} = F_{n,r} N$.

Let $B_1, B_2, \dots, B_d$ be the conormal nodes for $\lambda$ of residue $\alpha$ counted from left to right. Since the definitions of conormal for $GL(n)$ and $\Sigma_r$ are 'transpose to each other', there are exactly $d$ conormal $i$ with $\mathrm{res}(i, \lambda_i^t + 1) = -\alpha$. Let

$$\{s_1 < \cdots < s_d\} = \{s \mid 1 \le s \le n,\ s \text{ is conormal for } \lambda^t \text{ and } \mathrm{res}(s, \lambda_s + 1) = -\alpha\},$$

and let

$$(0) = N_0 < N_1 < \cdots < N_d = N$$

be the filtration from 2.7. We know from 2.7, 2.3(v) and the proof of Theorem 8.14 from [12] that

(a) $N_j/N_{j-1}$ is a non-zero quotient of $\Delta_n(\lambda^t + \epsilon_{s_j})$ $(1 \le j \le d)$;

(b) $[N_j/N_{j-1} : L_n(\lambda^t + \epsilon_k)] = 1$ $(1 \le j \le k \le d)$;

(c) there exists $\psi \in \mathrm{End}_{GL(n)}(N)$ such that $\{\mathrm{id}_N, \psi, \dots, \psi^{d-1}\}$ is a basis of $\mathrm{End}_{GL(n)}(N)$, $\psi(N_j) \subseteq N_{j-1}$, and the induced homomorphism

$$\overline{\psi^{k-j}} : N_j/N_{j-1} \to N_k/N_{k-1}$$

is a non-zero homomorphism whose image is properly contained in $N_k/N_{k-1}$ $(1 < k < j \le d)$.

Let $I_j := F_{n,r+1}(N_j) \otimes \mathrm{sgn},\ j = 0, 1, \dots, d$. Then

$$(0) = I_0 < I_1 < \cdots < I_d = I \tag{4}$$

is a filtration of $I$. It follows from 5.8(ii),(iii) and (a),(b) above that $I_j/I_{j-1}$ is a non-zero quotient of $S^{\lambda^{B_j}}$, $1 \le j \le d$, and $[I_j/I_{j-1} : D^{\lambda^{B_k}}] = 1$ if $1 \le j \le k \le d$ and $\lambda^{B_k}$ is $p$-regular. As $B_d$ is good, $\lambda^{B_d}$ is always $p$-regular. Note that $D^{\lambda^{B_d}}$

appears in every $I_j/I_{j-1}$ so the filtration (4) is strict. Let $\hat{\chi} = F_{n,r}(\psi)$. Then $\hat{\chi} = \chi \otimes \mathrm{id}_{\mathrm{sgn}}$ for some $\chi \in \mathrm{End}_{\Sigma_{r+1}}(I)$. It is proved in [48, Theorem B] and [12, section 3] that

$$\{\mathrm{id}_I, \chi, \ldots, \chi^{d-1}\}$$

is a basis of $\mathrm{End}_{\Sigma_{r+1}}(I)$. Moreover it follows from the corresponding properties of $\psi$ that $\chi(I_j) \subseteq I_{j-1}$ and that the induced homomorphism $\bar{\chi} : I_{j+1}/I_j \to I_j/I_{j-1}$ is non-zero provided $\lambda^{B_{j+1}}$ is $p$-regular. If, additionally, $\lambda^{B_j}$ is $p$-regular then the image of $\bar{\chi}$ is properly contained in $I_j/I_{j-1}$.

**Claim 1.** Let $1 < k < j \leq d$. Then the induced map $\overline{\chi^{j-k}} : I_j/I_{j-1} \to I_k/I_{k-1}$ is non-zero.

*Proof.* Since the image of $\overline{\chi^{j-k}}$ contains that of the map $\overline{\chi^{d-k}} : I_d/I_{d-1} \to I_k/I_{k-1}$, it is enough to prove the result for $j = d$. To prove that the latter map is non-zero, consider the corresponding map for $GL(n)$: $\overline{\psi^{d-k}} : N_d/N_{d-1} \to N_k/N_{k-1}$. By (c), this map is non-zero, and the head of $N_d/N_{d-1}$ is $p$-restricted. It follows that $F_{n,r}(\overline{\psi^{d-k}}) : I_d/I_{d-1} \otimes \mathrm{sgn} \to I_k/I_{k-1} \otimes \mathrm{sgn}$ is also non-zero, whence $\overline{\chi^{d-k}}$ is non-zero.

**Claim 2.** Let $1 < j < d$. The extension

$$0 \to I_j/I_{j-1} \to I_{j+1}/I_{j-1} \to I_{j+1}/I_j \to 0$$

does not split.

*Proof.* $\chi$ induces the map $\bar{\chi} : I_{j+1}/I_{j-1} \to I_{j+1}/I_{j-1}$. We know that this map is non-zero by Claim 1. On the other hand, $\bar{\chi}$ annihilates $I_j/I_{j-1}$ and $I_{j+1}/I_j$, and the claim follows.

**Claim 3.** Let $1 \leq j \leq d$. Assume that $\lambda^{B_k}$ is $p$-regular for all $k = j, j+1, \ldots, d$. Then the Loewy length of $I_j$ is at least $b + 1 - j$.

*Proof.* Put $L = I_j/I_{j-1}$, and for $k = j, j+1, \ldots, d$ pick $w_k$ to be any vector in $I_k$ such that $w_k + I_{k-1}$ is in the head of $I_k/I_{k-1}$. Note that this head is simple by assumption. Hence $w_k + I_{k-1}$ generates $I_k/I_{k-1}$ as an $F\Sigma_{r+1}$-module. Set

$$v_k := \chi^{k-j} w_k + I_{j-1} \in L \qquad (k = j, j+1, \ldots, d).$$

By Claim 1, $v_j, v_{j+1}, \ldots, v_b$ are non-zero vectors of $L$. Denote $L_k = F\Sigma_{r+1} v_k$. Then

$$(0) \gneqq L_d \gneqq L_{d-1} \gneqq \cdots \gneqq L_j = L,$$

the filtration being strict since $D^{B_k}$ is a composition factor of $L_k$ but not of $L_{k-1}$. Let $(0) = S_0 < S_1 < \ldots$ denote the socle series of $L$. We show that $v_k \notin S_{d-k}$ by downward induction on $k = d, d-1, \ldots, j$. In particular this will show that $v_j \notin S_{d-j}$ hence the length of the socle series is at least $d - j + 1$. The base of induction is clear as $v_d$ is non-zero. So let us take $k < d$, and assume that $v_{k+1} \notin S_{d-(k+1)}$. Suppose for a contradiction that $v_k \in S_{d-k}$. Let $\bar{v}_k = v_k + S_{d-(k+1)} \in S_{d-k}/S_{d-(k+1)}$. Since $v_{k+1} \in F\Sigma_{r+1} v_k$ we also have $v_{k+1} \in S_{d-k}$, and we get two non-zero vectors $\bar{v}_k$ and $\bar{v}_{k+1} := v_{k+1} + S_{d-(k+1)}$. We note that the submodule $F\Sigma_{r+1}\bar{v}_k \subseteq S_{d-k}/S_{d-(k+1)}$ is a quotient of $L_k$, so it has a simple head. But $S_{d-k}/S_{d-(k+1)}$ is semisimple, so this submodule must be simple and equal to $D^{\lambda^{B_k}}$. But $\bar{v}_{k+1}$ is non-zero vector of this submodule, and so it generates $D^{\lambda^{B_{k+1}}}$ by the same reason. Since $D^{\lambda^{B_k}} \not\cong D^{\lambda^{B_{k+1}}}$, we get a contradiction.

**Claim 4.** Assume that $\lambda^{B_j}$ is $p$-regular for all $j = 1, 2, \dots, d$. Then the Loewy length of $I$ is at least $2d - 1$.

*Proof.* The proof is similar to that of Corollary 8.15 of [12] but uses Claim 3 instead of Theorem 8.14 of that paper.

# 6. Group algebra of the finitary symmetric group

A permutation of the set $\mathbb{N} = \{1, 2, 3, \dots\}$ is called *finitary* if it fixes all but finitely many elements. The finitary symmetric group $\Sigma_\infty$ is the group of all finitary permutations of $\mathbb{N}$. It can be represented as a union $\Sigma_\infty = \bigcup_{n\geq 1} \Sigma_n$.

A. Zalesskii has shown (see [67] and the references there) that some ring theoretic questions on the group algebras of locally finite groups are closely related to the asymptotic behavior of the branching rules for finite groups. This fundamental observation accounts for the recent noticeable progress in the theory of group algebras of locally finite groups.

We illustrate the ideas for the case of the finitary symmetric group. Given a (two-sided) ideal $I$ in the group algebra $\mathbb{F}\Sigma_\infty$ we can form a family of ideals

$$I_n := I \cap \mathbb{F}\Sigma_n < \mathbb{F}\Sigma_n,\ n = 1, 2, \dots.$$

This family has the property

$$I_n \cap \mathbb{F}\Sigma_m = I_m, \quad \text{for any } 1 \leq m \leq n, \tag{5}$$

and $I$ can be reconstructed from it as $I = \bigcup_{n\geq 1} I_n$. On the other hand, given a family of ideals $I_n < \mathbb{F}\Sigma_n$ satisfying the property (5) we may form a union $I = \bigcup_{n\geq 1} I_n$, and then $I \cap \mathbb{F}\Sigma_n = I_n$ for any $n$. Thus the main problem is how to 'glue' a big ideal from small ones or how to produce and classify the families of ideals satisfying (5).

The right formalism for doing this comes from a sort of 'asymptotic' representation theory of symmetric groups.

**Definition 6.1. (A. Zalesskii).** Let $\Phi_n$ be a set of the isomorphism classes of irreducible $\mathbb{F}\Sigma_n$-modules, $n = 1, 2, \dots$. The collection $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is called an *inductive system* (for $\Sigma_\infty$) if for any $m, n \in \mathbb{N}$ with $m < n$ the following two properties hold:

(1) For any $D \in \Phi_n$, all composition factors of the restriction $D\downarrow_{\Sigma_m}$ belong to $\Phi_m$;

(2) For any $E \in \Phi_m$, there exists $D \in \Phi_n$ such that $E$ is a composition factor of $D\downarrow_{\Sigma_m}$.

**Theorem 6.2.** [67, 8.1,1.25] (see also [5, 2.6]). *There is an order reversing bijection of partially ordered sets between the inductive systems and the semi-primitive ideals of* $\mathbb{F}\Sigma_\infty$.

The ideals of $\mathbb{F}\Sigma_\infty$ are closely related with the theory of PI-rings (see [1, 68, 59, 58, 28, 27]). In [43], A. Kemer has shown that determining the prime ideals in $\mathbb{F}\Sigma_\infty$ would be a crucial step in a classification of the prime varieties of associative algebras.

The ideal structure of $\mathbb{F}\Sigma_\infty$ is very rich. In case where $\operatorname{char}\mathbb{F} = 0$ it was described in [58, 27]. Using Theorem 6.2 and the classical branching rule one can easily rederive such a description. On the other hand, the case of positive characteristic seems to be very difficult since a complete modular branching rule is not known. Recently Baranov and the second author [5] have described the *maximal* ideals in $\mathbb{F}\Sigma_\infty$ and $\mathbb{F}A_\infty$ (where $A_\infty$ is the finitary alternating group), provided $p > 2$ (see also [7, 50] for some relevant results).

The following problem has been raised in [68].

**Question (A. Zalesskii).** Do ideals of $\mathbb{F}\Sigma_\infty$ satisfy the ascending chains condition?

A little more special (but important) question is whether the semiprimitive ideals of $\mathbb{F}\Sigma_\infty$ satisfy A.C.C. By 6.2 this is equivalent to the D.C.C. for inductive systems.

It was observed in [6], that the ideal structure of $\mathbb{F}\Sigma_\infty$ is closely related with the ideal structure of some commutative algebra, which appeared in the papers of Georgiev and Mathieu [29, 30], and Andersen and Paradowski [3]. This observation is based on 5.10(iii) so we mention this as one of the applications of the results described in section 4.

First we note that if $\mu, \nu \in \Lambda^+(m)$ are $p$-regular and $(T_m(\mu) \otimes T_m(\nu) : T_m(\lambda)) \neq 0$ then $\lambda$ is also $p$-regular. This fact is a folklore (see [6] for one of the proofs). This allows one to define a commutative $\mathbb{Z}$-algebra $Q_m$ as follows. The elements of $Q_m$ are the formal finite $\mathbb{Z}$-linear combinations $\sum a_\lambda T_m(\lambda)$ of indecomposable tilting modules corresponding to the *p-regular* $\lambda$, and the multiplication is induced by tensor products. More precisely, we put

$$T_m(\mu)T_m(\nu) = \sum_\lambda a^\lambda_{\mu\nu} T_m(\lambda)$$

if

$$T_m(\mu) \otimes T_m(\nu) \cong \bigoplus_\lambda a^\lambda_{\mu\nu} T_m(\lambda).$$

An ideal $I$ of $Q_m$ is called a *tensor ideal* if $\sum a_\lambda T_m(\lambda) \in I$ and $a_\mu \neq 0$ imply $T_m(\lambda) \in I$. A tensor ideal is called *special* if $T_m(\lambda) \otimes V_m \in I$ implies $T_m(\lambda) \in I$.

Now we define an important invariant of inductive systems.

**Definition 6.3.** If $\lambda = (\lambda_1 \geq \cdots \geq \lambda_h > 0)$ we call $h$ the *height* of $\lambda$ and denote it by $h(\lambda)$. If $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ is an inductive system we define its height $h(\Phi)$ by setting

$$h(\Phi) = \sup\{h(\lambda) \mid D^\lambda \in \Phi_n \text{ for some } n \in \mathbb{N}\}.$$

The following result shows that all interesting inductive systems have finite height.

**Lemma 6.4.** [6, 5.2] *Let $\Phi = \{\Phi_n\}_{n\in\mathbb{N}}$ be an inductive system for $\Sigma_\infty$. Assume that $h(\Phi) = +\infty$. Then $\Phi_n = \mathrm{Irr}\ \Sigma_n$ for all $n \in \mathbb{N}$, where* Irr $\Sigma_n$ *denotes the set of all isomorphism classes of irreducible $F\Sigma_n$-modules.*

Now we are able to state the main result.

**Theorem 6.5.** [6] *There is an order reversing bijection between the special tensor ideals of the algebra $Q_m$ and the inductive systems for $\Sigma_\infty$ of height $m$.*

Motivated by the question of Zalesskii above, the following is raised in [6]:

**Question.** Is $Q_m$ noetherian?

We note though that Zalesskii's question is more subtle as, according to 6.5, it has to do with *tensor* ideals, which are the ideals 'respecting' the basis of tilting module

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