ON TRANSLATION FUNCTORS FOR GENERAL LINEAR AND SYMMETRIC GROUPS*

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1 Introduction

Let $\mathbb{F}$ be an algebraically closed field of characteristic p. A central problem in the representation theory of the general linear group $GL(n) = GL_n(\mathbb{F})$ is to understand the structure of the tensor space $V^{\otimes r}$, where V is the natural $GL(n)$-module. To do this (inductively) we would like information about the structure of tensor products of the form $M \otimes V$, where M is an irreducible (or Weyl or tilting) module.

Given $\alpha \in \mathbb{Z}/p\mathbb{Z}$, we will define functors Tr^α and Tr_α, which roughly speaking are given by tensoring with the natural $GL(n)$-module V and its dual V^* respectively, then projecting onto certain blocks determined by the residue α. In particular, for any rational $GL(n)$-module M,

$$M \otimes V \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Tr}^\alpha M \quad \text{and} \quad M \otimes V^* \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Tr}_\alpha M.$$

In fact, these functors can be viewed as special cases of Jantzen's translation functors. Our main results prove the following facts about Tr^α and Tr_α:

(1) Fix an irreducible rational $GL(n)$-module L. We give a precise combinatorial criterion for $N := \mathrm{Tr}^\alpha L$ (resp. $\mathrm{Tr}_\alpha L$) to be irreducible; whenever this occurs, the inverse

*1991 subject classification: 20G05, 20C05.

†Authors partially supported by the NSF (grant nos DMS-9801442 and DMS-9600124).

decomposition numbers for the irreducible module N can be computed from knowledge of the ones for L.

(2) Even when N is not irreducible, we show that it is always a (contravariantly) self-dual indecomposable module, and describe its (simple) socle and head, as well as the space of high weight vectors in N, precisely.

(3) We construct a natural filtration of N by precisely b high weight modules, for some combinatorially defined constant b (depending on L and α), compute certain composition multiplicities in N and describe the endomorphism ring of N using Casimir-type operators.

(4) In particular, we show that the dimension of the endomorphism ring of N is precisely b, and that the Loewy length of N is at least $2b - 1$.

Special cases of these results, with $b \leq 2$, follow from [**12**, II.7]. In general, the natural number b can take *any* value. We also obtain various related results describing the action of Tr^α and Tr_α on Weyl modules, and obtain combinatorial criteria for $L \otimes V$ and $L \otimes V^*$ to be completely reducible.

We also consider applications to the representation theory of the symmetric group Σ_n over $\mathbb{F}$. The functor Tr^α corresponds under the Schur functor to a certain 'α-induction' functor $\mathrm{Ind}^\alpha : \mathbb{F}\Sigma_n\text{-mod} \to \mathbb{F}\Sigma_{n+1}\text{-mod}$. Roughly speaking, the functor Ind^α can be defined as ordinary induction from Σ_n to Σ_{n+1} followed by projection onto certain blocks. This goes back to G. Robinson [**21**]. As consequences of the above results (1)–(4) for Tr^α, we will obtain analogous results for the functor Ind^α, which are the counterparts of the results proved in [**14**]–[**18**], [**1**], [**2**] for the 'α-restriction' functor Res_α.

In section 2 and section 3, we state our results precisely, for general linear and symmetric groups respectively. In the remaining sections 4–8, we prove the results, usually by relating the problems under consideration to certain branching problems studied in [**14**]–[**18**].

2 Statement of results for general linear groups

To describe our main results for $GL(n)$, we need a little notation. Let $X(n)$ denote the set of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ and $X^+(n) \subset X(n)$ denote all $\lambda \in X(n)$ satisfying $\lambda_1 \geq \cdots \geq \lambda_n$. As we explain in section 4, $X^+(n)$ can be identified with the dominant weights for the root system of $GL(n)$. So for $\lambda \in X^+(n)$, we have the (rational) $GL(n)$-modules $L_n(\lambda), \Delta_n(\lambda)$ and $\nabla_n(\lambda)$, which are the irreducible, standard (or Weyl) and costandard modules of highest weight λ, respectively. For $1 \leq i \leq n$, let ε_i denote the n-tuple $(0, \dots, 0, 1, 0, \dots, 0)$ where 1 is in the ith position. We also denote the natural $GL(n)$-module by V_n and its dual by V_n^*.

Given $(a, b) \in \mathbb{Z} \times \mathbb{Z}$, define the corresponding *p-residue* $\mathrm{res}(a, b)$ to be $(b - a)$ regarded as an element of the ring $\mathbb{Z}/p\mathbb{Z}$. For $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and $\lambda \in X(n)$, define the *α-content* of λ to be the integer:

$$\mathrm{cont}_\alpha(\lambda) := \left| \left\{ (a, b) \;\middle|\; \begin{array}{c} 1 \leq a \leq n, 0 < b \leq \lambda_a, \\ \mathrm{res}(a, b) = \alpha \end{array} \right\} \right| - \left| \left\{ (a, b) \;\middle|\; \begin{array}{c} 1 \leq a \leq n, \lambda_a \leq b < 0, \\ \mathrm{res}(a, b) = \alpha \end{array} \right\} \right|.$$

Say $\lambda, \mu \in X(n)$ are *linked*, written $\lambda \sim \mu$, if $\mathrm{cont}_\alpha(\lambda) = \mathrm{cont}_\alpha(\mu)$ for all $\alpha \in \mathbb{Z}/p\mathbb{Z}$. The *linkage principle* proved in [**5**] implies that if $\mathrm{Ext}^1_{GL(n)}(L_n(\lambda), L_n(\mu)) \neq 0$, for $\lambda, \mu \in X^+(n)$, then $\lambda \sim \mu$. We remark that in fact the blocks of $GL(n)$ (which can in general be smaller than the linkage classes) are explicitly known, see [**12**, II.7.2(3)].

Let $\mathcal{C}_n$ denote the category of all rational $GL(n)$-modules. For any $\lambda \in X(n)$, let $\mathcal{C}_n(\lambda)$ denote the full subcategory of $\mathcal{C}_n$ consisting of all $M \in \mathcal{C}_n$ such that all composition factors of M are of the form $L_n(\mu)$ for $\mu \sim \lambda$. By the linkage principle,

$$\mathcal{C}_n \cong \bigoplus_{\lambda} \mathcal{C}_n(\lambda)$$

where λ runs over the set of $\sim$-equivalence classes in $X(n)$.

Fix a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. We can now define the functors

$$\mathrm{Tr}^{\alpha} : \mathcal{C}_n \to \mathcal{C}_n \qquad \text{and} \qquad \mathrm{Tr}_{\alpha} : \mathcal{C}_n \to \mathcal{C}_n.$$

It suffices to define the restriction of Tr^{α} (resp. Tr_{α}) to $\mathcal{C}_n(\lambda)$, for any $\lambda \in X(n)$ (then we may extend the functors additively to all of $\mathcal{C}_n$). Given $M \in \mathcal{C}_n(\lambda)$, we let $\mathrm{Tr}^{\alpha} M$ (resp. $\mathrm{Tr}_{\alpha} M$) denote the largest submodule of $M \otimes V_n$ (resp. $M \otimes V_n^*$) all of whose composition factors are of the form $L_n(\mu)$ with

$$\mathrm{cont}_{\alpha}(\mu) = \mathrm{cont}_{\alpha}(\lambda) + 1 \qquad (\text{resp. } \mathrm{cont}_{\alpha}(\mu) = \mathrm{cont}_{\alpha}(\lambda) - 1),$$

and $\mathrm{cont}_{\beta}(\mu) = \mathrm{cont}_{\beta}(\lambda)$ for all $\alpha \neq \beta \in \mathbb{Z}/p\mathbb{Z}$. By the linkage principle, $\mathrm{Tr}^{\alpha} M$ (resp. $\mathrm{Tr}_{\alpha} M$) is a direct summand of $M \otimes V_n$ (resp. $M \otimes V_n^*$). Given a morphism $\theta : M \to N$, $\mathrm{Tr}^{\alpha} \theta$ is just the restriction to $\mathrm{Tr}^{\alpha} M$ of the natural map $\theta \otimes 1 : M \otimes V_n \to N \otimes V_n$, and similarly for Tr_{α}.

On any fixed block $\mathcal{C}_n(\lambda)$, the functor Tr^{α} (resp. Tr_{α}), for a suitable choice of α, coincides with the translation functor T_{λ}^{μ} defined in [**12**, II.7.6], for a weight $\mu \in X(n)$ such that the dominant conjugate of $(\mu - \lambda)$ is equal to the highest weight of V_n (resp. V_n^*). We note initially that the argument of [**12**, II.7.6] shows easily that the functors Tr^{α} and Tr_{α} are (left and right) adjoint to one another, and both are exact.

In the next combinatorial definitions, the notions of normal and good first appeared in [**15**]; the dual notions of conormal and cogood are new. The reader may be more familiar with normal and good *nodes*; we reserve this terminology for the symmetric group setting when definitions are 'transposed', see section 3. In the definitions, we call a map ψ from a set $M \subseteq \mathbb{Z}$ to a set $N \subseteq \mathbb{Z}$ *increasing* (resp. *decreasing*) if $\psi(m) > m$ (resp. $\psi(m) < m$) for all $m \in M$.

Fix $\lambda \in X^+(n)$ and $1 \leq i \leq n$. We say i is λ-*removable* if *either* $i = n$ *or* $1 \leq i < n$ and $\lambda_i > \lambda_{i+1}$; equivalently, i is λ-removable if $\lambda - \varepsilon_i \in X^+(n)$. We say i is λ-*addable* if *either* $i = 1$ *or* $1 < i \leq n$ and $\lambda_i < \lambda_{i-1}$; equivalently, i is λ-addable if $\lambda + \varepsilon_i \in X^+(n)$.

Say i is *normal for* λ if i is λ-removable and there is a decreasing injection from the set of

$$\lambda\text{-addable } j \text{ with } i < j \leq n \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(j, \lambda_j + 1)$$

into the set of

$$\lambda\text{-removable } j' \text{ with } i < j' \leq n \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(j', \lambda_{j'}).$$

Say i is *good for* λ if i is normal for λ and there is no j that is normal for λ with $1 \leq j < i$ and $\mathrm{res}(j, \lambda_j) = \mathrm{res}(i, \lambda_i)$.

Say i is *conormal for* λ if i is λ-addable and there is an increasing injection from the set of

$$\lambda\text{-removable } j \text{ with } 1 \leq j < i \text{ and } \mathrm{res}(j, \lambda_j) = \mathrm{res}(i, \lambda_i + 1)$$

into the set of

$$\lambda\text{-addable } j' \text{ with } 1 \leq j' < i \text{ and } \mathrm{res}(j', \lambda_{j'} + 1) = \mathrm{res}(i, \lambda_i + 1).$$

Say i is *cogood for* λ if i is conormal for λ and there is no j that is conormal for λ with $i < j \leq n$ and $\mathrm{res}(j, \lambda_j + 1) = \mathrm{res}(i, \lambda_i + 1)$.

We refer the reader to the combinatorial Lemma 5.8 for an explanation of the duality between these definitions. For a fixed $\alpha \in \mathbb{Z}/p\mathbb{Z}$, there is *at most one* i that is good for λ with $\mathrm{res}(i, \lambda_i) = \alpha$. Moreover, such an i exists if and only if there is at least one j that is normal for λ with $\mathrm{res}(j, \lambda_j) = \alpha$. A similar statement is true for conormal and cogood.

Our first result describes the effect of Tr^α on standard modules (the analogous result for costandard modules follows easily since Tr^α commutes with contravariant duality):

THEOREM A. *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\mathrm{Tr}^\alpha \Delta_n(\lambda)$ is zero unless there is at least one λ-addable i with $1 \leq i \leq n$ and $\mathrm{res}(i, \lambda_i + 1) = \alpha$. In that case,*

(i) *$\mathrm{Tr}^\alpha \Delta_n(\lambda)$ has a filtration with factors $\Delta_n(\lambda + \varepsilon_j)$ for all λ-addable j with $1 \leq j \leq n$ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$, each appearing with multiplicity one;*

(ii) *the head of $\mathrm{Tr}^\alpha \Delta_n(\lambda)$ is $\bigoplus_j L_n(\lambda + \varepsilon_j)$ where the sum is over all j with $1 \leq j \leq n$ such that j is normal for $\lambda + \varepsilon_j$ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$;*

Next we consider Tr^α applied to an irreducible module. Here and later $[M : L]$ denotes the multiplicity of an irreducible module L as a composition factor of a module M.

THEOREM B. *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\mathrm{Tr}^\alpha L_n(\lambda)$ is zero unless there is a (necessarily unique) i that is cogood for λ with $\mathrm{res}(i, \lambda_i + 1) = \alpha$. In that case,*

(i) *$\mathrm{Tr}^\alpha L_n(\lambda)$ is an indecomposable, contravariantly self-dual module, with simple socle and head isomorphic to $L_n(\lambda + \varepsilon_i)$;*

(ii) *$\mathrm{Tr}^\alpha L_n(\lambda)$ is irreducible if and only if there is a unique j with $1 \leq j \leq n$ such that j is conormal for λ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$; in particular, $L_n(\lambda) \otimes V_n$ is completely reducible if and only if for every $\alpha \in \mathbb{Z}/p\mathbb{Z}$, there is at most one j such that j is conormal for λ and $\mathrm{res}(j, \lambda_j + 1) = \alpha$;*

(iii) *for any $\mu \in X^+(n)$,*

$$\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), \mathrm{Tr}^\alpha L_n(\lambda)) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda + \varepsilon_j \text{ for some } j \text{ that is conormal for } \lambda, \\ & \text{with } \mathrm{res}(j, \lambda_j + 1) = \alpha; \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any λ-addable j with $1 \leq j \leq n$,*

$$[\mathrm{Tr}^\alpha L_n(\lambda) : L_n(\lambda + \varepsilon_j)] = \begin{cases} b_j & \textit{if } j \textit{ is conormal for } \lambda \textit{ and } \mathrm{res}(j, \lambda_j + 1) = \alpha, \\ 0 & \textit{otherwise} \end{cases}$$

where b_j denotes the number of k with $1 \leq k \leq j$ such that k is conormal for λ and $\mathrm{res}(k, \lambda_k + 1) = \alpha$*;*

(v) *the endomorphism ring* $\mathrm{End}_{GL(n)}(\mathrm{Tr}^\alpha L_n(\lambda))$ *is isomorphic to the truncated polynomial ring $\mathbb{F}[T]/(T^b)$, of dimension b, where b is the number of j with $1 \leq j \leq n$ such that j is conormal for λ and* $\mathrm{res}(j, \lambda_j + 1) = \alpha$*.*

If, in the language of [**12**, II.6.2], $\lambda + \varepsilon_i$ lies in the upper closure of the facet containing λ, where i is as in Theorem B, then one can easily deduce that $\mathrm{Tr}^\alpha L_n(\lambda)$ is irreducible directly from [**12**, II.7.15]. Theorem B(ii) gives a *necessary and sufficient* condition for $\mathrm{Tr}^\alpha L_n(\lambda)$ to be irreducible, and shows that there are many other more general circumstances when $\mathrm{Tr}^\alpha L_n(\lambda)$ is irreducible. The significance of this is the following corollary, which follows immediately from Theorem B(i) and (ii), by exactness of Tr^α:

Corollary 1. *Fix $\lambda \in X^+(n)$ and $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Suppose that there is a* unique *i with $1 \leq i \leq n$ such that i is conormal for λ and* $\mathrm{res}(i, \lambda_i + 1) = \alpha$*. If*

$$\mathrm{ch}\, L_n(\lambda) = \sum_{\mu \in X^+(n)} c_{\lambda,\mu}\, \mathrm{ch}\, \Delta_n(\mu)$$

with almost all coefficients $c_{\lambda,\mu}$ equal to zero, then

$$\mathrm{ch}\, L_n(\lambda + \varepsilon_i) = \sum_{\mu \in X^+(n)} c_{\lambda,\mu}\, \mathrm{ch}\, \mathrm{Tr}^\alpha\, \Delta_n(\mu),$$

which is known by Theorem A(i).

Our next result gives further information about the structure of $\mathrm{Tr}^\alpha L_n(\lambda)$:

Theorem C. *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let $1 = s_1 < s_2 < \cdots < s_b$ denote the set of all j with $1 \leq j \leq n$ such that j is conormal for λ and* $\mathrm{res}(j, \lambda_j + 1) = \alpha$. *Then, $N := \mathrm{Tr}^\alpha L_n(\lambda)$ has a filtration $0 = N_0 < N_1 < \cdots < N_b = N$ such that:*

(i) *for $1 \leq i \leq b$, N_i/N_{i-1} is a non-zero quotient of $\Delta_n(\lambda + \varepsilon_{s_i})$;*

(ii) *for $1 \leq i \leq j \leq b$,* $\dim \mathrm{Hom}_{GL(n)}(N_j/N_{j-1}, N_i/N_{i-1}) = [N_i/N_{i-1} : L_n(\lambda + \varepsilon_{s_j})] = 1$;

(iii) *for $1 \leq i < b$, the extension $0 \to N_i/N_{i-1} \to N_{i+1}/N_{i-1} \to N_{i+1}/N_i \to 0$ does not split;*

(iv) *the Loewy length of N_i/N_{i-1} is at least $b - i + 1$;*

(v) *the Loewy length of N is at least $2b - 1$.*

We state now the dual results to Theorems A, B and C, for the adjoint functor Tr_α. The dual statement to Corollary 1 is easily deduced from Theorem B′, and we leave the details to the reader.

Theorem A′. *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\mathrm{Tr}_\alpha \Delta_n(\lambda)$ is zero unless there is at least one λ-removable i with $1 \leq i \leq n$ and* $\mathrm{res}(i, \lambda_i) = \alpha$. *In that case,*

(i) $\mathrm{Tr}_\alpha \Delta_n(\lambda)$ *has a filtration with factors $\Delta_n(\lambda - \varepsilon_j)$ for all λ-removable j with $1 \leq j \leq n$ and* $\mathrm{res}(j, \lambda_j) = \alpha$, *each appearing with multiplicity one;*

(ii) *the head of $\mathrm{Tr}_\alpha \Delta_n(\lambda)$ is $\bigoplus_j L_n(\lambda - \varepsilon_j)$ where the sum is over all j with $1 \leq j \leq n$ such that j is conormal for $\lambda - \varepsilon_j$ and* $\mathrm{res}(j, \lambda_j) = \alpha$;

Theorem B′. *Fix $\lambda \in X^+(n)$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then, $\mathrm{Tr}_\alpha L_n(\lambda)$ is zero unless there is a (necessarily unique) i that is good for λ with* $\mathrm{res}(i, \lambda_i) = \alpha$. *In that case,*

(i) $\mathrm{Tr}_\alpha L_n(\lambda)$ *is an indecomposable, contravariantly self-dual module, with simple socle and head isomorphic to $L_n(\lambda - \varepsilon_i)$;*

(ii) $\mathrm{Tr}_\alpha L_n(\lambda)$ *is irreducible if and only if there is a unique j with $1 \leq j \leq n$ such that j is normal for λ and* $\mathrm{res}(j, \lambda_j) = \alpha$; *in particular, $L_n(\lambda) \otimes V_n^*$ is completely reducible if* *and only if for every* $\alpha \in \mathbb{Z}/p\mathbb{Z}$*, there is at most one* j *such that* j *is normal for* λ *and* $\mathrm{res}(j, \lambda_j) = \alpha$*;*

(iii) *for any* $\mu \in X^+(n)$*,*

$$\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), \mathrm{Tr}_\alpha L_n(\lambda)) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda - \varepsilon_j \text{ for some } j \text{ that is normal for } \lambda, \\ & \text{with } \mathrm{res}(j, \lambda_j) = \alpha; \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any* λ*-removable* j *with* $1 \leq j \leq n$*,*

$$[\mathrm{Tr}_\alpha L_n(\lambda) : L_n(\lambda - \varepsilon_j)] = \begin{cases} b_j & \text{if } j \text{ is normal for } \lambda \text{ and } \mathrm{res}(j, \lambda_j) = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

where b_j *is the number of* k *with* $j \leq k \leq n$ *such that* k *is normal for* λ *and* $\mathrm{res}(k, \lambda_k) = \alpha$*;*

(v) *the endomorphism ring* $\mathrm{End}_{GL(n)}(\mathrm{Tr}_\alpha L_n(\lambda))$ *is isomorphic to the truncated polynomial ring* $\mathbb{F}[T]/(T^b)$*, of dimension* b*, where* b *is the number of* j *with* $1 \leq j \leq n$ *such that* j *is normal for* λ *and* $\mathrm{res}(j, \lambda_j) = \alpha$*.*

THEOREM C$'$. *Fix* $\lambda \in X^+(n)$ *and a residue* $\alpha \in \mathbb{Z}/p\mathbb{Z}$*. Let* $n = s_1 > s_2 > \cdots > s_b$ *denote the set of all* j *with* $1 \leq j \leq n$ *such that* j *is normal for* λ *and* $\mathrm{res}(j, \lambda_j) = \alpha$*. Then,* $N := \mathrm{Tr}_\alpha L_n(\lambda)$ *has a filtration* $0 = N_0 < N_1 < \cdots < N_b = N$ *such that:*

(i) *for* $1 \leq i \leq b$*,* N_i/N_{i-1} *is a non-zero quotient of* $\Delta_n(\lambda - \varepsilon_{s_i})$*;*

(ii) *for* $1 \leq i \leq j \leq b$*,* $\dim \mathrm{Hom}_{GL(n)}(N_j/N_{j-1}, N_i/N_{i-1}) = [N_i/N_{i-1} : L_n(\lambda + \varepsilon_{s_j})] = 1$*;*

(iii) *for* $1 \leq i < b$*, the extension* $0 \to N_i/N_{i-1} \to N_{i+1}/N_{i-1} \to N_{i+1}/N_i \to 0$ *does not split.*

(iv) *the Loewy length of* N_i/N_{i-1} *is at least* $b - i + 1$*;*

(v) *the Loewy length of* N *is at least* $2b - 1$*.*

In the remainder of the paper, we will not mention the functors Tr^α and Tr_α again, preferring simply to work with the functors $? \otimes V_n$ and $? \otimes V_n^*$.

We now indicate where proofs of the above theorems can be found in the main body of the paper. Using the linkage principle and contravariant duality, Theorems A(i) and A$'$(i) are special cases of Lemma 4.8, and Theorems A(ii) and A$'$(ii) follow from Theorems 5.11(ii) and 5.9(ii) respectively. Theorems B(iii) and B$'$(iii) follow from the linkage principle and Theorems 5.11(i) and 5.9(i) respectively, while the description of the socles in Theorems B(i) and B$'$(i) follow from Theorems 5.11(iii) and 5.9(iii). In particular, since these socles are either simple (or zero), the modules $\mathrm{Tr}^\alpha L_n(\lambda)$ and $\mathrm{Tr}_\alpha L_n(\lambda)$ are indecomposable (or zero), and they are obviously self-dual as Tr^α and Tr_α commute with contravariant duality, giving the remaining parts of Theorems B(i) and B$'$(i). Theorem B(ii) follows from parts (i) and (iii) together with Proposition 4.7, applied to $N = \mathrm{Tr}^\alpha L_n(\lambda)$ and $M = \mathrm{Tr}^\alpha \nabla_n(\lambda)$, and Theorem B$'$(ii) follows similarly. Theorem B$'$(iv) and (v) are Theorem 7.7 and Theorem 8.6 respectively; Theorem B(iv) and (v) follow from these using the duality argument of the proof of Corollary 6.5 together with the combinatorial Lemma 5.8. Finally, Theorem C$'$(i) follows from Theorem 6.3 together with the linkage principle, as explained in (4). The remaining parts of Theorem C$'$ are proved in Theorem 8.9 and Corollary 8.10, and Theorem C follows from Theorem C$'$ by the duality argument again.

3 Statement of results for symmetric groups

Let Σ_r be the symmetric group on r letters. If λ is a partition of r we write $\lambda \vdash r$. We denote by S^λ the Specht module over $\mathbb{F}\Sigma_r$ corresponding to a partition $\lambda \vdash r$, and by D^λ the irreducible $\mathbb{F}\Sigma_r$-module, corresponding to a *p-regular* partition $\lambda \vdash r$ (the reader is referred to [**9**], [**10**] or [**11**] for these and other standard notions in the representation theory of symmetric groups).

Fix a partition $\lambda = (\lambda_1, \lambda_2, \dots) \vdash r$. We identify λ with its *Young diagram*

$$\lambda = \{(i,j) \in \mathbb{N} \times \mathbb{N} \mid j \leq \lambda_i\}.$$

The elements of $\mathbb{N} \times \mathbb{N}$ are called *nodes*. A node of the form (i, λ_i) is called a *removable node* (of λ) if $\lambda_i > \lambda_{i+1}$; a node of the form $(i, \lambda_i + 1)$ is called an *addable node* (for λ) if $i = 1$ or $i > 1$ and $\lambda_i < \lambda_{i-1}$. If $A = (i, \lambda_i)$ is a removable node, we let

$$\lambda_A := \lambda \setminus \{A\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i - 1, \lambda_{i+1}, \dots),$$

the partition of $(r-1)$ obtained by removing A from λ. If $B = (i, \lambda_i + 1)$ is an addable node, we let

$$\lambda^B := \lambda \cup \{B\} = (\lambda_1, \dots, \lambda_{i-1}, \lambda_i + 1, \lambda_{i+1}, \dots),$$

the partition of $(r+1)$ obtained by adding B to λ. The p-residue of a node $A = (i,j)$ is defined as in section 2: $\operatorname{res} A := (j - i) \in \mathbb{Z}/p\mathbb{Z}$. We say a node $A = (i,j)$ is to the right (resp. left) of $B = (k,l)$ if $j > l$ (resp. $j < l$).

A removable node A (of λ) is called a *normal node* if for every addable node B to the right of A with $\operatorname{res} B = \operatorname{res} A$ there exists a removable node $C(B)$ strictly between A and B with $\operatorname{res} C(B) = \operatorname{res} A$, and moreover $B \neq B'$ implies $C(B) \neq C(B')$. A removable node is called a *good node* if it is the leftmost among the normal nodes of a fixed residue.

An addable node B (for λ) is called a *conormal node* if for every removable node A to the left of B with $\operatorname{res} A = \operatorname{res} B$ there exists an addable node $C(A)$ strictly between B and A with $\operatorname{res} C(A) = \operatorname{res} B$, and moreover $A \neq A'$ implies $C(A) \neq C(A')$. An addable node is called a *cogood node* if it is the rightmost among the conormal nodes of a fixed residue.

In the next theorem, the last two equalities are new, while the first two equalities can be deduced from [**15**, 0.4,0.5] and Lemma 5.8(iv) using Frobenius reciprocity.

Theorem D. *Fix p-regular partitions $\lambda \vdash r, \mu \vdash (r+1)$. Then,*

$$\mathrm{Hom}_{\Sigma_{r+1}}(D^\mu, D^\lambda \uparrow^{\Sigma_{r+1}}) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some cogood node } B \text{ for } \lambda, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{Hom}_{\Sigma_{r+1}}(S^\lambda \uparrow^{\Sigma_{r+1}}, D^\mu) = \begin{cases} \mathbb{F} & \text{if } \lambda = \mu_A \text{ for some normal node } A \text{ for } \mu, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{Hom}_{\Sigma_{r+1}}(S^\mu, D^\lambda \uparrow^{\Sigma_{r+1}}) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some conormal node } B \text{ for } \lambda, \\ 0 & \text{otherwise;} \end{cases}$$

$$\mathrm{Hom}_{\Sigma_{r+1}}(S^\lambda \uparrow^{\Sigma_{r+1}}, (S^\mu)^*) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some addable node } B \text{ for } \lambda, \\ 0 & \text{otherwise.} \end{cases}$$

For $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and a partition λ, define the *α-content* of λ to be the integer

$$\mathrm{cont}_\alpha(\lambda) := |\{A \in \lambda \mid \mathrm{res}\, A = \alpha\}|$$

(which is a special case of the definition in section 2). For two partitions λ and μ we write $\lambda \sim \mu$, if $\mathrm{cont}_\alpha(\lambda) = \mathrm{cont}_\alpha(\mu)$ for all $\alpha \in \mathbb{Z}/p\mathbb{Z}$. The 'Nakayama Conjecture' (proved e.g. in [**11**]) claims that $\mathbb{F}\Sigma_r$-modules D^λ and D^μ are in the same block if and only if $\lambda \sim \mu$.

Fix a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. We define the functors

$$\mathrm{Ind}^\alpha : \mathbb{F}\Sigma_r\text{-mod} \to \mathbb{F}\Sigma_{r+1}\text{-mod} \qquad \text{and} \qquad \mathrm{Res}_\alpha : \mathbb{F}\Sigma_r\text{-mod} \to \mathbb{F}\Sigma_{r-1}\text{-mod},$$

by defining them first on a module M in any fixed block, and then extending additively to all of $\mathbb{F}\Sigma_r$-mod. Assume M belongs to the block corresponding to the residue contents $c_0, c_1, \dots, c_{p-1}$, that is, every composition factor of M is of the form D^λ with $\mathrm{cont}_\beta(\lambda) = c_\beta$ for all $\beta \in \mathbb{Z}/p\mathbb{Z}$. We let $\mathrm{Ind}^\alpha M$ (resp. $\mathrm{Res}_\alpha M$) denote the largest submodule of $M \uparrow^{\Sigma_{r+1}}$ (resp. $M \downarrow_{\Sigma_{r-1}}$) all of whose composition factors are of the form D^μ with $\mathrm{cont}_\alpha(\mu) = c_\alpha + 1$

(resp. $\mathrm{cont}_\alpha(\mu) = c_\alpha - 1$), and $\mathrm{cont}_\beta(\mu) = c_\beta$ for all $\alpha \neq \beta \in \mathbb{Z}/p\mathbb{Z}$. Given a morphism $\theta : M \to N$, $\mathrm{Ind}^\alpha \theta$ is just the restriction to $\mathrm{Ind}^\alpha M$ of the natural map $\hat{\theta} : M \uparrow^{\Sigma_{r+1}} \to N \uparrow^{\Sigma_{r+1}}$ induced by θ, and similarly for Res_α. We have

$$M \uparrow^{\Sigma_{r+1}} \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Ind}^\alpha M \quad \text{and} \quad M \downarrow_{\Sigma_{r-1}} \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{Res}_\alpha M.$$

The functors just defined are called *Robinson's α-induction* and *α-restriction functors* (cf. [**11**, 6.3.16]).

Our next result describes the effect of Ind^α on irreducible modules, and is the symmetric group analogue of Theorem B.

Theorem E. *Fix a p-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Then,* $\mathrm{Ind}^\alpha D^\lambda$ *is zero unless λ has at least one conormal node of residue α. In that case,*

(i) $\mathrm{Ind}^\alpha D^\lambda$ *is an indecomposable, self-dual module, with simple socle and head isomorphic to D^{λ^B} where B is the (unique) cogood node of residue α;*

(ii) $\mathrm{Ind}^\alpha D^\lambda$ *is irreducible if and only if there is a unique conormal node of residue α; in particular, the induced module $D^\lambda \uparrow^{\Sigma_{r+1}}$ is completely reducible if and only if all conormal nodes have different residues.*

(iii) *for any p-regular $\mu \vdash (r+1)$,*

$$\mathrm{Hom}_{\Sigma_{r+1}}(S^\mu, \mathrm{Ind}^\alpha D^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda^B \text{ for some conormal node } B \text{ with } \mathrm{res}\, B = \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any addable node B such that λ^B is p-regular,*

$$[\mathrm{Ind}^\alpha D^\lambda : D^{\lambda^B}] = \begin{cases} d_B & \text{if } B \text{ is conormal for } \lambda \text{ and } \mathrm{res}\, B = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

where d_B denotes the number of conormal nodes C to the left of B (counting B itself) such that $\mathrm{res}\, C = \alpha$*;*

(v) *the endomorphism ring* $\mathrm{End}_{\Sigma_{r+1}}(\mathrm{Ind}^\alpha D^\lambda)$ *is isomorphic to the truncated polynomial ring* $\mathbb{F}[T]/(T^d)$*, of dimension* d*, where* d *is the number of conormal nodes* B *with* $\mathrm{res}\, B = \alpha$*.*

We note that a criterion for complete reducibility of $D^\lambda \uparrow^{\Sigma_{r+1}}$ different from the one in Theorem E(ii) was found in [**17**, Theorem C]. It is difficult to see directly that the two combinatorial conditions are equivalent (see Corollary 8.7(iii)). Also, Theorem E(i) is immediate from [**16**, Theorem 3.2], using the combinatorial Lemma 5.8.

The next result is parallel to Corollary 1, and is useful in calculating inverse decomposition matrices. It follows immediately from Theorem E(ii) together with exactness of the functor Ind^α; in the second part of the statement, we have used the (known) fact that for any $\mu \vdash r$, $\mathrm{Ind}^\alpha S^\mu$ has a filtration with factors S^{μ^C} for all addable nodes C for μ with $\mathrm{res}\, C = \alpha$.

Corollary 2. *Let* $\lambda \vdash r$ *be* p*-regular, and* $\alpha \in \mathbb{Z}/p\mathbb{Z}$*. Suppose that* λ *has a* unique *conormal node* B *of residue* α*. If (in the Grothendieck group)*

$$D^\lambda = \sum_{\mu \vdash r} c_{\lambda,\mu} S^\mu \qquad \text{then} \qquad D^{\lambda^B} = \sum_{\mu \vdash r} c_{\lambda,\mu} \,\mathrm{Ind}^\alpha S^\mu = \sum_{\mu \vdash r} c_{\lambda,\mu} \left(\sum_C S^{\mu^C} \right),$$

where the last summation is over all addable nodes C *for* μ *with* $\mathrm{res}\, C = \alpha$*.*

We state now for completeness the dual result to Theorem E, for the functor Res_α. These results are known, see [**15**, 0.4,0.5], [**16**, 3.1], [**17**, Theorem B], [**18**, 1.4].

Theorem E′. *Fix a* p*-regular partition* $\lambda \vdash r$ *and a residue* $\alpha \in \mathbb{Z}/p\mathbb{Z}$*. Then,* $\mathrm{Res}_\alpha D^\lambda$ *is zero unless* λ *has at least one normal node of residue* α*. In that case,*

(i) $\mathrm{Res}_\alpha D^\lambda$ *is an indecomposable, self-dual module, with simple socle and head isomorphic to* D^{λ_A} *where* A *is the (unique) good node of residue* α*;*

(ii) $\operatorname{Res}_\alpha D^\lambda$ *is irreducible if and only if there is a unique normal node of residue* α*; in particular, the restriction* $D^\lambda \downarrow_{\Sigma_{r-1}}$ *is completely reducible if and only if all normal nodes have different residues.*

(iii) *for any* p*-regular* $\mu \vdash (r-1)$,

$$\operatorname{Hom}_{\Sigma_{r-1}}(S^\mu, \operatorname{Res}_\alpha D^\lambda) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda_A \text{ for some normal node } A \text{ with } \operatorname{res} A = \alpha, \\ 0 & \text{otherwise;} \end{cases}$$

(iv) *for any removable node* A *such that* λ_A *is* p*-regular,*

$$[\operatorname{Res}_\alpha D^\lambda : D^{\lambda_A}] = \begin{cases} d_A & \text{if } A \text{ is normal for } \lambda \text{ and } \operatorname{res} A = \alpha, \\ 0 & \text{otherwise} \end{cases}$$

where d_A *denotes the number of normal nodes* C *to the right of* A *(counting* A *itself) such that* $\operatorname{res} C = \alpha$*;*

(v) *the endomorphism ring* $\operatorname{End}_{\Sigma_{r-1}}(\operatorname{Res}_\alpha D^\lambda)$ *is isomorphic to the truncated polynomial ring* $\mathbb{F}[T]/(T^d)$*, of dimension* d*, where* d *is the number of normal nodes* A *with* $\operatorname{res} A = \alpha$*.*

Remark. We remark that an analogue of Theorem C can also be proved for the functor $\operatorname{Ind}^\alpha$. More precisely, fix a p-regular partition $\lambda \vdash r$ and a residue $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Let $B_1, B_2 \dots B_d$ be all conormal nodes of residue α counted from left to right. Then, one can show that $I := \operatorname{Ind}^\alpha D^\lambda$ has a filtration $0 = I_0 < I_1 < \dots < I_d = I$ such that:

(i) for $1 \le j \le d$, I_j/I_{j-1} is a non-zero quotient of $S^{\lambda^{B_j}}$;

(ii) for $1 \le j \le k \le d$ such that λ^{B_k} is p-regular, $[I_j/I_{j-1} : D^{\lambda^{B_k}}] = 1$;

(iii) the extension $0 \to I_j/I_{j-1} \to I_{j+1}/I_{j-1} \to I_{j+1}/I_j \to 0$ does not split;

(iv) if every λ^{B_j} is p-regular, then the Loewy length of I is at least $2d-1$.

We now indicate briefly how to deduce Theorems D and E from the results for $GL(n)$ stated in section 2, using Schur functors. The key point is that a Schur functor applied to a $GL(n)$-module of the form $M \otimes V_n$ (where M is polynomial of degree $r < n$) gives the $\mathbb{F}\Sigma_{r+1}$-module induced from the $\mathbb{F}\Sigma_r$-module obtained by applying the Schur functor to M; see [**3**, Theorem 4.17] for the precise statement. Now fix p-regular partitions $\lambda \vdash r$ and $\mu \vdash (r+1)$, and choose $n > r$. We can regard the transpose partitions λ^t and μ^t as elements of $X^+(n)$. In [**3**, Corollary 4.18] (cf. [**7**, Lemma 2.5]), we used the Schur functor to show that:

$$\operatorname{Hom}_{\Sigma_{r+1}}(D^\mu, D^\lambda \uparrow^{\Sigma_{r+1}}) \cong \operatorname{Hom}_{GL(n)}(L_n(\mu^t), L_n(\lambda^t) \otimes V_n),$$

$$\operatorname{Hom}_{\Sigma_{r+1}}(S^\mu, D^\lambda \uparrow^{\Sigma_{r+1}}) \cong \operatorname{Hom}_{GL(n)}(\Delta_n(\mu^t), L_n(\lambda^t) \otimes V_n),$$

$$\operatorname{Hom}_{\Sigma_{r+1}}(S^\lambda \uparrow^{\Sigma_{r+1}}, D^\mu) \cong \operatorname{Hom}_{GL(n)}(L_n(\mu^t), \nabla_n(\lambda^t) \otimes V_n),$$

$$\operatorname{Hom}_{\Sigma_{r+1}}(S^\lambda \uparrow^{\Sigma_{r+1}}, (S^\mu)^*) \cong \operatorname{Hom}_{GL(n)}(\Delta_n(\mu^t), \nabla_n(\lambda^t) \otimes V_n).$$

Theorem D follows from these isomorphisms, (5.1) and Theorem 5.11 (note for symmetric groups we have 'transposed' the combinatorial definitions of normal, conormal and cogood from section 2). Now Theorem E(iii) and the fact that $\operatorname{Ind}^\alpha D^\lambda$ has the socle as claimed in Theorem E(i) follow immediately from Theorem D on taking blocks. Since the socle is simple (whenever it is non-zero), it follows that $\operatorname{Ind}^\alpha D^\lambda$ is indecomposable, and it is self-dual as D^λ is and $\operatorname{Ind}^\alpha$ commutes with duality, proving the remaining parts of Theorem E(i). Theorem E(iv) follows from Theorem B(iv) together with the following fact proved in [**3**, Theorem 4.16] (where $\lambda \vdash r$ and $\mu \vdash (r+1)$ are p-regular and $n > r$):

$$[D^\lambda \uparrow^{\Sigma_{r+1}} : D^\mu] = [L_n(\lambda^t) \otimes V_n : L_n(\mu^t)].$$

To prove Theorem E(ii), we note that as $\operatorname{Ind}^\alpha D^\lambda$ has simple socle and head isomorphic to D^{λ^B} (where B is as in (i)), $\operatorname{Ind}^\alpha D^\lambda$ is irreducible if and only if $[\operatorname{Ind}^\alpha D^\lambda : D^{\lambda^B}] = 1$. So Theorem E(ii) follows from Theorem E(iv) (alternatively, one can deduce Theorem E(ii) directly from Theorem B(ii) using [**13**, 2.13] and a block argument). Finally, we observe that if $\lambda \vdash r < n$ is p-regular, then $L_n(\lambda^t) \otimes V_n$ has p-restricted socle and head, which follows from [**3**, Corollary 2.12]. So by [**3**, Theorem 4.17],

$$\mathrm{End}_{GL(n)}(L_n(\lambda^t) \otimes V_n) \cong \mathrm{End}_{\Sigma_{r+1}}(D^\lambda \uparrow^{\Sigma_{r+1}}).$$

This isomorphism is even an isomorphism of $\mathbb{F}$-algebras, as follows from the proof of [**3**, Theorem 4.17] (see [**3**, Lemma 2.17(ii)]), since the isomorphism is given by a natural restriction map. Theorem E(v) follows from Theorem B(v) by the same argument at the level of blocks.

4 A complete reducibility criterion

In the remainder of the paper, we prove the results stated in section 2 and section 3. We begin in this preliminary section with some general results in the setting of rational representations of an arbitrary reductive algebraic group as in [**12**]. Actually the results in this section (except for Lemma 4.8) are true for an arbitrary quasi-hereditary algebra with a duality fixing the simple modules. In later sections, we will only be concerned with $GL(n)$.

Fix a (connected) reductive algebraic group G over our algebraically closed field $\mathbb{F}$. A *G-module* always means a rational $\mathbb{F}G$-module as in [**12**]. Fix a maximal torus $T < G$ and a Borel subgroup $B^+ > T$. Let R denote the root system of G relative to T, and $R^+ \subset R$ denote the positive roots determined by the choice of positive Borel subgroup B^+. Let $X(T)$ denote the character group of T, and call elements of $X(T)$ *weights*. The Weyl group $W = N_G(T)/T$ acts on $X(T)$, and we let $\langle .,. \rangle$ denote some fixed W-invariant inner product on $X(T) \otimes_{\mathbb{Z}} \mathbb{R}$. For $\mu \in X(T) \otimes_{\mathbb{Z}} \mathbb{R}$, $\mu^\vee$ denotes $\frac{2\mu}{\langle \mu,\mu \rangle}$. Let w_0 denote the longest element of W. We have the usual *dominance order* on $X(T)$, defined by $\lambda > \mu$ if $(\lambda - \mu)$ is a sum of positive roots. The weight $\lambda \in X(T)$ is *dominant* (relative to B^+) if $\langle \lambda, \alpha^\vee \rangle \geq 0$ for all $\alpha \in R^+$; we let $X^+(T)$ denote the set of all dominant weights $\lambda \in X(T)$. For $\lambda \in X^+(T)$, we have the G-modules $\Delta(\lambda)$ (denoted $V(\lambda)$ in [**12**]), $\nabla(\lambda)$ (denoted $H^0(\lambda)$ in [**12**]) and $L(\lambda)$, which are the standard, costandard and irreducible G-modules of highest weight λ respectively.

In the special case that $G = GL(n)$, we always take $T = T(n)$ to be all diagonal invertible matrices and $B^+ = B^+(n)$ to be all upper triangular invertible matrices. The character group of $T(n)$ is the free abelian group with generators $\varepsilon_1, \dots, \varepsilon_n$, where ε_i denotes the standard character defined by $\mathrm{diag}(t_1, \dots, t_n) \mapsto t_i$ for all $t_1, \dots, t_n \in \mathbb{F}^\times$, and we identify this with the set $X(n)$ of all n-tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$, by letting $(\lambda_1, \dots, \lambda_n) \in X(n)$ correspond to the character $\sum_i \lambda_i \varepsilon_i$. For $\lambda = (\lambda_1, \dots, \lambda_n) \in X(n)$, we write:

$$\begin{array}{lll} |\lambda| & \text{for} & \lambda_1 + \cdots + \lambda_n \in \mathbb{Z}; \\ \bar{\lambda} & \text{for} & (\lambda_1, \dots, \lambda_{n-1}) \in X(n-1); \\ \tilde{\lambda} & \text{for} & (\lambda_1, \dots, \lambda_n, 0) \in X(n+1). \end{array}$$

The root system of $GL(n)$ is the set $\{\varepsilon_i - \varepsilon_j \mid 1 \leq i, j \leq n, i \neq j\}$, and the root $\varepsilon_i - \varepsilon_j$ is positive if $i < j$. A weight $(\lambda_1, \dots, \lambda_n) \in X(n)$ is dominant precisely when $\lambda_1 \geq \cdots \geq \lambda_n$, and we let $X^+(n) \subset X(n)$ denote the set of all such dominant weights. For $\lambda \in X^+(n)$, we then denote the irreducible, standard and costandard for $GL(n)$ of highest weight λ by $L_n(\lambda), \Delta_n(\lambda)$ and $\nabla_n(\lambda)$ respectively, as in section 2.

If M is a finite dimensional G-module, the *dual* M^* is defined as usual to be the dual vector space with action $(gf)(m) = f(g^{-1}m)$ for $g \in G, f \in M^*$ and $m \in M$. Recall that $L(\lambda)^* \cong L(\lambda^*)$ and $\Delta(\lambda)^* = \nabla(\lambda^*)$, where $\lambda^* := -w_0\lambda$. We also define the *contravariant dual* M^τ of M. This is the dual vector space M^* with new action defined by $(gf)(m) = f(\tau(g)m)$ for $g \in G, f \in M^*$ and $m \in M$, where τ is the Chevalley antiautomorphism of G as in [**12**, II.1.16]. For $\lambda \in X^+(T)$, we have $L(\lambda)^\tau \cong L(\lambda)$ and $\Delta(\lambda)^\tau \cong \nabla(\lambda)$. In the case $G = GL(n)$, τ is just matrix transposition, while for $\lambda \in X(n)$, $\lambda^* = -w_0\lambda = (-\lambda_n, \dots, -\lambda_1)$.

The following fact proved in [**12**, II.4.13] is of central importance:

4.1. *For $\lambda, \mu \in X^+(T)$, $\mathrm{Ext}^i_G(\Delta(\lambda), \nabla(\mu))$ is zero unless $i = 0$ and $\lambda = \mu$, when it is one dimensional.*

We say that a G-module M has a ∇*-filtration* (resp. a Δ*-filtration*) if it has an ascending filtration $0 = M_{(0)} < M_{(1)} < \dots$ with $\bigcup_{i \geq 0} M_{(i)} = M$ such that each factor $M_{(i)}/M_{(i-1)}$ is isomorphic to a (possibly infinite) direct sum of copies of $\nabla(\lambda^{(i)})$ (resp $\Delta(\lambda^{(i)})$) for some $\lambda^{(i)} \in X^+(n)$. We recall the Donkin-Mathieu theorem [**6**], [**20**]:

4.2. *If M, N are G-modules with ∇-filtrations, then $M \otimes N$ has a ∇-filtration.*

For the remainder of the section, fix G-modules W and N such that:

(P1) W has a ∇-filtration

$$0 = W_{(0)} < W_{(1)} < \cdots < W_{(s)} = W$$

where $W_{(t)}/W_{(t-1)} \cong \nabla(\lambda^{(t)})$ for some $\lambda^{(t)} \in X^+(T)$, $t = 1, 2, \dots, s$.

(P2) N is a submodule of W satisfying $N \cong N^\tau$. Define $N_{(t)}$ to be the intersection $N \cap W_{(t)}$; then N has the filtration

$$0 = N_{(0)} \leq N_{(1)} \leq \cdots \leq N_{(s)} = N,$$

and $N_{(t)}/N_{(t-1)}$ is a (possibly zero) submodule of $\nabla(\lambda^{(t)})$ for $1 \leq t \leq s$.

We have in mind the following example for the modules W and N. Take $W = \nabla(\lambda) \otimes \nabla(\nu)$ and $N = L(\lambda) \otimes L(\nu)$ (or any fixed block of these); then, (P1) is satisfied by (4.2), while (P2) is obvious. Another example is if $W = \nabla(\lambda) \downarrow_H$, $N = L(\lambda) \downarrow_H$ where H is a Levi subgroup of G.

We prove some general facts about modules W and N for use later in the paper. We will need the following known lemma:

4.3. Lemma. *For $\lambda \in X^+(T)$ and any finite dimensional G-module V all of whose composition factors are of the form $L(\nu)$ for $\nu \not> \lambda$, $\mathrm{Ext}^1_G(\Delta(\lambda), V) = 0$.*

Proof. We argue by induction on dimension. Pick a maximal submodule W of V such that $V/W \cong L(\nu)$ for $\nu \not> \lambda$. The long exact sequence of cohomology yields the exact sequence $\mathrm{Ext}^1_G(\Delta(\lambda), W) \to \mathrm{Ext}^1_G(\Delta(\lambda), V) \to \mathrm{Ext}^1_G(\Delta(\lambda), L(\nu))$. By induction, the left hand Ext is zero, so it suffices to show that $\mathrm{Ext}^1_G(\Delta(\lambda), L(\nu)) = 0$ which is done in the proof of [**12**, II.2.14].

Applying the lemma to $V = \Delta(\mu)$ and dualizing with [**12**, II.2.12(3)], we deduce the well-known:

4.4. *For $\lambda, \mu \in X^+(T)$ with $\mu \not> \lambda$, $\mathrm{Ext}^1_G(\Delta(\lambda), \Delta(\mu)) = \mathrm{Ext}^1_G(\nabla(\mu), \nabla(\lambda)) = 0$.*

We can reorder the ∇-filtration of W in (P1) if necessary using (4.4), to *assume from now on* that $\lambda^{(i)} > \lambda^{(j)}$ implies $i > j$ for all $1 \leq i, j \leq s$.

4.5. Lemma. *For $1 \leq t \leq s$ and $\lambda \in X^+(T)$, there is an exact sequence*

$$0 \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t-1)}) \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}) \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)}) \to 0.$$

Proof. We have the long exact sequence

$$0 \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t-1)}) \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}) \to \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)}) \\ \to \mathrm{Ext}^1_G(\Delta(\lambda), N_{(t-1)}) \to \dots.$$

Note that $\mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)}) \subseteq \mathrm{Hom}_G(\Delta(\lambda), \nabla(\lambda^{(t)}))$, and the latter is zero by (4.1) unless $\lambda = \lambda^{(t)}$. If $\mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)}) = 0$ the result is obvious, so we may assume that $\lambda = \lambda^{(t)}$. This means by the choice of ordering of the ∇-filtration of W that for $u < t$, $\lambda^{(u)} \not> \lambda$. All composition factors of $N_{(t-1)}$ are of the form $L(\nu)$ for $\nu \leq \lambda^{(u)}$ and some

$1 \leq u < t$, hence also satisfy $\nu \not> \lambda$. So $\mathrm{Ext}^1_G(\Delta(\lambda), N_{(t-1)}) = 0$ by Lemma 4.3, and the result follows from the long exact sequence.

4.6. Corollary. *For $\lambda \in X^+(T)$, $\dim \mathrm{Hom}_G(\Delta(\lambda), N)$ is equal to the number of t with $1 \leq t \leq s$ such that $\lambda = \lambda^{(t)}$ and $N_{(t)} \neq N_{(t-1)}$.*

Proof. Note that $\dim \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}) = \sum_{u=1}^{t} \dim \mathrm{Hom}_G(\Delta(\lambda), N_{(u)}/N_{(u-1)})$, which follows from lemma by induction on t. Now take $t = s$ to deduce that

$$\dim \mathrm{Hom}_G(\Delta(\lambda), N) = \sum_{t=1}^{s} \dim \mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)}).$$

Since $N_{(t)}/N_{(t-1)}$ is a submodule of $\nabla(\lambda^{(t)})$, $\mathrm{Hom}_G(\Delta(\lambda), N_{(t)}/N_{(t-1)})$ is zero unless $\lambda = \lambda^{(t)}$ and $N_{(t)} \neq N_{(t-1)}$, when it is one dimensional. The result follows.

Now we obtain our main result in the general setting:

4.7. Proposition. *The G-module N is completely reducible if and only if*

$$\mathrm{Hom}_G(L(\lambda), N) \cong \mathrm{Hom}_G(\Delta(\lambda), N)$$

for all $\lambda \in X^+(T)$.

Proof. Obviously, if N is completely reducible then $\mathrm{Hom}_G(L(\lambda), N) \cong \mathrm{Hom}_G(\Delta(\lambda), N)$ for all λ. Conversely, suppose $\mathrm{Hom}_G(L(\lambda), N) \cong \mathrm{Hom}_G(\Delta(\lambda), N)$ for all $\lambda \in X^+(T)$. Let $d := \dim \mathrm{Hom}_G(\Delta(\lambda), N)$; by Corollary 4.6, d is equal to the number of t with $1 \leq t \leq s$ such that $\lambda^{(t)} = \lambda$ and $N_{(t)} \neq N_{(t-1)}$.

We claim that $N_{(t)}/N_{(t-1)} \cong L(\lambda^{(t)})$ whenever it is non-zero, for all $1 \leq t \leq s$. Suppose for a contradiction that this is false, and take $u \leq s$ maximal such that $N_{(u)}/N_{(u-1)}$ is reducible. This implies that $\mathrm{Hom}_G(N_{(u)}/N_{(u-1)}, L(\lambda)) = 0$ as $N_{(u)}/N_{(u-1)}$ is a submodule of $\nabla(\lambda)$, where $\lambda := \lambda^{(u)}$. So, using contravariant duality,

$$\begin{aligned} d &= \dim \mathrm{Hom}_G(\Delta(\lambda), N) = \dim \mathrm{Hom}_G(L(\lambda), N) = \dim \mathrm{Hom}_G(N, L(\lambda)) \\ &\leq \sum_{t=1}^{s} \dim \mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda)) \\ &= \sum_{t=1}^{u-1} \dim \mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda)) + \sum_{t=u+1}^{s} \dim \mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda)). \end{aligned}$$

For $t < u$ such that $N_{(t)} \neq N_{(t-1)}$, the module $N_{(t)}/N_{(t-1)}$ is a submodule of $\nabla(\lambda^{(t)})$, so has highest weight $\lambda^{(t)} \not> \lambda$. Hence in this case, $\mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda))$ is zero unless $\lambda^{(t)} = \lambda$, when it is at most one dimensional. For $t > u$, $N_{(t)}/N_{(t-1)}$ is either zero or $L(\lambda^{(t)})$ by maximality of u, so $\mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda))$ is zero unless $\lambda^{(t)} = \lambda$ and $N_{(t)} \neq N_{(t-1)}$, when it is exactly one dimensional. This shows that

$$\sum_{t=1}^{u-1} \dim \mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda)) + \sum_{t=u+1}^{s} \dim \mathrm{Hom}_G(N_{(t)}/N_{(t-1)}, L(\lambda)) \leq d - 1.$$

Comparing this with our previous inequality gives the desired contadiction.

We have now shown that $N_{(t)}/N_{(t-1)} \cong L(\lambda^{(t)})$ whenever it is non-zero. Hence by Corollary 4.6 again, $\dim \mathrm{Hom}_G(L(\lambda), N) = [N : L(\lambda)]$ for all $\lambda \in X^+(T)$. Hence, N coincides with its socle, so is completely reducible.

Now, let $\varepsilon \in X^+(n)$ be a *miniscule* weight (see [**8**, §13 ex. 13] where such weights are called minimal). Then $L(\varepsilon) = \Delta(\varepsilon) = \nabla(\varepsilon)$. It is of importance that the ∇-filtration of $W = \nabla(\lambda) \otimes \nabla(\varepsilon)$ in (P1) is in fact multiplicity-free:

4.8. Lemma. *Let $\varepsilon \in X^+(T)$ be a miniscule weight. For any $\lambda \in X^+(T)$, the G-module $\nabla(\lambda) \otimes L(\varepsilon)$ has a ∇-filtration with factors*

$$\{\nabla(\lambda + w\varepsilon) \mid \textit{for all } w \in W \textit{ such that } \lambda + w\varepsilon \in X^+(T)\},$$

each appearing with multiplicity one. In particular by (4.1), for any $\mu \in X^+(T)$

$$\mathrm{Hom}_G(\Delta(\mu), \nabla(\lambda) \otimes L(\varepsilon)) = \begin{cases} \mathbb{F} & \text{if } \mu \in \lambda + W\varepsilon, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The fact that $\nabla(\lambda) \otimes L(\varepsilon)$ has a ∇-filtration follows from (4.2) (or [**6**, Proposition 4.2.1] or [**22**]). To compute the factors appearing in a ∇-filtration, it suffices to work at the level of characters. Consequently, the multiplicities are the same as in characteristic 0. Now use [**8**, §26 ex. 9]. We note that the lemma can also be deduced as a special case of [**12**, II.7.13].

5 Primitive vectors and the socle of $L_n(\lambda) \otimes V_n^*$

As observed in section 4, the $GL(n)$-modules $W = \nabla_n(\lambda) \otimes V_n$ (resp. $\nabla_n(\lambda) \otimes V_n^*$) and $N = L_n(\lambda) \otimes V_n$ (resp. $L_n(\lambda) \otimes V_n^*$) satisfy the hypotheses (P1) and (P2) of section 4, for any $\lambda \in X^+(n)$. In this section, we will relate the combinatorial notions from section 2 to the general properties of the modules W and N considered in section 4.

We point out that the weights ε_1 and $-\varepsilon_n$ are miniscule for $G = GL(n)$, so we have the following special cases of Lemma 4.8:

5.1. *Fix* $\lambda, \mu \in X^+(n)$.

(i) *The space* $\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), \nabla_n(\lambda) \otimes V_n)$ *is zero unless* $\mu = \lambda + \varepsilon_i$ *for some* λ*-addable* i*, when it is one dimensional.*

(ii) *The space* $\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), \nabla_n(\lambda) \otimes V_n^*)$ *is zero unless* $\mu = \lambda - \varepsilon_i$ *for some* λ*-removable* i*, when it is one dimensional.*

We work now with the hyperalgebra $U(n)$ of $GL(n)$ (denoted $\mathrm{Dist}(GL(n))$ in [**12**]), taking notation from [**4**, Section 2]. The hyperalgebra can be defined by base change starting from the universal enveloping algebra $U(n,\mathbb{C})$ of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$. For $1 \le i,j \le n$, let $x_{i,j}$ denote the element of $\mathfrak{gl}(n,\mathbb{C})$ corresponding to the $n \times n$ matrix with 1 in the ij-entry and zeros elsewhere. Let $U(n,\mathbb{Z})$ be the $\mathbb{Z}$-subalgebra of $U(n,\mathbb{C})$ generated by

$$\left\{ x_{i,j}^{(r)}, \begin{pmatrix} x_{i,i} \\ r \end{pmatrix} \;\middle|\; 1 \le i,j \le n, i \ne j, r \ge 0 \right\},$$

where $x^{(r)}$ denotes the divided power $\frac{x^r}{r!}$ and $\begin{pmatrix} x \\ r \end{pmatrix}$ denotes $\frac{x(x-1)\dots(x-r+1)}{r!}$. The *hyperalgebra* $U(n) = U(n,\mathbb{F})$ is then $U(n,\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}$. For $1 \le i < j \le n$, we denote the image of $x_{i,j}^{(r)}$ and $x_{j,i}^{(r)}$ in $U(n)$ by $E_{i,j}^{(r)}$ and $F_{i,j}^{(r)}$ respectively, and the image of $\begin{pmatrix} x_{i,i} \\ r \end{pmatrix}$ by $\begin{pmatrix} H_i \\ r \end{pmatrix}$. Letting $U^-(n), U^0(n)$ and $U^+(n)$ denote the subalgebras of $U(n)$ generated by

$$\left\{ F_{i,j}^{(r)} \right\}_{1 \le i < j \le n, r \ge 0}, \left\{ \begin{pmatrix} H_i \\ r \end{pmatrix} \right\}_{1 \le i \le n, r \ge 0} \quad \text{and} \quad \left\{ E_{i,j}^{(r)} \right\}_{1 \le i < j \le n, r \ge 0}$$

respectively, we have the usual 'triangular decomposition' $U(n) = U^-(n)U^0(n)U^+(n)$.

Any rational $GL(n)$-module M is naturally a $U(n)$-module (see [**12**, I.8]). We say that a vector $v \in M$ is a *primitive vector* if it is annihilated by $E_{i,j}^{(k)}$ for all $1 \le i < j \le n$ and $k \ge 1$. We say M is a *high weight module* of high weight λ if it is generated by a primitive vector of weight λ. By [**12**, II.2.13], the standard module $\Delta_n(\lambda)$ is universal amongst all high weight modules of high weight λ. This implies:

5.2. *For any $GL(n)$-module M and $\lambda \in X^+(T)$, the dimension of the space of all primitive vectors in M of weight λ is equal to* $\dim \operatorname{Hom}_{GL(n)}(\Delta_n(\lambda), M)$.

Let $f_1, \dots, f_n$ denote the basis for V_n^* that is dual to the natural basis for V_n. The action of the generators of $U(n)$ on V_n^* is given by:

$$E_{i,j}^{(l)} f_k = -\delta_{l,1}\delta_{i,k} f_j,\ F_{i,j}^{(l)} f_k = -\delta_{l,1}\delta_{j,k} f_i,\ \begin{pmatrix} H_i \\ l \end{pmatrix} f_k = -\delta_{l,1}\delta_{i,k} f_k \tag{1}$$

for $1 \leq i, j, k \leq n$ and $l \geq 1$.

For the remainder of the section, let $P = LY$ denote the standard parabolic subgroup of $GL(n)$, where $L \cong GL(n-1) \times G(1)$ (embedded diagonally) and Y is the unipotent radical generated by the root subgroups corresponding to the roots $\varepsilon_i - \varepsilon_n$, for $i = 1, 2, \ldots n-1$. Note that for any $GL(n)$-module M, the Y-fixed points M^Y of M are L-invariant, so we can regard M^Y as a $GL(n-1)$-module in a natural way.

5.3. Lemma. *For any $GL(n)$-module M, the linear map $e : M \to M \otimes V_n^*$ defined for $v \in M$ by*

$$e(v) := v \otimes f_n + \sum_{h=1}^{n-1} E_{h,n} v \otimes f_h$$

is an injective $GL(n-1)$-homomorphism such that $e(M) \supseteq (M \otimes V_n^*)^Y$.

Proof. The map e is obviously injective as $f_1, \ldots, f_n$ are linearly independent. To prove that it is a $GL(n-1)$-homomorphism, we work with the naturally embedded hyperalgebra $U(n-1) < U(n)$. Note that for any vector $v \in M$ that is a weight vector for the maximal torus $T(n-1)$ of $GL(n-1)$, the weights of v and $e(v)$ relative to $T(n-1)$ are equal. So, to prove that e is a $GL(n-1)$-homomorphism, it suffices to show that $e(Xv) = Xe(v)$ for any $v \in M$ and any $X \in U(n-1)$ of the form $E_{i,j}^{(k)}$ or $F_{i,j}^{(k)}$ with $1 \leq i < j < n$ and $k > 0$. We have (interpreting $E_{n,n}$ as 1):

$$\begin{aligned} E_{i,j}^{(k)} e(v) = E_{i,j}^{(k)} \left(\sum_{h=1}^{n} E_{h,n} v \otimes f_h \right) &= \sum_{h=1}^{n} (E_{i,j}^{(k)} E_{h,n} v) \otimes f_h + (E_{i,j}^{(k-1)} E_{i,n} v) \otimes (E_{i,j} f_i) \\ &= \sum_{h=1}^{n} E_{h,n} E_{i,j}^{(k)} v \otimes f_h + E_{i,n} E_{i,j}^{(k-1)} v \otimes f_j + E_{i,n} E_{i,j}^{(k-1)} v \otimes (-f_j) = e(E_{i,j}^{(k)} v), \end{aligned}$$

and

$$F_{i,j}^{(k)}e(v) = F_{i,j}^{(k)}\left(\sum_{h=1}^{n} E_{h,n}v \otimes f_h\right) = \sum_{h=1}^{n}(F_{i,j}^{(k)}E_{h,n}v) \otimes f_h + (F_{i,j}^{(k-1)}E_{j,n}v) \otimes (F_{i,j}f_j)$$
$$= \sum_{h=1}^{n} E_{h,n}F_{i,j}^{(k)}v \otimes f_h + E_{j,n}F_{i,j}^{(k-1)}v \otimes f_i + E_{j,n}F_{i,j}^{(k-1)}v \otimes (-f_i) = e(F_{i,j}^{(k)}v).$$

For the last part, take any element $w = \sum_{h=1}^{n} v_h \otimes f_h$ of $(M \otimes V_n^*)^Y$. Then $E_{i,n}w = 0$ for any $1 \leq i < n$. That is,

$$0 = E_{i,n}(\sum_{h=1}^{n} v_h \otimes f_h) = \sum_{h=1}^{n} E_{i,n}v_h \otimes f_h + v_i \otimes E_{i,n}f_i = \sum_{h=1}^{n} E_{i,n}v_h \otimes f_h - v_i \otimes f_n.$$

Since $f_1, \dots, f_n$ are linearly independent, we conclude that $v_i = E_{i,n}v_n$ for $i = 1, \dots, n-1$. So $w = e(v_n)$ and we have shown that $(M \otimes V_n^*)^Y \subseteq e(M)$.

We recall the notion of level introduced by Seitz. For $\lambda \in X^+(n)$, we define the *ℓth level* $M^{(\ell)}$ of any submodule M of $\nabla_n(\lambda)$ to be the sum of the weight spaces M_ν for all $\nu \in X(n)$ satisfying $\nu_n = \lambda_n + \ell$. Observe that $M^{(\ell)}$ is a $GL(n-1)$-submodule of M and $M = \bigoplus_{\ell \geq 0} M^{(\ell)}$.

5.4. Proposition. *For any $\lambda \in X^+(n)$ and any submodule M of $\nabla_n(\lambda)$, the restriction of the map e from Lemma 5.3 to $M^{(0)} \oplus M^{(1)}$ gives an isomorphism*

$$\bar{e} : M^{(0)} \oplus M^{(1)} \to (M \otimes V_n^*)^Y$$

as $GL(n-1)$-modules.

Proof. We first check that for $v \in M^{(0)} \oplus M^{(1)}$, $e(v)$ lies in $(M \otimes V_n^*)^Y$. We have to prove that $E_{i,n}^{(k)}e(v) = 0$ for any $1 \leq i < n$ and $k > 0$. We have

$$E_{i,n}^{(k)}\left(\sum_{h=1}^{n} E_{h,n}v \otimes f_h\right) = \sum_{h=1}^{n} E_{i,n}^{(k)}E_{h,n}v \otimes f_h + E_{i,n}^{(k-1)}E_{i,n}v \otimes E_{i,n}f_i,$$

which is clearly zero by weights if $k > 1$. For $k = 1$, it equals $E_{i,n}v \otimes f_n + E_{i,n}v \otimes (-f_n)$ which is again 0.

Applying Lemma 5.3, this shows that $\bar{e}$ is an injective $GL(n-1)$-homomorphism from $M^{(0)} \oplus M^{(1)}$ to $(M \otimes V_n^*)^Y$. To show that $\bar{e}$ is surjective take $w = \sum_{h=1}^n v_h \otimes f_h \in (M \otimes V_n^*)^Y$. We may assume that w is a weight vector of weight $\nu \in X(n)$. By Lemma 5.3, we can write $w = e(v)$ for some weight vector $v \in M$ of weight $\nu + \varepsilon_n$. We now prove by downward induction on the dominance order on ν that v lies in $M^{(0)} \oplus M^{(1)}$, to complete the proof.

If v is a $GL(n-1)$-primitive vector in M, then $e(v) = w$ is a $GL(n)$-primitive vector, since we have already shown that it is Y-invariant. So (5.1) and (5.2) imply that the weight ν of w is $\lambda - \varepsilon_j$ for some $1 \leq j \leq n$. But then, the weight of v is $\lambda - \varepsilon_j + \varepsilon_n$ so v lies in $M^{(0)} \oplus M^{(1)}$ as required.

Otherwise, if v is not $GL(n-1)$-primitive, then we can find $1 \leq i < j < n$ and $k > 0$ such that $E_{i,j}^{(k)} v \neq 0$. By induction $E_{i,j}^{(k)} v$ lies in $M^{(0)} \oplus M^{(1)}$ so v does too by weights.

Recall the notation $\bar{\lambda}$ from section 4.

5.5. Corollary. *For $\lambda, \mu \in X^+(n)$ and a submodule M of $\nabla_n(\lambda)$, the restriction of the map $\bar{e}$ from Proposition 5.4 gives a bijection between the $GL(n-1)$-primitive vectors in $M^{(0)} \oplus M^{(1)}$ of weight $\mu + \varepsilon_n$ and the $GL(n)$-primitive vectors in $M \otimes V_n^*$ of weight μ. In particular, if $\mu_n = \lambda_n$,*

$$\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), M \otimes V_n^*) \cong \mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\bar{\mu}), M^{(1)}).$$

Proof. The first statement follows immediately from Proposition 5.4, since a vector $v \in M \otimes V_n^*$ is $GL(n)$-primitive if and only if it is $GL(n-1)$-primitive and lies in $(M \otimes V_n^*)^Y$. If $\mu_n = \lambda_n$, a vector $v \in M^{(0)} \oplus M^{(1)}$ has $GL(n)$-weight $\mu + \varepsilon_n$ if and only if it lies in $M^{(1)}$ and has $GL(n-1)$-weight $\bar{\mu}$. So the second statement now follows using (5.2).

5.6. Remark. A functorial proof of the Hom-space isomorphism in Corollary 5.5 is given in [**3**, Corollary 2.10]. The elementary argument given here provides more information.

Now we can prove the main results of the section. In fact, using Corollary 5.5, these will follow from the following theorem first proved in [**15**, Theorem 4.2] (see also [**1**, Theorem 5.3]) concerning the branching rule from $GL(n)$ to $GL(n-1)$:

5.7. *Suppose $\lambda \in X^+(n)$, $\mu \in X^+(n-1)$. Then, the space* $\mathrm{Hom}_{GL(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)^{(1)})$ *is zero unless $\mu = \bar{\lambda} - \varepsilon_i$ for some i with $1 \leq i < n$ that is normal for λ, in which case it is one dimensional.*

Recall the combinatorial notions from section 2 and the notation $\lambda^* = (-\lambda_n, \dots, -\lambda_1)$ from section 4. For $1 \leq i \leq n$, we let $i^* := n + 1 - i$. Obviously, given $\lambda \in X^+(n)$, i is λ-removable if and only if i^* is λ^*-addable, and dually i is λ-addable if and only if i^* is λ^*-removable. We have the following combinatorial lemma:

5.8. Lemma. (i) *i is normal for λ if and only if i^* is conormal for λ^*;*

(ii) *i is good for λ if and only if i^* is cogood for λ^*;*

(iii) *i is good for λ if and only if i is normal for λ and conormal for $\lambda - \varepsilon_i$;*

(iii)$'$ *i is cogood for λ if and only if i is conormal for λ and normal for $\lambda + \varepsilon_i$.*

(iv) *i is cogood for λ if and only if i is λ-addable and i is good for $\lambda + \varepsilon_i$;*

(iv)$'$ *i is good for λ if and only if i is λ-removable and i is cogood for $\lambda - \varepsilon_i$.*

Proof. Note that the primed statements are equivalent to the unprimed statements, applying $*$. Also, parts (i) and (ii) follow immediately from the definitions.

Now consider (iii). We first prove that if i is both normal and conormal for λ then i is good for λ. Assume that i is normal for λ but not good for λ. Then, there is a j with $1 \leq j < i$, j normal for λ and $\mathrm{res}(i, \lambda_i) = \mathrm{res}(j, \lambda_j)$. Since j is normal, there is a decreasing injection from the set of

$$\lambda\text{-addable } k \text{ with } j < k < i \text{ and } \mathrm{res}(j, \lambda_j) = \mathrm{res}(k, \lambda_k + 1)$$

into the set of λ-removable k' with $j < k' < i, \mathrm{res}(j, \lambda_j) = \mathrm{res}(k', \lambda_{k'})$. So, there are strictly more

$$\lambda\text{-removable } h \text{ with } j \leq h < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(h, \lambda_h)$$

than there are

$$\lambda\text{-addable } h' \text{ with } j < h' < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(h', \lambda_{h'} + 1).$$

This means that there cannot be an increasing injection from the set of

$$\lambda\text{-removable } h \text{ with } 1 \leq h < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(h, \lambda_h)$$

into the set of

$$\lambda\text{-addable } h' \text{ with } 1 \leq h' < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(h', \lambda_{h'} + 1).$$

Hence, i is not conormal for $\lambda - \varepsilon_i$.

Conversely, if i is not normal for λ then it is not good, so suppose that i is normal for λ but not conormal for $\lambda - \varepsilon_i$. Then, we can find some λ-removable $j < i$ such that there is no increasing injection from the set of

$$\lambda\text{-removable } k \text{ with } j \leq k < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(k, \lambda_k)$$

into the set of

$$\lambda\text{-addable } k' \text{ with } j < k' < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(k', \lambda_{k'} + 1).$$

If we take the largest such j, then there must be an increasing *bijection* θ_1 from the set of

$$\lambda\text{-removable } k \text{ with } j < k < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(k, \lambda_k)$$

into the set of

$$\lambda\text{-addable } k' \text{ with } j < k' < i \text{ and } \mathrm{res}(i, \lambda_i) = \mathrm{res}(k', \lambda_{k'} + 1).$$

Since i is normal, there is a decreasing injection θ_2 from the set of

$$\lambda\text{-addable } k \text{ with } i < k \leq n \text{ and } \operatorname{res}(i, \lambda_i) = \operatorname{res}(k, \lambda_k + 1)$$

into the set of

$$\lambda\text{-removable } k' \text{ with } i < k' \leq n \text{ and } \operatorname{res}(i, \lambda_i) = \operatorname{res}(k', \lambda_{k'}).$$

Now combining θ_1^{-1} and θ_2 gives a decreasing injection θ from the set of

$$\lambda\text{-addable } k \text{ with } j < k \leq n \text{ and } \operatorname{res}(i, \lambda_i) = \operatorname{res}(k, \lambda_k + 1)$$

into the set of

$$\lambda\text{-removable } k' \text{ with } j < k' \leq n \text{ and } \operatorname{res}(i, \lambda_i) = \operatorname{res}(k', \lambda_{k'}).$$

Noting $\operatorname{res}(j, \lambda_j) = \operatorname{res}(i, \lambda_i)$, this shows that j is also normal for λ. Therefore i is not good for λ. This proves (iii) (hence (iii)$'$).

Now we deduce (iv). By (iii), i is λ-addable and good for $\lambda + \varepsilon_i$ if and only if i is conormal for λ and normal for $\lambda + \varepsilon_i$. By (iii)$'$, this is if and only if i is cogood for λ, as required for (iv).

5.9. Theorem. *Fix $\lambda, \mu \in X^+(n)$.*

(i) *The space $\operatorname{Hom}_{GL(n)}(\Delta_n(\mu), L_n(\lambda) \otimes V_n^*)$ is zero unless $\mu = \lambda - \varepsilon_i$ for some $1 \leq i \leq n$ that is normal for λ, in which case it is one dimensional.*

(ii) *The space $\operatorname{Hom}_{GL(n)}(L_n(\mu), \nabla_n(\lambda) \otimes V_n^*)$ is zero unless $\mu = \lambda - \varepsilon_i$ for some $1 \leq i \leq n$ that is conormal for $\lambda - \varepsilon_i$, in which case it is one dimensional.*

(iii) *The space $\operatorname{Hom}_{GL(n)}(L_n(\mu), L_n(\lambda) \otimes V_n^*)$ is zero unless $\mu = \lambda - \varepsilon_i$ for some $1 \leq i \leq n$ that is good for λ, in which case it is one dimensional.*

Proof. We note initially that by (5.1), we may assume that $\mu = \lambda - \varepsilon_i$ for some $1 \leq i \leq n$ that is λ-removable. Moreover, all three Hom spaces are certainly either zero or one dimensional. Part (i) is now immediate from (5.7) and Corollary 5.5 providing $i < n$. If $i = n$, note that $\mathrm{Hom}_{GL(n)}(\Delta_n(\lambda - \varepsilon_n), L_n(\lambda) \otimes V_n^*)$ is one dimensional as $\lambda - \varepsilon_n$ is the highest weight of $L_n(\lambda) \otimes V_n^*$, while n is always normal for λ according to the definition in section 2. Hence, (i) is true for all i. Part (ii) follows easily from (i) and Lemma 5.8(i), because $\mathrm{Hom}_{GL(n)}(L_n(\lambda - \varepsilon_i), \nabla_n(\lambda) \otimes V_n^*) \cong \mathrm{Hom}_{GL(n)}(\Delta_n(\lambda^*), L_n(\lambda^* + \varepsilon_{i^*}) \otimes V_n^*)$.

Finally, we prove (iii). By Lemma 5.8(iii), it suffices to show that

$$\mathrm{Hom}_{GL(n)}(L_n(\lambda - \varepsilon_i), L_n(\lambda) \otimes V_n^*)$$

is non-zero if and only if i is normal for λ and conormal for $\lambda - \varepsilon_i$. The forward implication is obvious because of (i) and (ii). For the converse, assume that i is normal for λ and conormal for $\lambda - \varepsilon_i$. Take a generator θ of the one dimensional space $\mathrm{Hom}_{GL(n)}(\Delta_n(\lambda - \varepsilon_i), \nabla_n(\lambda) \otimes V_n^*)$. Observe that by (ii), θ factors through the quotient $L_n(\lambda - \varepsilon_i)$ of $\Delta_n(\lambda - \varepsilon_i)$. Also, by (i), the image of θ lies in the subspace $L_n(\lambda) \otimes V_n^*$ of $\nabla_n(\lambda) \otimes V_n^*$. So θ induces a non-zero homomorphism from $L_n(\lambda - \varepsilon_i)$ to $L_n(\lambda) \otimes V_n^*$, as required.

Applying Proposition 4.7, we obtain the following complete reducibility criterion:

5.10. Corollary. *For $\lambda \in X^+(n)$, the module $L_n(\lambda) \otimes V_n^*$ is completely reducible if and only if every i with $1 \leq i \leq n$ that is normal for λ is good for λ.*

We also have the dual statements:

5.11. Theorem. *Fix $\lambda, \mu \in X^+(n)$.*

(i) *The space $\mathrm{Hom}_{GL(n)}(\Delta_n(\mu), L_n(\lambda) \otimes V_n)$ is zero unless $\mu = \lambda + \varepsilon_i$ for some $1 \leq i \leq n$ that is conormal for λ, in which case it is one dimensional.*

(ii) *The space* $\mathrm{Hom}_{GL(n)}(L_n(\mu), \nabla_n(\lambda) \otimes V_n)$ *is zero unless* $\mu = \lambda + \varepsilon_i$ *for some* $1 \leq i \leq n$ *that is normal for* $\lambda + \varepsilon_i$*, in which case it is one dimensional.*

(iii) *The space* $\mathrm{Hom}_{GL(n)}(L_n(\mu), L_n(\lambda) \otimes V_n)$ *is zero unless* $\mu = \lambda + \varepsilon_i$ *for some* $1 \leq i \leq n$ *that is cogood for* λ*, in which case it is one dimensional.*

Proof. Note that by contravariant duality,

$$\begin{aligned}\mathrm{Hom}_{GL(n)}(L_n(\mu), L_n(\lambda) \otimes V_n) &\cong \mathrm{Hom}_{GL(n)}(L_n(\lambda) \otimes V_n, L_n(\mu)) \\ &\cong \mathrm{Hom}_{GL(n)}(L_n(\lambda), L_n(\mu) \otimes V_n^*).\end{aligned}$$

So (iii) follows easily from Theorem 5.9(iii) and Lemma 5.8. Parts (i) and (ii) follow similarly from Theorem 5.9(ii) and (i) respectively.

Applying Proposition 4.7 again we deduce:

5.12. Corollary. *For* $\lambda \in X^+(n)$*, the module* $L_n(\lambda) \otimes V_n$ *is completely reducible if and only if every* i *with* $1 \leq i \leq n$ *that is conormal for* λ *is cogood for* λ*.*

6 A filtration of $L_n(\lambda) \otimes V_n^*$

We now construct a filtration of the module $L_n(\lambda) \otimes V_n^*$ by high weight modules. This will allow us in the first place to prove that certain composition multiplicities are zero.

Fix $\lambda \in X^+(n)$ and let w_λ (resp. v_λ) denote a non-zero high weight vector in $\Delta_n(\lambda)$ (resp. $L_n(\lambda)$) throughout the section. Let $r_a < \cdots < r_1 = n$ denote the set of all r with $1 \leq r \leq n$ which are λ-removable. Let $s_b < \cdots < s_1 = n$ denote the set of all s with $1 \leq s \leq n$ which are normal for λ.

6.1. Lemma. *The module* $\Delta_n(\lambda) \otimes V_n^*$ *is generated as a* $U^-(n)$*-module by the vectors*

$$\{w_\lambda \otimes f_{r_i} \mid 1 \leq i \leq a\}.$$

Proof. We first prove by induction on i that any vector $w \otimes f_i \in \Delta_n(\lambda) \otimes V_n^*$ lies in the $U^-(n)$-module generated by $w_\lambda \otimes f_1, \dots, w_\lambda \otimes f_i$. Since w_λ generates $\Delta_n(\lambda)$ as a $U^-(n)$-module, we can find $y \in U^-(n)$ such that $yw_\lambda = w$. Then,

$$y(w_\lambda \otimes f_i) = (yw_\lambda) \otimes f_i + X$$

where X is a linear combination of terms of the form $w' \otimes f_{i'}$ for $w' \in \Delta_n(\lambda)$ and $i' < i$. By induction, all such terms lie in the $U^-(n)$-submodule generated by $w_\lambda \otimes f_1, \dots, w_\lambda \otimes f_{i-1}$, hence $w \otimes f_i$ lies in the submodule generated by $w_\lambda \otimes f_1, \dots, w_\lambda \otimes f_i$ as required.

This shows that $\Delta_n(\lambda) \otimes V_n^*$ is generated by $\{w_\lambda \otimes f_i \mid 1 \leq i \leq n\}$. Now suppose that i is not λ-removable. Let j be minimal such that j is λ-removable with $j > i$. Since $\lambda_i = \lambda_j$, the weight $\lambda - \varepsilon_i + \varepsilon_j$ is not a weight of $\Delta_n(\lambda)$, so $F_{i,j} w_\lambda = 0$. Hence,

$$F_{i,j}(w_\lambda \otimes f_j) = w_\lambda \otimes (F_{i,j} f_j) = -w_\lambda \otimes f_i.$$

Hence, $w_\lambda \otimes f_i$ lies in the $U^-(n)$-submodule generated by $w_\lambda \otimes f_j$.

6.2. Lemma. *For $1 \leq i \leq a$, let $M_{(i)}$ be the $U^-(n)$-submodule of $\Delta_n(\lambda) \otimes V_n^*$ generated by $M_{(i-1)}$ and $w_\lambda \otimes f_{r_i}$. Then, each $M_{(i)}$ is $GL(n)$-stable and*

$$0 = M_{(0)} < M_{(1)} < \cdots < M_{(a)} = M$$

is a Δ-filtration of $M := \Delta_n(\lambda) \otimes V_n^*$ with $M_{(i)}/M_{(i-1)} \cong \Delta_n(\lambda - \varepsilon_{r_i})$ for each i. Moreover, the image of $w_\lambda \otimes f_{r_i}$ is a non-zero primitive vector in $M_{(i)}/M_{(i-1)}$.

Proof. We first prove by induction on $i \leq a$ that $w_\lambda \otimes f_{r_i} + M_{(i-1)}$ is either zero or a primitive vector in $M_{(i)}/M_{(i-1)}$, hence that each $M_{(i)}$ is a $GL(n)$-module. The induction starts trivially with $i = 0$. So take $i > 0$ and assume the claim for all smaller i. Moreover, if $M_{(i-1)} = M$ then $M_{(i)} = M_{(i-1)} = M$ and the result is trivial, so we can assume that

$M_{(i-1)} \neq M$. Observe then that by Lemma 6.1, $M/M_{(i-1)}$ is generated as a $U^-(n)$-module by the vectors $\{w_\lambda \otimes f_{r_j} + M_{(i-1)} \,|\, j \geq i\}$. If $w_\lambda \otimes f_{r_i} + M_{(i-1)}$ is zero then $M_{(i)} = M_{(i-1)}$ and the conclusion is immediate. So we may assume that $w_\lambda \otimes f_{r_i} \notin M_{(i-1)}$. But that means that $w_\lambda \otimes f_{r_i} + M_{(i-1)}$ is a non-zero vector of maximal weight $\lambda - \varepsilon_{r_i}$ in $M/M_{(i-1)}$. Hence, $w_\lambda \otimes f_{r_i}$ is a primitive vector as required.

Noting that $M_{(a)} = M$ by Lemma 6.1, we have now constructed a filtration

$$0 = M_{(0)} \leq \cdots \leq M_{(a)} = M$$

with each factor $M_{(i)}/M_{(i-1)}$ isomorphic to a (possible zero) quotient of $\Delta_n(\lambda - \varepsilon_{r_i})$. It just remains to show that in fact $M_{(i)}/M_{(i-1)} \cong \Delta_n(\lambda - \varepsilon_{r_i})$ for each i. This follows immediately by dimension from Lemma 4.8 or (5.1).

6.3. Theorem. *For $1 \leq i \leq b$, let $N_{(i)}$ be the $U^-(n)$-submodule of $L_n(\lambda) \otimes V_n^*$ generated by $N_{(i-1)}$ and $v_\lambda \otimes f_{s_i}$. Then, each $N_{(i)}$ is $GL(n)$-stable and*

$$0 = N_{(0)} < N_{(1)} < \cdots < N_{(b)} = N$$

is a filtration of $N := L_n(\lambda) \otimes V_n^*$ such that $N_{(i)}/N_{(i-1)}$ is a non-zero quotient of $\Delta_n(\lambda - \varepsilon_{s_i})$ for each i. Moreover, the image of $v_\lambda \otimes f_{r_i}$ is a non-zero primitive vector in $N_{(i)}/N_{(i-1)}$.

Proof. For $1 \leq i \leq a$, let $\bar{M}_{(i)}$ be the image of $M_{(i)}$ in the filtration of Lemma 6.2 under the quotient $\Delta_n(\lambda) \otimes V_n^* \to L_n(\lambda) \otimes V_n^*$. This gives a filtration

$$0 = \bar{M}_{(0)} \leq \bar{M}_{(1)} \leq \cdots \leq \bar{M}_{(a)} = N.$$

By contravariant duality, Corollary 4.6 and Theorem 5.9(i), $\bar{M}_{(i)} \neq \bar{M}_{(i-1)}$ if and only if r_i is normal for λ. The theorem follows immediately on deleting all trivial factors $\bar{M}_{(i)}/\bar{M}_{(i-1)}$ with r_i not normal for λ from this filtration.

Our first application of this high weight filtration is the following result, which is the analogue for tensor products of [**18**, Lemma 7.4]:

6.4. Lemma. *Let* $\mu \in X^+(n)$ *be any weight of the form* $\mu = \lambda - \varepsilon_j - \sum_{i=1}^{j-2} a_i(\varepsilon_i - \varepsilon_{i+1})$ *for some* $1 \leq j \leq n$ *and non-negative coefficients* a_i *not all of which are zero. Then,* $[L_n(\lambda) \otimes V_n^* : L_n(\mu)] = 0$.

Proof. Let $\kappa = \sum_{i=1}^{j-2} a_i(\varepsilon_i - \varepsilon_{i+1})$. Suppose that $L_n(\mu)$ is a composition factor of $L_n(\lambda) \otimes V_n^*$. Then, it is a composition factor of some factor $V := N_{(i)}/N_{(i-1)}$ of the high weight filtration constructed in Theorem 6.3, for some $1 \leq i \leq b$. Note that V is a high weight module of high weight $\lambda - \varepsilon_{s_i}$, generated by the primitive vector $v_\lambda \otimes f_{s_i} + N_{(i-1)}$. We can find a filtration $0 \leq V'' < V' \leq V$ such that $V'/V'' \cong L_n(\mu)$. Let $v \in V'$ be a vector of weight μ such that $v + V''$ is a non-zero primitive vector in $L_n(\mu)$.

Since $\mu = \lambda - \varepsilon_j - \kappa$ and V is generated as a $U^-(n)$-module by the vector $v_\lambda \otimes f_{s_i} + N_{(i-1)}$, which has weight $\lambda - \varepsilon_{s_i}$, we must have that $\lambda - \varepsilon_j - \kappa \leq \lambda - \varepsilon_{s_i}$. Hence, $0 \leq \kappa + \varepsilon_j - \varepsilon_{s_i}$ so that $j \leq s_i$ by the definition of κ. We can find operators y_1 and y_2 lying in the $-(\varepsilon_j - \varepsilon_{s_i})$- and $-\kappa$-weight spaces of $U^-(n)$ respectively, such that

$$v = y_2 y_1 (v_\lambda \otimes f_{s_i}) + N_{(i-1)}.$$

Observe that y_1 and y_2 commute and so

$$v = y_2 y_1 (v_\lambda \otimes f_{s_i}) + N_{(i-1)} = y_1(y_2(v_\lambda \otimes f_{s_i})) + N_{(i-1)} = y_1((y_2 v_\lambda) \otimes f_{s_i}) + N_{(i-1)}.$$

Since this is non-zero, $y_2 v_\lambda$ must be a non-zero vector in $L_n(\lambda)$, so we can find x_2 lying in the κ-weight space of $U^+(n)$ such that $x_2 y_2 v_\lambda = v_\lambda$. Since v is a primitive vector modulo V'' and $\kappa \neq 0$, $x_2 v \in V''$. But x_2 and y_1 commute, so

$$x_2 v = x_2 y_1((y_2 v_\lambda) \otimes f_{s_i}) + N_{(i-1)} = y_1((x_2 y_2 v_\lambda) \otimes f_{s_i}) + N_{(i-1)} = y_1(v_\lambda \otimes f_{s_i}) + N_{(i-1)}.$$

But then $y_2 x_2 v = v$, which contradicts the fact that $x_2 v \in V''$.

We formulate the dual statement for completeness:

6.5. Corollary. *Let $\mu \in X^+(n)$ be any weight of the form $\mu = \lambda + \varepsilon_j - \sum_{i=j+1}^{n-1} a_i(\varepsilon_i - \varepsilon_{i+1})$ for some $1 \leq j \leq n$ and non-negative coefficients a_i not all of which are zero. Then, $[L_n(\lambda) \otimes V_n : L_n(\mu)] = 0$.*

Proof. Given a $GL(n)$-module M, let $d(M)$ denote the $GL(n)$-module M but with action twisted by the automorphism $g \mapsto (g^{-1})^t$ of $GL(n)$. Note that $d(L_n(\lambda)) = L_n(\lambda^*)$ and $d(V_n) = V_n^*$. Now

$$[L_n(\lambda) \otimes V_n : L_n(\mu)] = [d(L_n(\lambda) \otimes V_n) : d(L_n(\mu))] = [L_n(\lambda^*) \otimes V_n^* : L_n(\mu^*)],$$

and the corollary follows immediately from Lemma 6.4.

7 Some composition multiplicities in $L_n(\lambda) \otimes V_n^*$

We calculate the multiplicity $[L_n(\lambda) \otimes V_n^* : L_n(\lambda - \varepsilon_i)]$ for any $\lambda \in X^+(n)$ and any $1 \leq i \leq n$. To do this, we will relate the latter multiplicities to branching multiplicities in the first level $L_n(\lambda)^{(1)}$ of the restriction of $L_n(\lambda)$ to $GL(n-1)$, as defined in section 5. The latter multiplicities are known from [**18**, Corollary 9.4(ii)]:

7.1. *Fix $\lambda \in X^+(n)$ and any λ-removable i with $1 \leq i < n$. Then, $[L_n(\lambda)^{(1)} : L_{n-1}(\bar{\lambda} - \varepsilon_i)]$ is zero unless i is normal for λ, in which case it equals the number of j with $i \leq j < n$ such that j is normal for λ and $\mathrm{res}(i, \lambda_i) = \mathrm{res}(j, \lambda_j)$.*

We let Y denote the subgroup of $GL(n)$ generated by root subgroups corresponding to the roots $\varepsilon_1 - \varepsilon_n, \ldots, \varepsilon_{n-1} - \varepsilon_n$ as in section 5. As noted earlier, taking Y-fixed points gives a functor from the category of $GL(n)$-modules to the category of $GL(n-1)$-modules. We state [**12**, II.2.11]:

7.2. *For $\lambda \in X^+(n)$, $L_{n-1}(\bar{\lambda}) \cong L_n(\lambda)^Y = \bigoplus_\nu L_n(\lambda)_\nu$ where the sum is over all $\nu \in X(n)$ with $\nu_n = \lambda_n$.*

We say a $GL(n)$-module M is *polynomial of degree r* if all non-zero weight spaces of M are of weight $\mu \in X(n)$ with $|\mu| = r$ and $\mu_i \geq 0$ for all $1 \leq i \leq n$.

7.3. LEMMA. *Fix $\lambda \in X^+(n)$ with $\lambda_n = 0$. Let M be a finite dimensional $GL(n)$-module that is polynomial of degree $r := |\lambda|$. Then,*

$$[M : L_n(\lambda)] = [M^Y : L_{n-1}(\bar{\lambda})].$$

Proof. Let $0 = M_{(0)} < \cdots < M_{(s)} = M$ be a composition series of M. We prove by induction on i that $[M_{(i)} : L_n(\lambda)] = [M^Y_{(i)} : L_{n-1}(\bar{\lambda})]$, starting from $i = 0$. Assume the result is true for $i-1$. Let $M_{(i)}/M_{(i-1)} \cong L_n(\mu)$ for $\mu \in X^+(n)$. Since $|\mu| = r = |\lambda|$, we note that $\mu = \lambda$ if and only if $\bar{\mu} = \bar{\lambda}$. We have the natural exact sequence

$$0 \to M^Y_{(i-1)} \to M^Y_{(i)} \to (M_{(i)}/M_{(i-1)})^Y.$$

By (7.2), $(M_{(i)}/M_{(i-1)})^Y \cong L_n(\mu)^Y \cong L_{n-1}(\bar{\mu})$, so the exact sequence implies

$$[M^Y_{(i-1)} : L_{n-1}(\bar{\lambda})] \leq [M^Y_{(i)} : L_{n-1}(\bar{\lambda})] \leq [M^Y_{(i-1)} : L_{n-1}(\bar{\lambda})] + [L_{n-1}(\bar{\mu}) : L_{n-1}(\bar{\lambda})].$$

To prove the induction step, we need to show that right hand inequality is always equality. This is obvious if $\bar{\mu} \neq \bar{\lambda}$, or equivalently if $\mu \neq \lambda$. So suppose that $\mu = \lambda$. Then we need to show that the right hand map in the above exact sequence is surjective. Take $v + M_{(i-1)} \in (M_{(i)}/M_{(i-1)})^Y$. By (7.2)

$$L_n(\lambda)^Y = \bigoplus_\nu L_n(\lambda)_\nu,$$

where the sum is over $\nu \in X(n)$ with $\nu_n = 0$. So we may assume that v is a weight vector of weight ν with $\nu_n = 0$. But then $v \in M_{(i)}$ is Y-invariant by weights. So $v \in M_{(i)}^Y$, as required for surjectivity.

Recall the equivalence relation $\sim$ from section 2. Now we can relate $[L_n(\lambda) \otimes V_n^* : L_n(\mu)]$ to branching multiplicities in the first level in almost all cases:

7.4. Lemma. *Fix $\lambda, \mu \in X^+(n)$ with $\mu_n = \lambda_n = 0$ and $|\mu| = |\lambda| - 1$. Suppose that $\mu \not\sim \lambda - \varepsilon_n$. Then,*

$$[L_n(\lambda) \otimes V_n^* : L_n(\mu)] = [L_n(\lambda)^{(1)} : L_{n-1}(\bar{\mu})].$$

Proof. Let $N = L_n(\lambda) \otimes V_n^*$. By the linkage principle, we may write $N = N' \oplus N''$ for unique submodules N', N'' such that all composition factors of N' (resp. N'') are of the form $L_n(\nu)$ for $\nu \sim \lambda - \varepsilon_n$ (resp. $\nu \not\sim \lambda - \varepsilon_n$). Note that $[N : L_n(\mu)] = [N'' : L_n(\mu)]$ since $\mu \not\sim \lambda - \varepsilon_n$. By Proposition 5.4, $N^Y \cong L_n(\lambda)^{(0)} \oplus L_n(\lambda)^{(1)}$. But $L_{n-1}(\bar{\mu})$ is not a composition factor of $L_n(\lambda)^{(0)}$ by degree. So,

$$[L_n(\lambda)^{(1)} : L_{n-1}(\bar{\mu})] = [N^Y : L_{n-1}(\bar{\mu})] = [(N')^Y : L_{n-1}(\bar{\mu})] + [(N'')^Y : L_{n-1}(\bar{\mu})].$$

As $L_n(\mu)$ is not a composition factor of N' and $|\nu| = |\lambda| - 1 = |\mu|$ for any composition factor $L_n(\nu)$ of N', (7.2) implies that $[(N')^Y : L_{n-1}(\bar{\mu})] = 0$. It just remains to show that $[N'' : L_n(\mu)] = [(N'')^Y : L_{n-1}(\bar{\mu})]$. This will follow immediately from Lemma 7.3 once we have shown that N'' is polynomial.

Note that the submodule $N_{(1)}$ of N in the filtration of Theorem 6.3 is a quotient of $\Delta_n(\lambda - \varepsilon_n)$, so $N_{(1)} \leq N'$ and N'' is a quotient of $N/N_{(1)}$. By Theorem 6.3, $N/N_{(1)}$ has a filtration by high weight modules all of whose high weights are of the form $\lambda - \varepsilon_i$ for λ-removable $i < n$. Hence, $N/N_{(1)}$ is polynomial, so N'' is too.

To give a similar result in the case that $\mu \sim \lambda - \varepsilon_n$, we need to allow n to vary. Recall the notation $\tilde{\lambda}$ from section 4.

7.5. LEMMA. *Fix $\lambda, \mu \in X^+(n)$. Then,*

$$[L_n(\lambda) \otimes V_n^* : L_n(\mu)] = [L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^* : L_{n+1}(\tilde{\mu})] \\ + [L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)} : L_n(\mu)] - [L_{n+1}(\tilde{\lambda})^{(1)} : L_n(\mu)].$$

Proof. Let S denote the exact functor from the category of $GL(n+1)$-modules to the category of $GL(n)$-modules defined by sending a $GL(n+1)$-module N to the $GL(n)$-submodule of N consisting of the sum of all weight spaces N_γ with $\gamma_{n+1} = 0$, and by restriction on morphisms. Let $N = L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^*$. We compute $[S(N) : L_n(\mu)]$ in two different ways.

First, take a composition series $0 = N^{(0)} < N^{(1)} < \cdots < N^{(s)} = N$ for N. As S is exact,

$$[S(N) : L_n(\mu)] = \sum_{i=1}^{s} [S(N_{(i)}/N_{(i-1)}) : L_n(\mu)].$$

Observe that $L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})$ appears in the composition series precisely once as $\tilde{\lambda} - \varepsilon_{n+1}$ is the highest weight of N. Moreover, $S(L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1}))$ is the first level $L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)}$ of $L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})$. So, this unique composition factor $L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})$ of N contributes $[L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)} : L_n(\mu)]$ to the right hand side of this expression. By Lemma 6.4, all *other* composition factors of N are of the form $L_{n+1}(\nu)$ for $\nu \in X^+(n+1)$ with $\nu_{n+1} \geq 0$. For such ν, $S(L_{n+1}(\nu))$ is nonzero only if $\nu_{n+1} = 0$, in which case it is $L_n(\bar{\nu})$ by (7.2). Hence $[S(L_{n+1}(\nu)) : L_n(\mu)]$ is zero unless $\nu = \tilde{\mu}$, when it is one. So we obtain a total contribution of $[N : L_{n+1}(\tilde{\mu})]$ to $[S(N) : L_n(\bar{\mu})]$ from such composition factors. We have shown:

$$[S(N) : L_n(\mu)] = [L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^* : L_{n+1}(\tilde{\mu})] + [L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)} : L_n(\mu)].$$

Second, we note $N = L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^* \cong L_{n+1}(\tilde{\lambda}) \otimes V_n^* \oplus L_{n+1}(\tilde{\lambda}) \otimes V_1^*$ as a $GL(n)$-module, where $V_{n+1}^* = V_n^* \oplus V_1^*$ is a decomposition of V_{n+1}^* with respect to $GL(n)$. So, $S(N)$, which just amounts to picking certain weight spaces, is equal to $(L_{n+1}(\tilde{\lambda})^{(0)} \otimes V_n^*) \oplus L_{n+1}(\tilde{\lambda})^{(1)}$. So using (7.2) again,

$$[S(N) : L_n(\mu)] = [L_n(\lambda) \otimes V_n^* : L_n(\mu)] + [L_{n+1}(\tilde{\lambda})^{(1)} : L_n(\mu)].$$

Comparing these two different expressions for $[S(N) : L_n(\mu)]$ gives the result.

7.6. Proposition. *Fix $\lambda, \mu \in X^+(n)$ with $\lambda_n = \mu_n = 0$. Then,*

$$[L_n(\lambda) \otimes V_n^* : L_n(\mu)] = \begin{cases} [L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)} : L_n(\mu)] & \text{if } \mu \sim \lambda - \varepsilon_n. \\ [L_n(\lambda)^{(1)} : L_{n-1}(\bar{\mu})] & \text{if } \mu \not\sim \lambda - \varepsilon_n \end{cases}$$

Proof. If $\mu \not\sim \lambda - \varepsilon_n$, this is just Lemma 7.4. So suppose that $\mu \sim \lambda - \varepsilon_n$. By Lemma 7.5,

$$\begin{aligned}[L_n(\lambda) \otimes V_n^* : L_n(\mu)] = {} & [L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^* : L_{n+1}(\tilde{\mu})] \\ & + [L_{n+1}(\tilde{\lambda} - \varepsilon_{n+1})^{(1)} : L_n(\mu)] - [L_{n+1}(\tilde{\lambda})^{(1)} : L_n(\mu)].\end{aligned}$$

We can compute the first term on the right hand side using Lemma 7.4, as $\mathrm{res}(n, \tilde{\lambda}_n) \neq \mathrm{res}(n+1, \tilde{\lambda}_{n+1})$. Hence, $[L_{n+1}(\tilde{\lambda}) \otimes V_{n+1}^* : L_{n+1}(\tilde{\mu})] = [L_{n+1}(\tilde{\lambda})^{(1)} : L_n(\mu)]$. Now the right hand side simplifies to give the required result.

Now we can deduce the main result of the section:

7.7. Theorem. *Fix $\lambda \in X^+(n)$ and a λ-removable i with $1 \leq i \leq n$. Then,*

$$[L_n(\lambda) \otimes V_n^* : L_n(\lambda - \varepsilon_i)]$$

is zero unless i is normal for λ, when it equals the number of j with $i \leq j \leq n$ such that j is normal for λ and $\mathrm{res}(i, \lambda_i) = \mathrm{res}(j, \lambda_j)$.

Proof. By tensoring with a power of determinant if necessary, we may assume that $\lambda_n = 0$. If $i = n$, the composition multiplicity is always 1, and the statement is true. So we also assume $1 \leq i < n$. Now use Proposition 7.6 together with (7.1) and the combinatorial definition of normal.

8 The endomorphism ring of $L_n(\lambda) \otimes V_n^*$

Now we describe the endomorphism ring of $L_n(\lambda) \otimes V_n^*$ using Casimir-type operators. This gives further information about the structure of the high weight filtration of Lemma 6.2.

Throughout the section, fix $\lambda \in X^+(n)$ and let $\alpha_i \in \mathbb{Z}/p\mathbb{Z}$ denote the residue $\mathrm{res}(i, \lambda_i)$, for $1 \leq i \leq n$. Let

$$Z_n(\lambda) := X_n + \sum_{1 \leq i < j \leq n} [(H_i - i)(H_j - j) - (\lambda_i - i)(\lambda_j - j)] + \sum_{1 \leq i \leq n} (\lambda_i - i) \tag{2}$$

where

$$X_n := - \sum_{1 \leq i < j \leq n} F_{i,j} E_{i,j}.$$

This element $Z_n(\lambda)$ differs from the central element C_2 constructed in [**5**, 2.2] by a scalar, so $Z_n(\lambda)$ lies in the centre of $U(n)$.

8.1. Lemma. *$Z_n(\lambda)v = (X_n + \alpha_i)v$, for any vector v of weight $\lambda - \varepsilon_i$ (in some $GL(n)$-module) and $1 \leq i \leq n$.*

Proof. It suffices to check that

$$\sum_{1 \leq h < k \leq n} [(\lambda_h - \delta_{hi} - h)(\lambda_k - \delta_{ki} - k) - (\lambda_h - h)(\lambda_k - k)] + \sum_{1 \leq h \leq n} (\lambda_h - h) = \lambda_i - i$$

which is routine.

8.2. Lemma. *Fix $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Suppose M is a $GL(n)$-module with a filtration*

$$0 = M_{(0)} < M_{(1)} < \cdots < M_{(a)} = M$$

such that each factor $M_{(i)}/M_{(i-1)}$ is a quotient of $\Delta_n(\lambda - \varepsilon_k)$ for some $1 \leq k \leq n$ with $\alpha_k = \alpha$. Then, for any $1 \leq j \leq i \leq a$, $(Z_n(\lambda) - \alpha)^j M_{(i)} \subseteq M_{(i-j)}$. In particular, $(Z_n(\lambda) - \alpha)^a M = 0$.

Proof. The general case follows by induction from the special case $i = j = 1$. So suppose $i = j = 1$ and let $v \in M_{(1)}$ be a non-zero primitive vector of weight $\lambda - \varepsilon_k$. Since $\alpha_k = \alpha$, Lemma 8.1 implies that $(Z_n(\lambda) - \alpha)v = X_n v$, which is zero as v is primitive. Since v generates $M_{(1)}$ and $(Z_n(\lambda) - \alpha)$ lies in the centre of $U(n)$, it follows that $(Z_n(\lambda) - \alpha)M_{(1)} = 0$ as required.

Now let $n = s_1 > \cdots > s_b$ denote the set of all j which are normal for λ with $1 \leq j \leq n$. Let $N = L_n(\lambda) \otimes V_n^*$ and fix a non-zero primitive vector v_λ in $L_n(\lambda)$. Let

$$0 = N_{(0)} < N_{(1)} < \cdots < N_{(b)} = N \tag{3}$$

be the high weight filtration of Theorem 6.3, where $N_{(i)}/N_{(i-1)}$ is a (nonzero) high weight module generated by the image of $v_\lambda \otimes f_{s_i}$. For $\alpha \in \mathbb{Z}/p\mathbb{Z}$, let

$$s_1^\alpha > \cdots > s_{b_\alpha}^\alpha$$

denote the set of all s_j with $1 \leq j \leq b$ and $\alpha_{s_j} = \alpha$. Note that given any $1 \leq i, j \leq n$, $\lambda - \varepsilon_i \sim \lambda - \varepsilon_j$ if and only if $\alpha_i = \alpha_j$. So by the linkage principle, we can decompose N as

$$N = \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} N^\alpha$$

where N^α is the largest submodule of N all of whose composition factors are of the form $L_n(\nu)$ with $\mathrm{cont}_\alpha(\nu) = \mathrm{cont}_\alpha(\lambda) - 1$ and $\mathrm{cont}_\gamma(\nu) = \mathrm{cont}_\gamma(\lambda)$ for $\gamma \neq \alpha$. The high weight filtration of N in (3) yields a high weight filtration of N^α,

$$0 = N^\alpha_{(0)} < N^\alpha_{(1)} < \cdots < N^\alpha_{(b_\alpha)} = N^\alpha \tag{4}$$

such that $N^\alpha_{(i)}/N^\alpha_{(i-1)}$ is a non-zero quotient of $\Delta_n(\lambda - \varepsilon_{s_i^\alpha})$ and $N^\alpha_{(i)}/N^\alpha_{(i-1)} \cong N_{(j)}/N_{(j-1)}$, where j is uniquely determined by $s_j = s_i^\alpha$. Indeed, if π^α is the projection from N onto N^α then $N^\alpha_{(i)}$ is just $\pi^\alpha(N_{(j)})$.

8.3. LEMMA. *For any $\alpha \in \mathbb{Z}/p\mathbb{Z}$,*

(i) $(Z_n(\lambda) - \alpha)^{b_\alpha} N^\alpha = 0$*;*

(ii) $(Z_n(\lambda) - \gamma)N^\alpha = N^\alpha$ *for any* $\alpha \neq \gamma \in \mathbb{Z}/p\mathbb{Z}$*;*

(iii) $N^\alpha = \prod_{\gamma\neq\alpha}(Z_n(\lambda) - \gamma)^{b_\gamma} N$*.*

(iv) $N^\alpha_{(i)}/N^\alpha_{(i-1)}$ *is generated as a high weight by the primitive vector*

$$\prod_{\gamma\neq\alpha}(Z_n(\lambda) - \gamma)^{b_\gamma}(v_\lambda \otimes f_{s_i^\alpha}) + N^\alpha_{(i-1)}.$$

Proof. Part (i) follows immediately from Lemma 8.2. For (ii), suppose $\gamma \neq \alpha$. Then $(Z_n(\lambda) - \gamma) = (Z_n(\lambda) - \alpha) + (\alpha - \gamma)$. Since $(Z_n(\lambda) - \alpha)$ induces a nilpotent endomorphism of N^α by (i) and $\alpha - \gamma \neq 0$, it follows that $(Z_n(\lambda) - \gamma)$ induces an automorphism of N^α. For (iii), note that by (i), $\prod_{\gamma\neq\alpha}(Z_n(\lambda) - \gamma)^{b_\gamma}$ annihilates all blocks N_γ of N for $\gamma \neq \alpha$, while by (ii) it induces an automorphism of N^α. Finally, (iv) follows from (iii) and the description of $N^\alpha_{(i)}$ in the paragraph preceding this lemma.

The same argument applies to the $GL(n-1)$-module $Q := L_n(\lambda)^{(1)}$. Let $t_1 > \cdots > t_c$ denote the set of all j which are normal for λ with $1 \leq j < n$ (actually, $c = b - 1$, $t_i = s_{i+1}$, $i = 1, \dots, b-1$). Then, by [**4**, Lemma 3.7, Proposition A.2], Q has a high weight filtration

$$0 = Q_{(0)} < Q_{(1)} < \cdots < Q_{(c)} = Q \tag{5}$$

where $Q_{(i)}/Q_{(i-1)}$ is a (non-zero) high weight module generated by the image of $F_{t_i,n}v_\lambda$, which is a primitive vector in $Q_{(i)}/Q_{(i-1)}$ of weight $\lambda - \varepsilon_{t_i} + \varepsilon_n$. For $\alpha \in \mathbb{Z}/p\mathbb{Z}$, let

$$t_1^\alpha > \cdots > t_{c_\alpha}^\alpha$$

denote the set of all t_j with $1 \leq j \leq c$ and $\alpha_{t_j} = \alpha$. By the linkage principle for $GL(n-1)$, we can decompose Q as

$$Q = \bigoplus_{\alpha\in\mathbb{Z}/p\mathbb{Z}} Q^\alpha$$

where Q^α is the largest submodule of Q all of whose composition factors are of the form $L_{n-1}(\nu)$ with $\mathrm{cont}_\alpha(\nu) = \mathrm{cont}_\alpha(\lambda) - 1$ and $\mathrm{cont}_\gamma(\nu) = \mathrm{cont}_\gamma(\lambda)$ for $\gamma \neq \alpha$. Now we can state the following result proved in [**17**, Theorem 2.14] (the element ϕ_α defined here is precisely the element φ_Γ defined in [**17**, Definition 2.9], as follows from the definitions using Lemma 8.1):

8.4. *For any $\alpha \in \mathbb{Z}/p\mathbb{Z}$, let ϕ_α denote the image of $(Z_{n-1}(\bar{\lambda}) - \alpha)$ in $\mathrm{End}_{\mathbb{F}}(Q^\alpha)$. Then, the endomorphisms*

$$\{1, \phi_\alpha, \dots, (\phi_\alpha)^{c_\alpha - 1}\}$$

form a basis for $\mathrm{End}_{GL(n-1)}(Q^\alpha)$.

To deduce the analogous result for N^α, we need one computational lemma:

8.5. Lemma. *Let $e : L_n(\lambda) \to L_n(\lambda) \otimes V_n^*$ be the map defined in Lemma 5.3.*

(i) *$e(Z_{n-1}(\bar{\lambda})v) = Z_n(\lambda)e(v)$, for any $v \in L_n(\lambda)^{(0)} \oplus L_n(\lambda)^{(1)}$ of weight $\lambda - \varepsilon_i + \varepsilon_n$ and $1 \leq i < n$.*

(ii) *$e(F_{i,n}v_\lambda) = (Z_n(\lambda) - \alpha_n)(v_\lambda \otimes f_i)$, for $1 \leq i < n$.*

Proof. (i) Note that $X_n = X_{n-1} - \sum_{1 \leq i < n} F_{i,n}E_{i,n}$. Now e is a $GL(n-1)$-map, so X_{n-1} commutes with e, while every $E_{i,n}$ annihilates $e(v)$ as $e(v)$ is fixed by the unipotent radical Y from Proposition 5.4. Hence, $e(X_{n-1}v) = X_n e(v)$. Now, to prove (i) note that if v has $GL(n)$-weight $\lambda - \varepsilon_i + \varepsilon_n$, then $e(v)$ has weight $\lambda - \varepsilon_i$. So, applying Lemma 8.1 twice,

$$Z_n(\lambda)e(v) = X_n e(v) + \alpha_i e(v) = e((X_{n-1} + \alpha_i)v) = e(Z_{n-1}(\bar{\lambda})v),$$

as required.

(ii) We calculate

$$e(F_{i,n}v_\lambda) = \sum_{j=i}^{n} (E_{j,n}F_{i,n}v_\lambda) \otimes f_j = (\lambda_i - \lambda_n)v_\lambda \otimes f_j + \sum_{j=i+1}^{n} (F_{i,j}v_\lambda) \otimes f_j,$$

and using Lemma 8.1,

$$\begin{aligned}(Z_n(\lambda)-\alpha_n)(v_\lambda\otimes f_i) &= (\alpha_i-\alpha_n)(v_\lambda\otimes f_i)+X_n(v_\lambda\otimes f_i)\\ &= (\alpha_i-\alpha_n)(v_\lambda\otimes f_i)+\sum_{j=i+1}^{n}F_{i,j}(v_\lambda\otimes f_j)\\ &= (\alpha_i-\alpha_n)(v_\lambda\otimes f_i)+\sum_{j=i+1}^{n}((F_{i,j}v_\lambda)\otimes f_j-v_\lambda\otimes f_i)\\ &= (\alpha_i-\alpha_n-n+i)(v_\lambda\otimes f_i)+\sum_{j=i+1}^{n}(F_{i,j}v_\lambda)\otimes f_j.\end{aligned}$$

The result follows since $\alpha_i+i-\alpha_n-n=\lambda_i-\lambda_n$ in $\mathbb{F}$.

8.6. THEOREM. *For any $\alpha\in\mathbb{Z}/p\mathbb{Z}$, let ψ_α denote the image of $(Z_n(\lambda)-\alpha)$ in $\mathrm{End}_{\mathbb{F}}(N^\alpha)$. Then, the endomorphisms*

$$\{1,\psi_\alpha,\ldots,(\psi_\alpha)^{b_\alpha-1}\}$$

form a basis for $\mathrm{End}_{GL(n)}(N^\alpha)$, and $(\psi_\alpha)^{b_\alpha}=0$.

Proof. We first show that $\{1,\psi_\alpha,\ldots,(\psi_\alpha)^{b_\alpha-1}\}$ are linearly independent endomorphisms. This is obvious if $b_\alpha\le 1$, so we assume that $b_\alpha>1$. Note first that the argument of Lemma 8.3(iii) applies equally well to the $GL(n-1)$-module Q, to show that:

$$Q^\alpha=\prod_{\gamma\neq\alpha}(Z_{n-1}(\bar\lambda)-\gamma)^{c_\gamma}Q.$$

Consequently, the high weight filtration of Q in (5) yields a high weight filtration of Q^α,

$$0=Q^\alpha_{(0)}<Q^\alpha_{(1)}<\cdots<Q^\alpha_{(c_\alpha)}=Q^\alpha \tag{6}$$

where $Q^\alpha_{(i)}/Q^\alpha_{(i-1)}$ is a high weight module generated by the image of

$$\prod_{\gamma\neq\alpha}(Z_{n-1}(\bar\lambda)-\gamma)^{c_\gamma}F_{t_i^\alpha,n}v_\lambda,$$

precisely as in Lemma 8.3(iv). By (8.4), $(Z_{n-1}(\bar{\lambda}) - \alpha)^{c_\alpha - 1} Q^\alpha \neq 0$. However, $(Z_{n-1}(\bar{\lambda}) - \alpha)^{c_\alpha - 1}$ annihilates the submodule $Q^\alpha_{(c_\alpha - 1)}$ of Q^α by Lemma 8.2. Hence,

$$(Z_{n-1}(\bar{\lambda}) - \alpha)^{c_\alpha - 1} \prod_{\gamma \neq \alpha} (Z_{n-1}(\bar{\lambda}) - \gamma)^{c_\gamma} F_{t,n} v_\lambda \neq 0$$

where $t = t^\alpha_{c_\alpha}$. By definition, $b_\alpha = c_\alpha$ unless $\alpha = \alpha_n$, when $b_\alpha = c_\alpha + 1$. So, on applying the injective map e to this non-zero vector, Lemma 8.5 implies:

$$(Z_n(\lambda) - \alpha)^{b_\alpha - 1} \prod_{\gamma \neq \alpha} (Z_n(\lambda) - \gamma)^{b_\gamma} v_\lambda \otimes f_t \neq 0.$$

By Lemma 8.3(iii), this shows that $(\psi_\alpha)^{b_\alpha - 1} \neq 0$. But by Lemma 8.3(i), $(\psi_\alpha)^{b_\alpha} = 0$, so $1, \psi_\alpha, \dots, \psi_\alpha^{b_\alpha - 1}$ are certainly linearly independent endomorphisms.

It now just remains to check that $\dim \mathrm{End}_{GL(n)}(N^\alpha) = b_\alpha$ for all $\alpha \in \mathbb{Z}/p\mathbb{Z}$. Since

$$\mathrm{End}_{GL(n)}(N) \cong \bigoplus_{\alpha \in \mathbb{Z}/p\mathbb{Z}} \mathrm{End}_{GL(n)}(N^\alpha),$$

it suffices to show that $\dim \mathrm{End}_{GL(n)}(N) \leq b$. Let M denote $\Delta_n(\lambda) \otimes V_n^*$, with the notation of Lemma 6.2 for the Δ-filtration of M constructed there. Then,

$$\dim \mathrm{End}_{GL(n)}(N) \leq \dim \mathrm{Hom}_{GL(n)}(M, N) \leq \sum_{i=1}^{a} \dim \mathrm{Hom}_G(\Delta_n(\lambda - \varepsilon_{r_i}), N).$$

The right hand side is precisely the number b of j with $1 \leq j \leq n$ that are normal for λ, thanks to Theorem 5.9(i). This completes the proof.

8.7. Corollary. (i) *The dimension of* $\mathrm{End}_{GL(n)}(L_n(\lambda) \otimes V_n^*)$ *is equal to the number of* j *with* $1 \leq j \leq n$ *which are normal for* λ.

(ii) *The dimension of* $\mathrm{End}_{GL(n)}(L_n(\lambda) \otimes V_n)$ *is equal to the number of* j *with* $1 \leq j \leq n$ *which are conormal for* λ.

(iii) *The number of* j *with* $1 \leq j \leq n$ *that are normal for* λ *is equal to the number of* k *with* $1 \leq k \leq n$ *which are conormal for* λ.

Proof. Part (i) follows immediately Theorem 8.6. Note that

$$\mathrm{End}_{GL(n)}(L_n(\lambda) \otimes V_n) \cong \mathrm{End}_{GL(n)}(L_n(\lambda^*) \otimes V_n^*).$$

Now apply (i) and Lemma 5.8(i) to deduce (ii). For (iii) use the isomorphism

$$\mathrm{End}_{GL(n)}(L_n(\lambda) \otimes V_n^*) \cong \mathrm{End}_{GL(n)}(L_n(\lambda) \otimes V_n)$$

together with (i) and (ii).

8.8. Remark. We do not know a combinatorial proof of the purely combinatorial statement in Corollary 8.7(iii).

In (iv) of the next theorem, we refer to the Loewy length of a finite dimensional $GL(n)$-module M. We recall briefly the definition. We let $0 = S_0(M) \le S_1(M) \le \dots$ denote the *socle series* of M, and $M = J^0(M) \ge J^1(M) \ge \dots$ denote the *radical series* of M as in [**19**, section I.8]. Then, the *Loewy length* of M is the smallest ℓ such that $S_\ell(M) = M$; by [**19**, I.8.2(i)], this is also the smallest ℓ such that $J^\ell(M) = 0$.

8.9. Theorem. *Fix $\alpha \in \mathbb{Z}/p\mathbb{Z}$ and let $0 = N^\alpha_{(0)} < N^\alpha_{(1)} < \dots < N^\alpha_{(b_\alpha)} = N^\alpha$ be the high weight filtration of (4).*

(i) *For $1 \le i \le k \le b_\alpha$, the vector $(Z_n(\lambda) - \alpha)^{k-i} \prod_{\gamma \neq \alpha} (Z_n(\lambda) - \gamma)^{b_\gamma} (v_\lambda \otimes f_{s^\alpha_k}) + N^\alpha_{(i-1)}$ is a non-zero primitive vector in $N^\alpha_{(i)}/N^\alpha_{(i-1)}$ of weight $\lambda - \varepsilon_{s^\alpha_k}$.*

(ii) *for $1 \le i \le k \le b_\alpha$,*

$$\dim \mathrm{Hom}_{GL(n)}(N^\alpha_{(k)}/N^\alpha_{(k-1)}, N^\alpha_{(i)}/N^\alpha_{(i-1)}) = [N^\alpha_{(i)}/N^\alpha_{(i-1)} : L_n(\lambda + \varepsilon_{s^\alpha_k})] = 1;$$

(iii) *For $1 \le i < b_\alpha$, the extension $0 \to N^\alpha_{(i)}/N^\alpha_{(i-1)} \to N^\alpha_{(i+1)}/N^\alpha_{(i-1)} \to N^\alpha_{(i+1)}/N^\alpha_{(i)} \to 0$ does not split.*

(iv) *The Loewy length of $N^\alpha_{(i)}/N^\alpha_{(i-1)}$ is at least $b_\alpha - i + 1$.*

Proof. Throughout the proof, write $Z = Z_n(\lambda)$ for short.

(i) By Lemma 8.3(iv), the module $N^\alpha_{(k)}$ is generated by $N^\alpha_{(k-1)}$ and the vector $\prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_k})$, which is primitive modulo $N^\alpha_{(k-1)}$. Therefore, since $(Z-\alpha)^{k-i}N^\alpha_{(k-1)} \subset N^\alpha_{(i-1)}$ by Lemma 8.2, it follows that

$$v = (Z-\alpha)^{k-i}\prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_k}) + N^\alpha_{(i-1)}$$

is either primitive or zero. Also by Lemma 8.2, v lies in $N^\alpha_{(i)}/N^\alpha_{(i-1)}$, so it just remains to show that v is non-zero. Suppose now for a contradiction that $v = 0$. Then

$$(Z-\alpha)^{k-i}N^\alpha_{(k)} \subseteq N^\alpha_{(i-1)}.$$

Since $(Z-\alpha)^{b_\alpha-k}N^\alpha \subseteq N^\alpha_{(k)}$ and $(Z-\alpha)^{i-1}N^\alpha_{(i-1)} = 0$ by Lemma 8.2, it follows that $(Z-\alpha)^{b_\alpha-1}N^\alpha = 0$, which contradicts Theorem 8.6.

(ii) By (i), left multiplication by $(Z-\alpha)^{k-i}$ induces a non-zero homomorphism between $N^\alpha_{(k)}/N^\alpha_{(k-1)}$ and $N^\alpha_{(i)}/N^\alpha_{(i-1)}$. Hence both the Hom-dimension and the composition multiplicity are at least one. So, since $[N^\alpha_{(i)}/N^\alpha_{(i-1)} : L_n(\lambda+\varepsilon_{s^\alpha_k})] \geq 1$ for all $1 \leq i \leq k$, $[N^\alpha : L_n(\lambda+\varepsilon_{s^\alpha_k})] \geq k$ with equality only if $[N^\alpha_{(i)}/N^\alpha_{(i-1)} : L_n(\lambda+\varepsilon_{s^\alpha_k})] = 1$ for all $i \leq k$. But Theorem 7.7 shows that equality does hold. So $[N^\alpha_{(i)}/N^\alpha_{(i-1)} : L_n(\lambda+\varepsilon_{s^\alpha_k})] = 1$ and this also implies that the Hom-dimension is at most one.

(iii) If the extension is split, Lemma 8.2 implies that $(Z-\alpha)$ annihilates $N^\alpha_{(i+1)}/N^\alpha_{(i-1)}$. But by Lemma 8.3(iv), the vector $w := \prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_{i+1}})$ belongs to $N^\alpha_{(i+1)}$, and by (i), $(Z-\alpha)w + N^\alpha_{(i-1)}$ is a non-zero vector of $N^\alpha_{(i+1)}/N^\alpha_{(i-1)}$. Hence $(Z-\alpha)$ does not annihilate $N^\alpha_{(i+1)}/N^\alpha_{(i-1)}$.

(iv) Let $L = N^\alpha_{(i)}/N^\alpha_{(i-1)}$. For $i \leq k \leq b_\alpha$, let

$$v_k = (Z-\alpha)^{k-i}\prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_k}) + N^\alpha_{(i-1)}$$

which by (i) is a non-zero primitive vector in L.

We claim first that for $k > i$, v_k lies in the $GL(n)$-submodule of L generated by v_{k-1}. By Lemma 8.3(iv), we can write

$$(Z-\alpha)\prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_k}) + N^\alpha_{(k-2)} = y\prod_{\gamma\neq\alpha}(Z-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_{k-1}}) + N^\alpha_{(k-2)}$$

for some $y \in U^-(n)$. Applying the central element $(Z-\alpha)^{k-i-1}$, which sends $N^\alpha_{(k-2)}$ into $N^\alpha_{(i-1)}$, this shows that $v_k = yv_{k-1}$ as required.

Now let $0 = S_0 \leq S_1 \leq \dots$ denote the socle series of L. We show that for $i \leq k \leq b_\alpha$, $v_k \notin S_{b_\alpha - k}$, by downward induction on k. In particular, this will show that $v_i \notin S_{b_\alpha - i}$, hence that the socle series of L has length at least $b_\alpha - i + 1$. The induction starts trivially with $k = b_\alpha$, since $v_{b_\alpha} \neq 0$. So now take $k < b_\alpha$ and assume by induction that $v_{k+1} \notin S_{b_\alpha-k-1}$. Suppose for a contradiction that $v_k \in S_{b_\alpha-j}$. Let $\bar v_k = v_k + S_{b_\alpha-k-1} \in S_{b_\alpha-k}/S_{b_\alpha-k-1}$. Since $v_{k+1} \in U(n)v_k$ and $v_{k+1} \notin S_{b_\alpha-k-1}$, we conclude that $\bar v_k$ and $\bar v_{k+1} = v_{k+1} + S_{b_\alpha-k-1}$ are two non-zero primitive vectors in $S_{b_\alpha-k}/S_{b_\alpha-k-1}$, which have different weights. Since $\bar v_{k+1} \in U(n)\bar v_k$, this contradicts the semisimplicity of $S_{b_\alpha-k}/S_{b_\alpha-k-1}$.

8.10. Corollary. *For $\alpha \in \mathbb{Z}/p\mathbb{Z}$, the Loewy length of N^α is at least $2b_\alpha - 1$, where b_α is the number of j with $1 \leq j \leq n$ which are normal for λ and $\alpha_j = \alpha$.*

Proof. Let ℓ denote the Loewy length of N^α. Consider the submodule $N^\alpha_{(1)} \subset N^\alpha$ generated by $\prod_{\gamma\neq\alpha}(Z_n(\lambda)-\gamma)^{b_\gamma}(v_\lambda\otimes f_{s^\alpha_1})$ as in Theorem 8.9. Note that $N^\alpha_{(1)}$ has simple head $L_n(\mu)$ where μ is the highest weight $\lambda - \varepsilon_{s^\alpha_1}$ of N, hence that $[N^\alpha : L_n(\mu)] = 1$.

By Theorem 8.9(iv), the Loewy length of $N^\alpha_{(1)}$ is at least b_α, so as $N^\alpha_{(1)}$ has simple head $L_n(\mu)$, we conclude that $[S_{b_\alpha-1}(N^\alpha_{(1)}) : L_n(\mu)] = 0$, whence $L_n(\mu)$ appears with multiplicity 1 in $N^\alpha_{(1)}/S_{b_\alpha-1}(N^\alpha_{(1)})$. By [**19**, Lemma 8.5(i)], $S_{b_\alpha-1}(N^\alpha_{(1)}) = N^\alpha_{(1)} \cap S_{b_\alpha-1}(N^\alpha)$. So $N^\alpha_{(1)}/S_{b_\alpha-1}(N^\alpha_{(1)})$ embeds into $N^\alpha/S_{b_\alpha-1}(N^\alpha)$, which implies $[N^\alpha/S_{b_\alpha-1}(N^\alpha) : L_n(\mu)] = 1$.

Therefore

$$[S_{b_\alpha - 1}(N^\alpha) : L_n(\mu)] = 0.$$

Note by contravariant duality that $N^\alpha / J^i(N^\alpha) \cong (S_i(N^\alpha))^\tau$. Hence, as N^α contains $L_n(\mu)$ with multiplicity one,

$$[J^{b_\alpha - 1}(N^\alpha) : L_n(\mu)] = 1.$$

Since $J^{b_\alpha - 1}(N^\alpha) \subseteq S_{\ell-(b_\alpha - 1)}(N^\alpha)$ by [**19**, Lemma I.8.2(ii)], this shows that

$$[S_{\ell - b_\alpha + 1}(N^\alpha) : L_n(\mu)] = 1.$$

Hence, $S_{\ell - b_\alpha + 1}(N^\alpha)$ strictly contains $S_{b_\alpha - 1}(N^\alpha)$. So, $\ell - b_\alpha + 1 > b_\alpha - 1$, as required.

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