SUPER KAC-MOODY 2-CATEGORIES

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Abstract. We introduce generalizations of Kac-Moody 2-categories in which the quiver Hecke algebras of Khovanov, Lauda and Rouquier are replaced by the quiver Hecke superalgebras of Kang, Kashiwara and Tsuchioka.

1. Introduction

Overview. Kac-Moody 2-categories were introduced by Khovanov and Lauda [KL3] and Rouquier [R]. They have rapidly become accepted as fundamental objects in representation theory, with intimate connections especially to quantum groups, canonical bases and knot invariants. Rouquier gave a seemingly different definition to Khovanov and Lauda:

- Rouquier's presentation starts from generators and relations for certain underlying quiver Hecke algebras, adjoins right duals of all the generating 1-morphisms, then imposes one more "inversion relation" at the level of 2-morphisms.
- The Khovanov-Lauda presentation incorporates various additional generating 2-morphisms, and extra relations including biadjointness and cyclicity. These additional generators and relations are useful for various applications, e.g. they are needed in order to extract a candidate for a basis in each space of 2-morphisms.

In [B], the first author has shown that the two versions are actually equivalent. The main purpose of this article is to extend the computations made in [B] to include *super Kac-Moody 2-categories*. We will define these shortly following Rouquier's approach, starting from certain underlying *quiver Hecke superalgebras* which were introduced already by Kang, Kashiwara and Tsuchioka [KKT]. For the quiver with one odd vertex, the quiver Hecke superalgebra is the *odd nilHecke algebra* defined independently in [EKL]; see also [Wa, §3.3] which introduced the closely related degenerate spin affine Hecke algebras. In this case, a super analog of the Kac-Moody 2-category was defined and studied already in [EL]. We will work here in the setting of *2-supercategories* following [BE], since it leads to some conceptual simplifications compared to the approach of [EL].

2-Supercategories. We proceed to the definitions. Fix once and for all a supercommutative ground ring $\Bbbk = \Bbbk_{\bar 0} \oplus \Bbbk_{\bar 1}$. We are mainly interested in the situation that $\Bbbk$ is a field concentrated in even parity.

Definition 1.1. A *superspace* is a $\mathbb{Z}/2$-graded $(\Bbbk, \Bbbk)$-bimodule in which the left and right actions are related by $cv = (-1)^{|c||v|}vc$; here and subsequently, $|x|$ denotes the parity of a homogeneous vector in a superspace. An *even linear map* between superspaces is a parity-preserving $\Bbbk$-module homomorphism.

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Let $\underline{\mathcal{SV}ec}$ be the Abelian category of all (small) superspaces and even linear maps. It is a symmetric monoidal category with tensor functor

$$-\otimes - : \underline{\mathcal{SV}ec} \times \underline{\mathcal{SV}ec} \rightarrow \underline{\mathcal{SV}ec}$$

being the usual tensor product over $\Bbbk$, and symmetric braiding defined on objects by $u \otimes v \mapsto (-1)^{|u||v|} v \otimes u$. (Our notation here follows [BE]: $\underline{\mathcal{SV}ec}$ is the underlying category to the monoidal *super*category $\mathcal{SV}ec$ whose morphisms are not necessarily homogeneous linear maps.)

Definition 1.2. A *supercategory* means a $\underline{\mathcal{SV}ec}$-enriched category, i.e. each morphism space is a superspace and composition induces an even linear map. A *superfunctor* between supercategories is a $\underline{\mathcal{SV}ec}$-enriched functor, i.e. a functor $F : \mathcal{A} \rightarrow \mathcal{B}$ such that the function $\mathrm{Hom}_{\mathcal{A}}(\lambda, \mu) \rightarrow \mathrm{Hom}_{\mathcal{B}}(F\lambda, F\mu), f \mapsto Ff$ is an even linear map for all $\lambda, \mu \in \mathrm{ob}\,\mathcal{A}$.

Let $\mathcal{SC}at$ be the category of all (small) supercategories, with morphisms being superfunctors. Given two supercategories $\mathcal{A}$ and $\mathcal{B}$, we define $\mathcal{A} \boxtimes \mathcal{B}$ to be the supercategory whose objects are ordered pairs (λ, μ) of objects of $\mathcal{A}$ and $\mathcal{B}$, respectively, and

$$\mathrm{Hom}_{\mathcal{A}\boxtimes\mathcal{B}}((\lambda, \mu), (\sigma, \tau)) := \mathrm{Hom}_{\mathcal{A}}(\lambda, \sigma) \otimes \mathrm{Hom}_{\mathcal{B}}(\mu, \tau).$$

Composition in $\mathcal{A} \boxtimes \mathcal{B}$ is defined using the symmetric braiding in $\underline{\mathcal{SV}ec}$, so that $(f \otimes g) \circ (h \otimes k) = (-1)^{|g||h|}(f \circ h) \otimes (g \circ k)$. Given superfunctors $F : \mathcal{A} \rightarrow \mathcal{A}'$ and $G : \mathcal{B} \rightarrow \mathcal{B}'$, there is a superfunctor $F \boxtimes G : \mathcal{A} \boxtimes \mathcal{B} \rightarrow \mathcal{A}' \boxtimes \mathcal{B}'$ sending $(\lambda, \mu) \mapsto (F\lambda, G\mu)$ and $f \otimes g \mapsto Ff \otimes Gg$. We have now defined a functor

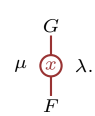
$$-\boxtimes - : \mathcal{SC}at \times \mathcal{SC}at \rightarrow \mathcal{SC}at$$

which makes $\mathcal{SC}at$ into a monoidal category.

Definition 1.3. A *2-supercategory* is a category enriched in $\mathcal{SC}at$. See also [BE, Definition 2.2] for the definition of a *2-superfunctor* between 2-supercategories.

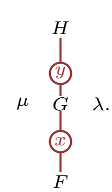
Remark 1.4. In [BE, Definition 2.1], the 2-supercategories of Definition 1.3 are called *strict* 2-supercategories. Since we will not encounter any 2-supercategories below that are not strict, we have suppressed the adjective from the outset. On the other hand, we will occasionally meet 2-superfunctors that are not strict.

According to Definition 1.3, for objects λ, μ in a 2-supercategory $\mathfrak{A}$, there is given a supercategory $\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$ of morphisms from λ to μ. Elements of $\mathrm{Hom}_{\mathfrak{A}}(\lambda, \mu) := \mathrm{ob}\,\mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)$ are *1-morphisms* in $\mathfrak{A}$. For 1-morphisms $F, G \in \mathrm{Hom}_{\mathfrak{A}}(\lambda, \mu)$, we also use the shorthand $\mathrm{Hom}_{\mathfrak{A}}(F, G)$ for the superspace $\mathrm{Hom}_{\mathcal{H}om_{\mathfrak{A}}(\lambda,\mu)}(F, G)$. Its elements are *2-morphisms*. We often represent $x \in \mathrm{Hom}_{\mathfrak{A}}(F, G)$ by the picture



(1.1)

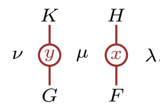
The composition $y \circ x$ of x with another 2-morphism $y \in \mathrm{Hom}_{\mathfrak{A}}(G, H)$ is obtained by vertically stacking pictures:



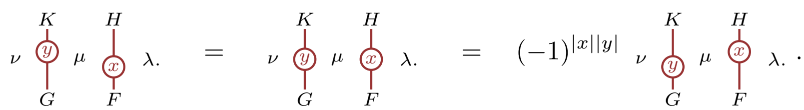
The composition law in $\mathfrak{A}$ gives a coherent family of superfunctors

$$T_{\nu,\mu,\lambda} : \mathcal{H}om_{\mathfrak{A}}(\mu,\nu) \boxtimes \mathcal{H}om_{\mathfrak{A}}(\lambda,\mu) \to \mathcal{H}om_{\mathfrak{A}}(\lambda,\nu)$$

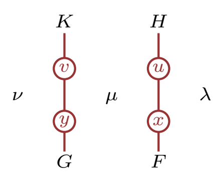
for $\lambda, \mu, \nu \in \mathrm{ob}\,\mathfrak{A}$. Given 2-morphisms $x : F \to H, y : G \to K$ between 1-morphisms $F, H : \lambda \to \mu, G, K : \mu \to \nu$, we denote $T_{\nu,\mu,\lambda}(y \otimes x) : T_{\nu,\mu,\lambda}(G,F) \to T_{\nu,\mu,\lambda}(K,H)$ simply by $yx : GF \to KH$, and represent it by horizontally stacking pictures:



When confusion seems unlikely, we will use the same notation for a 1-morphism F as for its identity 2-morphism. With this convention, we have that $yH \circ Gx = yx = (-1)^{|x||y|} Kx \circ yF$, or in pictures:



This identity is the *super interchange law.* The presence of the sign here means that a 2-supercategory is *not* a 2-category in the usual sense. In particular, diagrams for 2-morphisms in 2-supercategories are only invariant under rectilinear isotopy modulo signs. Consequently, care is needed with horizontal levels when working with odd 2-morphisms diagrammatically: a more complicated diagram such as



should be interpreted by *first* composing horizontally *then* composing vertically. The example just given represents $(vu) \circ (yx)$ *not* $(v \circ y)(u \circ x)$.

Super Kac-Moody 2-categories. With these foundational definitions behind us, we are ready to introduce the main object of study. We need to fix some additional data:

- Let I be a (possibly infinite) index set equipped with a parity function $I \to \mathbb{Z}/2, i \mapsto |i|$; we will say that $i \in I$ is *even* or *odd* according to whether $|i| = \bar{0}$ or $\bar{1}$, respectively. If I has odd elements, we make the additional assumption that 2 is invertible in the ground ring $\Bbbk$.
- Let $(-d_{ij})_{i,j \in I}$ be a generalized Cartan matrix, so $d_{ii} = -2$, $d_{ij} \geq 0$ for $i \neq j$, and $d_{ij} = 0 \Leftrightarrow d_{ji} = 0$. We make the additional assumption that

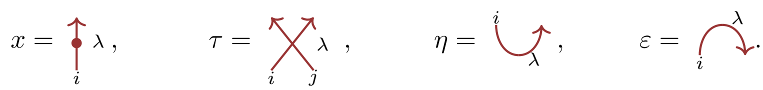
$$|i| = \bar{1} \Rightarrow d_{ij} \text{ is even.} \tag{1.2}$$

- Pick a complex vector space $\mathfrak{h}$ and linearly independent subsets $\{\alpha_i \mid i \in I\}$ and $\{h_i \mid i \in I\}$ of $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, such that $\langle h_i, \alpha_j \rangle = -d_{ij}$ for all $i, j \in I$. Let $P := \{\lambda \in \mathfrak{h}^* \mid \langle h_i, \lambda \rangle \in \mathbb{Z}$ for all $i \in I\}$ be the *weight lattice* and $Q := \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ be the *root lattice.*
- Let $\mathfrak{g}$ be the *Kac-Moody algebra* associated to this data with Chevalley generators $\{e_i, f_i, h_i \mid i \in I\}$ and Cartan subalgebra $\mathfrak{h}$.

- Finally fix units $t_{ij} \in \Bbbk_{\bar 0}^\times$ such that
$$t_{ii} = 1, \quad d_{ij} = 0 \Rightarrow t_{ij} = t_{ji}, \tag{1.3}$$
and scalars $s_{ij}^{pq} \in \Bbbk_{\bar 0}$ for $0 < p < d_{ij}$, $0 < q < d_{ji}$ such that
$$s_{ij}^{pq} = s_{ji}^{qp}, \quad p|i| = \bar 1 \Rightarrow s_{ij}^{pq} = 0. \tag{1.4}$$

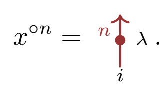
In case all elements of I are even, the following is the same as the Rouquier's definition of Kac-Moody 2-category from [R] (viewing the latter as a 2-supercategory by declaring that all of its 2-morphisms are even).

Definition 1.5. The *Kac-Moody 2-supercategory* is the 2-supercategory $\mathfrak{U}(\mathfrak{g})$ with objects P, generating 1-morphisms $E_i 1_\lambda : \lambda \to \lambda + \alpha_i$ and $F_i 1_\lambda : \lambda \to \lambda - \alpha_i$ for each $i \in I$ and $\lambda \in P$, and generating 2-morphisms $x : E_i 1_\lambda \to E_i 1_\lambda$ of parity $|i|$, $\tau : E_i E_j 1_\lambda \to E_j E_i 1_\lambda$ of parity $|i||j|$, $\eta : 1_\lambda \to F_i E_i 1_\lambda$ of parity $\bar 0$ and $\varepsilon : E_i F_i 1_\lambda \to 1_\lambda$ of parity $\bar 0$, subject to certain relations. To record the relations among these generators, we switch to diagrams, representing the identity 2-morphisms of $E_i 1_\lambda$ and $F_i 1_\lambda$ by ${}^{\lambda+\alpha_i}\!\uparrow_i{}^{\lambda}$ and ${}^{\lambda-\alpha_i}\!\downarrow_i{}^{\lambda}$, respectively, and the other generators by


(1.5)

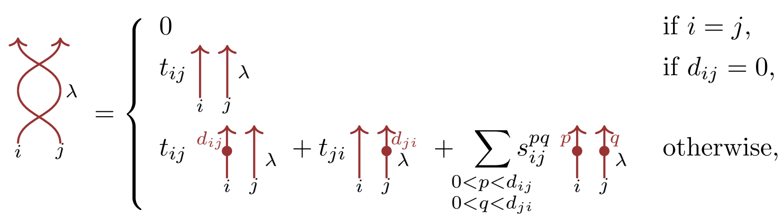
(parity $|i|$) (parity $|i||j|$) (parity $\bar 0$) (parity $\bar 0$)

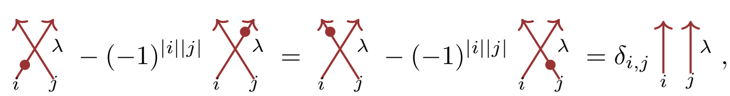
We denote the nth power of x (under vertical composition) by

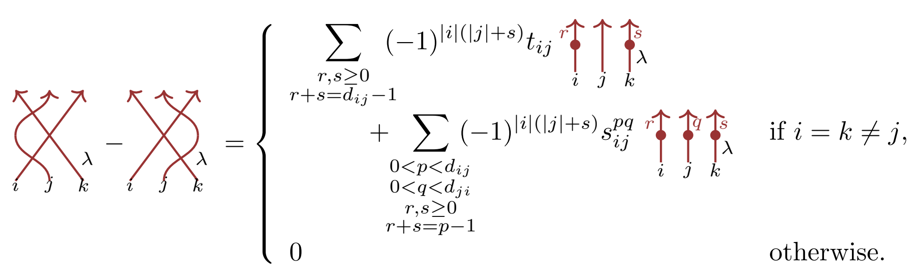

(1.6)

(parity $|i|n$)

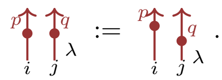
First, we have the *quiver Hecke superalgebra relations* from [KKT]:


(1.7)

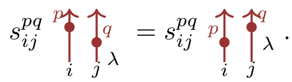

(1.8)


(1.9)

In (1.7), we have drawn multiple dots on the same horizontal level, which is potentially ambiguous: our convention for this is that it means the horizontal composition of $x^{\circ p}$ and $x^{\circ q}$, so that

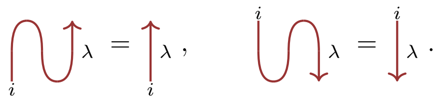


Note further by the assumption (1.4) that



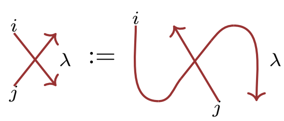
Similar remarks apply to (1.9) and all other such situations below.

Next we have the *right adjunction relations*:


(1.10)

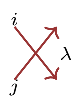
These imply that $F_i 1_{\lambda+\alpha_i}$ is a right dual of $E_i 1_\lambda$.

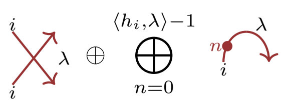
Finally there are some *inversion relations*. To formulate these, we first introduce a new 2-morphism

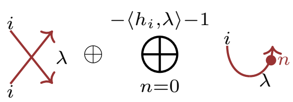

$: E_j F_i 1_\lambda \to F_i E_j 1_\lambda.$ (1.11)

(parity $|i||j|$)

Then we require that the following (not necessarily homogeneous) 2-morphisms are isomorphisms:


$: E_j F_i 1_\lambda \xrightarrow{\sim} F_i E_j 1_\lambda$ if $i \neq j$, (1.12)


$: E_i F_i 1_\lambda \xrightarrow{\sim} F_i E_i 1_\lambda \oplus 1_\lambda^{\oplus \langle h_i, \lambda\rangle}$ if $\langle h_i, \lambda\rangle \geq 0$, (1.13)


$: E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus -\langle h_i, \lambda\rangle} \xrightarrow{\sim} F_i E_i 1_\lambda$ if $\langle h_i, \lambda\rangle \leq 0$. (1.14)

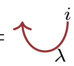
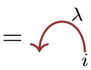
Note that (1.13)–(1.14) are 2-morphisms in the additive envelope of $\mathfrak{U}(\mathfrak{g})$. Nevertheless this defines some genuine relations for $\mathfrak{U}(\mathfrak{g})$ itself (rather than its additive envelope): we mean that there are some as yet unnamed generating 2-morphisms in $\mathfrak{U}(\mathfrak{g})$ which are the matrix entries of two-sided inverses to (1.13)–(1.14).

Second adjunction. Let

$$|i, \lambda| := |i|(\langle h_i, \lambda\rangle + 1). \tag{1.15}$$

Since $|i, \lambda| = |i, \lambda \pm \alpha_j|$ for any $j \in I$, this only depends on the coset of λ modulo Q. In section 2, we will define some additional 2-morphisms $\eta' : 1_\lambda \to E_i F_i 1_\lambda$ and

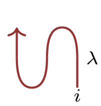
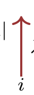
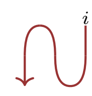
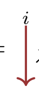
$\varepsilon' : F_iE_i1_\lambda \to 1_\lambda$ represented diagrammatically by leftward cups and caps:

$\eta' =$  , (parity $|i,\lambda|$) $\qquad \varepsilon' =$  . (parity $|i,\lambda|$) (1.16)

Following the idea of [BHLW], we will normalize these in a different way to [CL, B], in order to salvage some cyclicity. Consequently, our definitions of ε' and η' depend on the additional choice of units $c_{\lambda;i} \in \Bbbk_{\bar 0}^\times$ for each $i \in I$ and $\lambda \in P$ such that

$$c_{\lambda+\alpha_j;i} = t_{ij} c_{\lambda;i}. \tag{1.17}$$

In section 6, we will show that η' and ε' satisfy the following *left adjunction relations*:

 $= (-1)^{|i,\lambda|}$  ,  $=$  . (1.18)

Consequently, $\Pi^{|i,\lambda|}F_i1_{\lambda+\alpha_i}$ is a left dual of E_i1_λ, working now in the Π-envelope $\mathfrak{U}_\pi(\mathfrak{g})$ of $\mathfrak{U}(\mathfrak{g})$ from [BE, Definition 4.4]; cf. Definition 1.6 below.

Further relations. In sections 3–7, we also derive various other relations from the defining relations, enough to see in particular that the inverses of the 2-morphisms (1.12)–(1.14) can be written as certain horizontal and vertical compositions of $x, \tau, \varepsilon, \eta, \varepsilon'$ and η', i.e. the 2-morphisms named so far are enough to generate all other 2-morphisms in $\mathfrak{U}(\mathfrak{g})$. Some of our extra relations are as follows.

- The super analog of Lauda's *infinite Grassmannian relation*: Let Sym be the algebra of symmetric functions over $\Bbbk$. Recall Sym is generated both by the elementary symmetric functions e_r ($r \geq 0$) and by the complete symmetric functions h_s ($s \geq 0$); we view it as a superalgebra by declaring that all these generators are even. By [M, (I.2.6′)], elementary and complete symmetric functions are related by the equations

$$\mathrm{e}_0 = \mathrm{h}_0 = 1, \qquad \sum_{r+s=n} (-1)^s \mathrm{e}_r \mathrm{h}_s = 0 \text{ for all } n > 0.$$

Take $i \in I$, $\lambda \in P$ and set $h := \langle h_i, \lambda\rangle$. If i is even, Lauda [L] observed already that there exists a unique homomorphism

$$\beta_{\lambda;i} : \mathrm{Sym} \to \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda) \tag{1.19}$$

such that

$\mathrm{e}_n \mapsto c_{\lambda;i}^{-1}$ $n+h-1$  λ if $n > -h$, $\quad \mathrm{h}_n \mapsto (-1)^n c_{\lambda;i}$ λ  $n-h-1$ if $n > h$,

bearing in mind the new normalization of bubbles. The analog of this when i is odd is as follows. Let Sym[d] be the supercommutative superalgebra obtained from Sym by adjoining an odd generator $\mathtt{d}$ with $\mathtt{d}^2 = 0$. Then there exists a unique homomorphism

$$\beta_{\lambda;i} : \mathrm{Sym}[\mathtt{d}] \to \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda) \tag{1.20}$$

such that

$\mathrm{e}_n \mapsto c_{\lambda;i}^{-1}$ $2n+h-1$  λ if $n > -\frac{h}{2}$, $\quad \mathrm{h}_n \mapsto (-1)^n c_{\lambda;i}$ λ  $2n-h-1$ if $n > \frac{h}{2}$,

$\mathtt{d}\mathrm{e}_n \mapsto c_{\lambda;i}^{-1}$ $2n+h$  λ if $n \geq -\frac{h}{2}$, $\quad \mathtt{d}\mathrm{h}_n \mapsto (-1)^n c_{\lambda;i}$ λ  $2n-h$ if $n \geq \frac{h}{2}$.

Furthermore, letting

$$\mathrm{SYM} := \bigotimes_{i\,\text{even}} \mathrm{Sym} \otimes \bigotimes_{i\,\text{odd}} \mathrm{Sym}[\mathtt{d}] \tag{1.21}$$

where the tensor products are taken in some fixed order, there is *surjective* homomorphism

$$\beta_\lambda : \mathrm{SYM} \twoheadrightarrow \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda), \tag{1.22}$$

defined by taking the product of the maps $\beta_{\lambda;i}$ applied to the ith tensor factor of SYM for all $i \in I$.

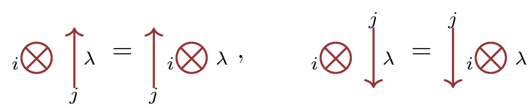
- *Centrality of odd bubbles*: Assuming $i \in I$ is odd, we introduce the odd 2-morphism

$$\lambda \bigotimes_{i} := \beta_{\lambda;i}(\mathtt{d}). \tag{1.23}$$

We call this the *odd bubble* of color i. By the super interchange law it squares to zero:

$$\left({}_i\bigotimes^{\lambda} \right)^2 = 0. \tag{1.24}$$

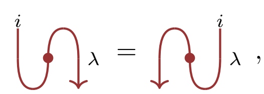
We show moreover that odd bubbles are central in $\mathfrak{U}(\mathfrak{g})$ in the sense that



(1.25)

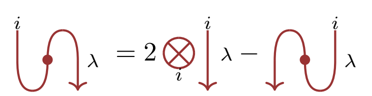
for all $j \in I$. (This means that it would be reasonable to set odd bubbles to zero, imposing additional relations $\lambda \bigotimes_i = 0$ for all odd $i \in I$ and $\lambda \in P$.)

- *Cyclicity properties*: If i is even then



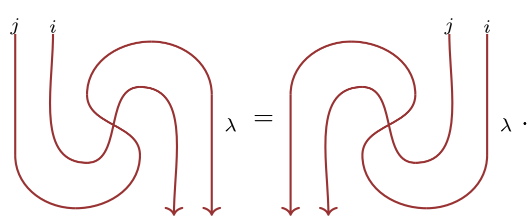
(1.26)

i.e. even dots are cyclic. However if i is odd we have that



(1.27)

In all cases, crossings satisfy



(1.28)

Nondegeneracy Conjecture. Let $F, G : \lambda \to \mu$ be some 1-morphisms in $\mathfrak{U}(\mathfrak{g})$. In section 8, we construct an explicit set $\{ f(\sigma) \mid \sigma \in \widehat{M}(F,G) \}$ of 2-morphisms which generates $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(F,G)$ as a right SYM-module; here the action of $p \in \mathrm{SYM}$ is by horizontally composing on the right with $\beta_\lambda(p)$. This puts us in position to formulate the following conjecture, which is the appropriate generalization of the nondegeneracy condition formulated by Khovanov and Lauda in [KL3, §3.2.3]; for example, taking $F = G = 1_\lambda$, it implies that the homomorphism β_λ from (1.22) is an isomorphism.

Conjecture: $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(F,G)$ *is a free* SYM*-module with basis* $\left\{f(\sigma)\,\middle|\,\sigma\in\widehat{M}(F,G)\right\}$.

We cannot prove this at present. We will discuss its signficance and some possible approaches to its proof later on in the introduction.

Gradings. By a *graded superspace*, we mean a superspace equipped with an additional $\mathbb{Z}$-grading $V=\bigoplus_{n\in\mathbb{Z}}V_n=\bigoplus_{n\in\mathbb{Z}}V_{n,\bar 0}\oplus V_{n,\bar 1}$. Let $\underline{\mathcal{GSV}ec}$ be the symmetric monoidal category of graded superspaces and degree-preserving even linear maps. Mimicking Definition 1.2, a *graded supercategory* means a $\underline{\mathcal{GSV}ec}$-enriched category. Let $\mathcal{GSC}at$ be the monoidal category of all (small) graded supercategories. Finally, mimicking Definition 1.3, a *graded 2-supercategory* means a category enriched in $\mathcal{GSC}at$. Thus, it is a 2-supercategory whose 2-morphism spaces are graded superspaces, and horizontal and vertical composition respect these gradings. We will soon need the following universal construction from [BE, Definition 6.10]:

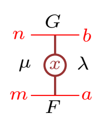
Definition 1.6. Suppose that $\mathfrak{A}$ is a graded 2-supercategory. Its (Q,Π)-*envelope* $\mathfrak{A}_{q,\pi}$ is the graded 2-supercategory with the same objects as $\mathfrak{A}$, 1-morphisms defined from

$$\mathrm{Hom}_{\mathfrak{A}_{q,\pi}}(\lambda,\mu):=\left\{Q^m\Pi^aF\,\middle|\,\text{for all }F\in\mathrm{Hom}_{\mathfrak{A}}(\lambda,\mu),\ m\in\mathbb{Z}\text{ and }a\in\mathbb{Z}/2\right\}$$

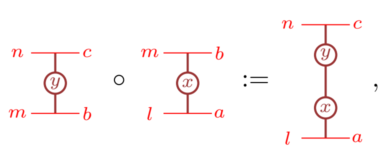
with the horizontal composition law $(Q^n\Pi^bG)(Q^m\Pi^aF):=Q^{m+n}\Pi^{a+b}(GF)$, and 2-morphisms defined from

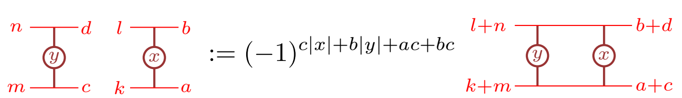
$$\mathrm{Hom}_{\mathfrak{A}_{q,\pi}}(Q^m\Pi^aF,Q^n\Pi^bG):=\left\{x_{m,a}^{n,b}\,\middle|\,\text{for all }x\in\mathrm{Hom}_{\mathfrak{A}}(F,G)\right\}$$

viewed as a superspace with operations $x_{m,a}^{n,b}+y_{m,a}^{n,b}:=(x+y)_{m,a}^{n,b}$, $c(x_{m,a}^{n,b}):=(cx)_{m,a}^{n,b}$ for $c\in\Bbbk$, and grading $\deg(x_{m,a}^{n,b}):=\deg(x)+n-m$, $\left|x_{m,a}^{n,b}\right|:=|x|+a+b$. Representing $x_{m,a}^{n,b}$ by the picture



for x as in (1.1), the vertical and horizontal composition laws for 2-morphisms in $\mathfrak{A}_{q,\pi}$ are defined in terms of the ones in $\mathfrak{A}$ as follows:


(1.29)



(1.30)

For each object λ, there are distinguished 1-morphisms $q_\lambda:=Q^1\Pi^{\bar 0}1_\lambda$, $q_\lambda^{-1}:=Q^{-1}\Pi^{\bar 0}1_\lambda$ and $\pi_\lambda:=Q^0\Pi^{\bar 1}1_\lambda$ in $\mathrm{End}_{\mathfrak{A}_{q,\pi}}(\lambda)$. Moreover, there are 2-isomorphisms $\sigma_\lambda:q_\lambda\overset{\sim}{\Rightarrow}1_\lambda$, $\bar\sigma_\lambda:q_\lambda^{-1}\overset{\sim}{\Rightarrow}1_\lambda$ and $\zeta_\lambda:\pi_\lambda\overset{\sim}{\Rightarrow}1_\lambda$, all induced by the identity 2-morphism 1_{1_λ}. These give the required structure maps to make $\mathfrak{A}_{q,\pi}$ into a *graded* (Q,Π)-*2-supercategory* in the sense of [BE, Definition 6.5].

Assume for the remainder of the introduction that the Cartan matrix A is symmetrizable, so that there exist positive integers $(d_i)_{i\in I}$ such that $d_id_{ij}=d_jd_{ji}$ for all $i, j \in I$. Assume moreover that $\Bbbk$ is a field, and that the parameters chosen above satisfy the following *homogeneity condition*:

$$s_{ij}^{pq} \neq 0 \Rightarrow pd_{ji} + qd_{ij} = d_{ij}d_{ji}. \tag{1.31}$$

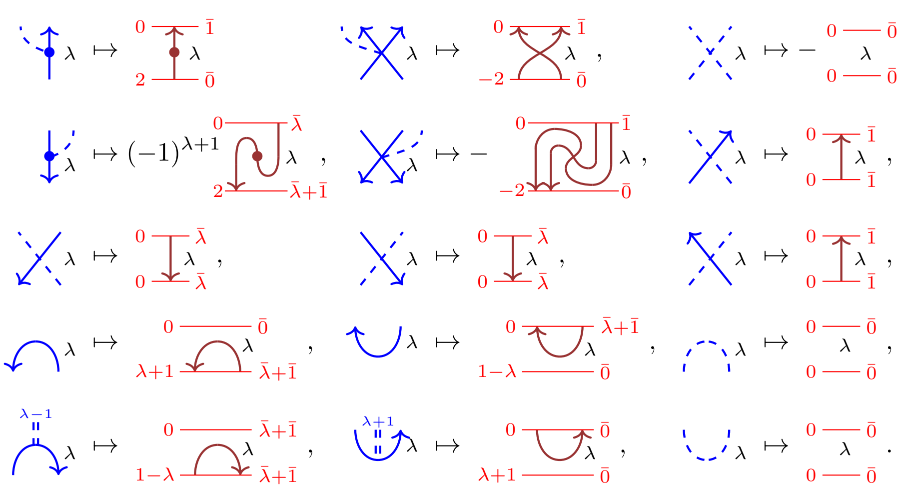
Then we can put an additional $\mathbb{Z}$-grading on $\mathfrak{U}(\mathfrak{g})$ making it into a graded 2-supercategory, by declaring that the generators from (1.5) and (1.16) are of the degrees listed in the following table:

x	τ	η	ε	η'	ε'
$2d_i$	$d_i d_{ij}$	$d_i(1+\langle h_i, \lambda\rangle)$	$d_i(1-\langle h_i, \lambda\rangle)$	$d_i(1-\langle h_i, \lambda\rangle)$	$d_i(1+\langle h_i, \lambda\rangle)$

Let $\mathfrak{U}_{q,\pi}(\mathfrak{g})$ denote the (Q,Π)-envelope of $\mathfrak{U}(\mathfrak{g})$ in the sense of Definition 1.6. The *underlying 2-category* $\underline{\mathfrak{U}}_{q,\pi}(\mathfrak{g})$ consists of the same objects and 1-morphisms as $\mathfrak{U}_{q,\pi}(\mathfrak{g})$ but only its even 2-morphisms of degree zero. Also let $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ be the idempotent completion of the additive envelope of $\underline{\mathfrak{U}}_{q,\pi}(\mathfrak{g})$. Both of $\underline{\mathfrak{U}}_{q,\pi}(\mathfrak{g})$ and $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ are (Q,Π)-*2-categories* in the sense of [BE, Definition 6.14]. In particular, they are equipped with distinguished objects $q = (q_\lambda)$ and $\pi = (\pi_\lambda)$ in their Drinfeld centers.

Relation to the Ellis-Lauda 2-category. Suppose that $\mathfrak{g}$ is *odd* $\mathfrak{sl}_2$, i.e. I is an odd singleton. Then the 2-category $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ is 2-equivalent to the 2-category introduced [EL]. We do not think that this is an important result going forward, so we will only give a rough sketch of its proof in the next paragraph. Our new approach to the definition seems to be both conceptually more satisfactory and less prone to errors when working with the relations. So our point of view really is that, henceforth, one should simply replace the object in [EL] with the one here.

Briefly, the idea is simply to construct quasi-inverse 2-functors between the Ellis-Lauda 2-category $\mathfrak{U}_{EL}$ and our $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ by verifying relations. Let us write simply E, F and h for E_i, F_i and h_i for the unique $i \in I$. Also we take $d_i := 1$ and identify $P \leftrightarrow \mathbb{Z}$ so $\lambda \leftrightarrow \langle h, \lambda\rangle$. Then, the appropriate 2-functor in the direction $\mathfrak{U}_{EL} \to \dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ is the identity on the object set P. It sends the generating 1-morphisms $\mathtt{E}1_\lambda, \mathtt{F}1_\lambda$ and $\Pi 1_\lambda$ from [EL, §3.2.1] to our 1-morphisms $E1_\lambda, \Pi^{\bar\lambda+\bar 1}F1_\lambda$ and π_λ, respectively. On the generating 2-morphisms from [EL, §3.2.2], it goes as follows:



We leave it to the reader to compare the relations in [EL] with our relations, and to construct a quasi-inverse 2-functor in the other direction. In fact, when doing this carefully, one uncovers some inconsistencies in the relations of [EL]; e.g. the relation [EL, (3.1)] is wrong in the case $\lambda = 0$ (due to an error in the last sentence of the proof of [EL, Lemma 5.1] related to the nilpotency of the odd bubble).

Decategorification Conjecture. Recall finally that the *Grothendieck ring* of an additive 2-category $\mathfrak{A}$ is

$$K_0(\mathfrak{A}) := \bigoplus_{\lambda,\mu \in \operatorname{ob}\mathfrak{A}} K_0(\mathcal{H}om_{\mathfrak{A}}(\lambda,\mu)) \tag{1.32}$$

where the K_0 on the right hand side is the usual split Grothendieck group of the additive category $\mathcal{H}om_{\mathfrak{A}}(\lambda,\mu)$. It is a locally unital ring with distinguished idempotents $\{1_\lambda \mid \lambda \in \operatorname{ob}\mathfrak{A}\}$. If $\mathfrak{A}$ is a (Q,Π)-2-category, then $K_0(\mathfrak{A})$ is also linear over $\mathcal{L} := \mathbb{Z}[q,q^{-1},\pi]/(\pi^2-1)$, with q and π acting by multiplication by the classes of the distinguished objects q and π of the Drinfeld center.

This discussion applies in particular to the (Q,Π)-2-category $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$, so that $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ is a locally unital $\mathcal{L}$-algebra with idempotents $\{1_\lambda \mid \lambda \in P\}$. Also let $\dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$ be the $\mathcal{L}$-form of the idempotented version of the covering quantized enveloping algebra associated to $\mathfrak{g}$ introduced by Clark, Hill and Wang in [CHW1]; see section 9. By similar arguments to those of [KL3], using also some results from [HW], we will show in section 11 that there is a *surjective* homomorphism of locally unital $\mathcal{L}$-algebras

$$\gamma : \dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}} \twoheadrightarrow K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})) \tag{1.33}$$

sending $e_i 1_\lambda$ and $f_i 1_\lambda$ to $[E_i 1_\lambda]$ and $[F_i 1_\lambda]$, respectively. Moreover, also just like in [KL3], we will show in section 12 that the Nondegeneracy Conjecture formulated above, together with an additional assumption of bar-consistency on the Cartan datum, implies the truth of the following:

Conjecture: *γ is an isomorphism.*

Discussion. In the purely even case, i.e. when all $i \in I$ are even, the Nondegeneracy Conjecture (hence, the Decategorification Conjecture) was established by Khovanov and Lauda in [KL3, §6.4] in case $\mathfrak{g} = \mathfrak{sl}_n$. In [W], Webster has proposed a proof of the Nondegeneracy Conjecture for all purely even types. There is also a completely different proof of the Decategorification Conjecture based on results of [KK], which is valid in all finite types; see e.g. [BD, Corollary 4.21].

Turning to the odd case, the Decategorification Conjecture for odd $\mathfrak{sl}_2$ is proved in [EL, Theorem 8.4]. The only additional finite type possibilities come from *odd* $\mathfrak{b}_n$, i.e. type $\mathfrak{b}_n$ with the element of I corresponding to the short simple root chosen to be odd. For these, the Decategorification Conjecture may be deduced from [KKO1, KKO2]. We hope that Webster's methods from [W] can be extended to the super case to prove the Nondegeneracy Conjecture in general, but there is a great deal of work still to be done in order to see this through. As a first step, we would like to see the proof of the Nondegeneracy Conjecture from [KL3] extended in order to include all odd $\mathfrak{b}_n$, and hope to address this in subsequent work.

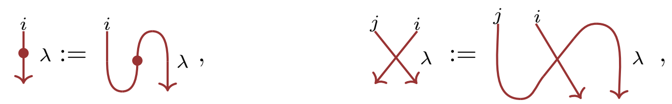
Assuming the Decategorification Conjecture, one gets an interesting basis for the covering quantum group $\dot{U}_{q,\pi}(g)_{\mathcal{L}}$ coming from the isomorphism classes of the indecomposable objects of $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$. In symmetric types, this should coincide (up to parity shift) with the canonical basis from [C, Theorem 4.14]. For odd $\mathfrak{b}_1$, this assertion follows already from the results of [EL].

In a different direction, it should now be possible to develop super analogs of many of the foundational structural results proved by Chuang-Rouquier and Rouquier in [CR, R]. Various applications, e.g. to spin representations of symmetric groups and to representations of the Lie superalgebra $\mathfrak{q}(n)$, are expected.

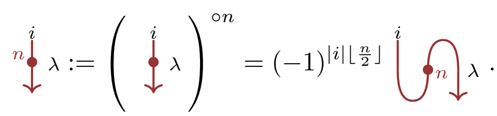
2. More generators

In sections 2–8, we assume that the ground ring $\Bbbk$ is as in Definition 1.1, and let $\mathfrak{U}(\mathfrak{g})$ be the Kac-Moody 2-supercategory from Definition 1.5. We begin by defining various additional 2-morphisms in $\mathfrak{U}(\mathfrak{g})$.

Definition 2.1. We have the *downward dots and crossings*, which are the right mates of the upward dots and crossings:

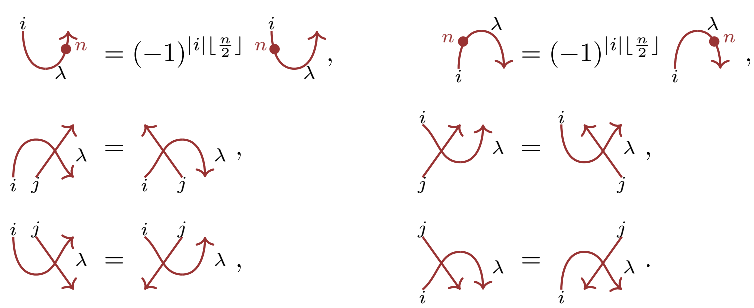


(parity $|i|$) (parity $|i||j|$) (2.1)



(parity $|i|n$) (2.2)

The sign in (2.2) is easily checked using the diagrammatics; see also [KKO2, Proposition 7.14]. Using (1.10) and (1.11), we deduce:

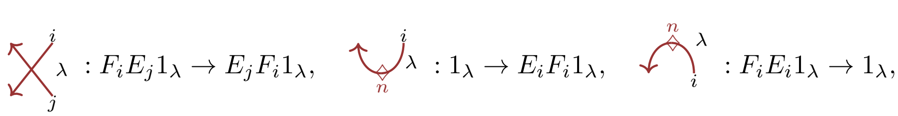


(2.3)

(2.4)

(2.5)

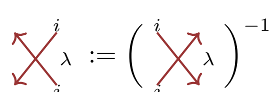
Definition 2.2. We define the *leftward crossing* and various *leftward cups and caps*. First define



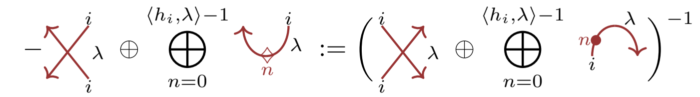
(2.6)

(parity $|i||j|$) (parity $|i|n$) (parity $|i|n$)

by declaring that

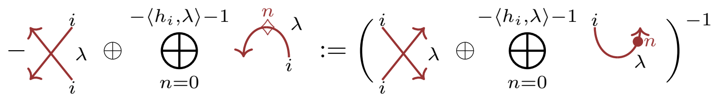


if $i \neq j$, (2.7)



if $\langle h_i, \lambda \rangle \geq 0$,

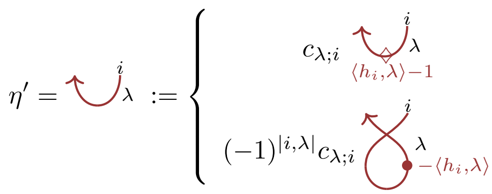
(2.8)



if $\langle h_i, \lambda \rangle \leq 0$,

(2.9)

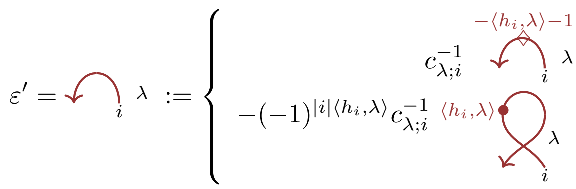
working in the additive envelope of $\mathfrak{U}(\mathfrak{g})$. Then, remembering the scalars $c_{\lambda;i}$ chosen for (1.17), we set



if $\langle h_i, \lambda \rangle > 0$,

if $\langle h_i, \lambda \rangle \leq 0$,

(2.10)

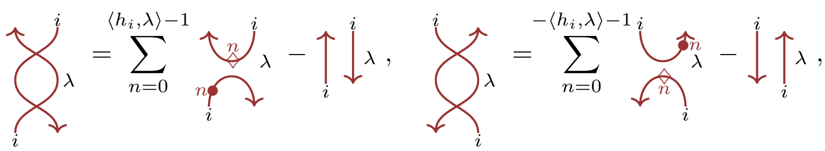


if $\langle h_i, \lambda \rangle < 0$,

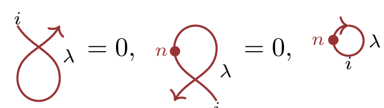
if $\langle h_i, \lambda \rangle \geq 0$,

(2.11)

both of which are of parity $|i, \lambda|$. The following are immediate from these definitions.

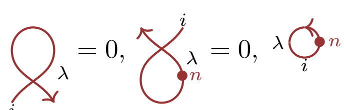


(2.12)



$= \delta_{n,\langle h_i,\lambda\rangle -1} c_{\lambda;i} 1_{1_\lambda}$ all for $0 \leq n < \langle h_i, \lambda \rangle$,

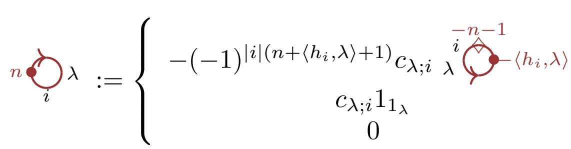
(2.13)



$= \delta_{n,-\langle h_i,\lambda\rangle -1} c_{\lambda;i}^{-1} 1_{1_\lambda}$ all for $0 \leq n < -\langle h_i, \lambda \rangle$.

(2.14)

Definition 2.3. We give meaning to *negatively dotted bubbles* by making the following definitions for $n < 0$:

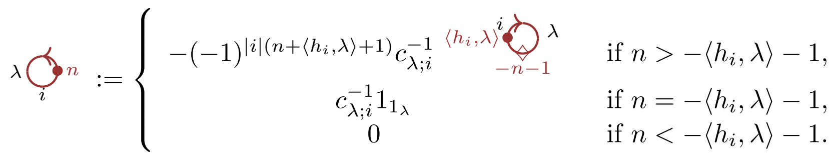


if $n > \langle h_i, \lambda \rangle - 1$,

if $n = \langle h_i, \lambda \rangle - 1$,

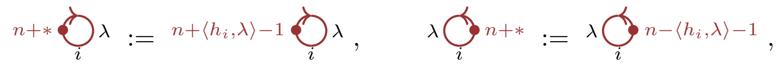
if $n < \langle h_i, \lambda \rangle - 1$,

(2.15)



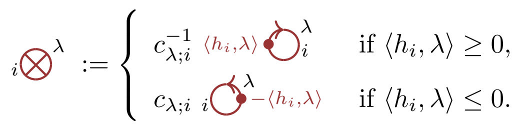
(2.16)

Sometimes we will use the following convenient shorthand for dotted bubbles for any $n \in \mathbb{Z}$:



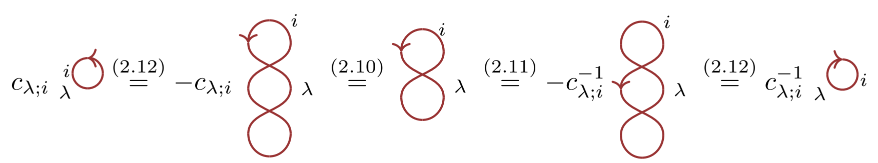
(2.17)

both of which are of parity $|i|n$. Also, assuming that $i \in I$ is odd, we introduce the *odd bubble*



(2.18)

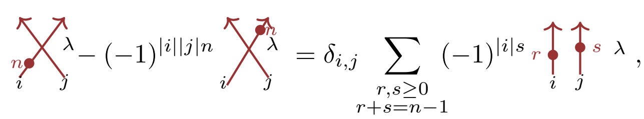
There is no ambiguity in this definition in the case $\langle h_i, \lambda\rangle = 0$ thanks to the following calculation:



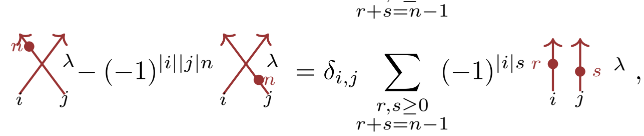
3. The Chevalley involution

The next task is to construct an important symmetry of $\mathfrak{U}(\mathfrak{g})$. For this, we need some preliminary lemmas.

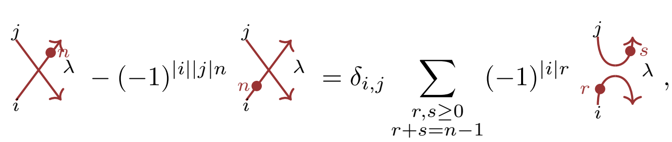
Lemma 3.1. *The following relations hold for all $n \geq 0$:*



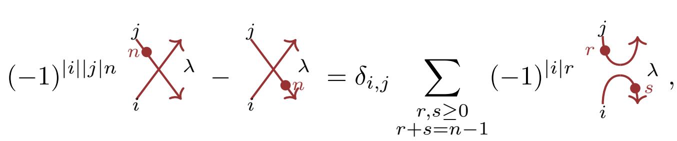
(3.1)



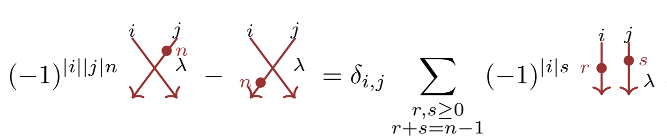
(3.2)



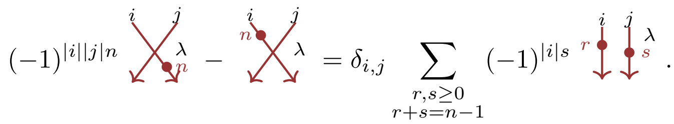
(3.3)



(3.4)

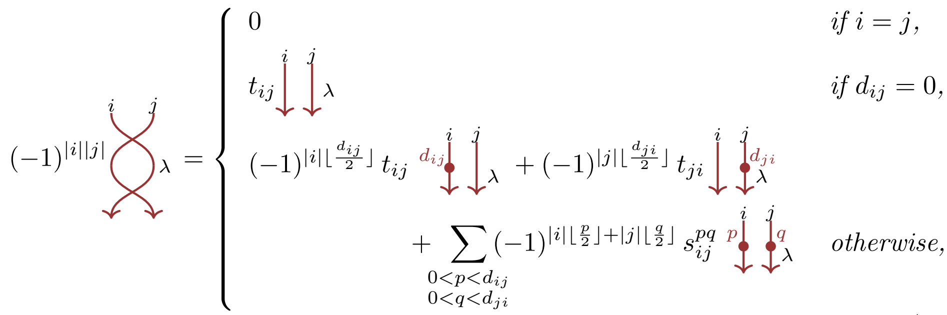


(3.5)


(3.6)

Proof. The first two relations follow inductively from (1.8). The rest then follow by rotating clockwise, i.e. attach rightward caps to the top right strands and rightward cups to the bottom left strands then use (2.3)–(2.5). ☐

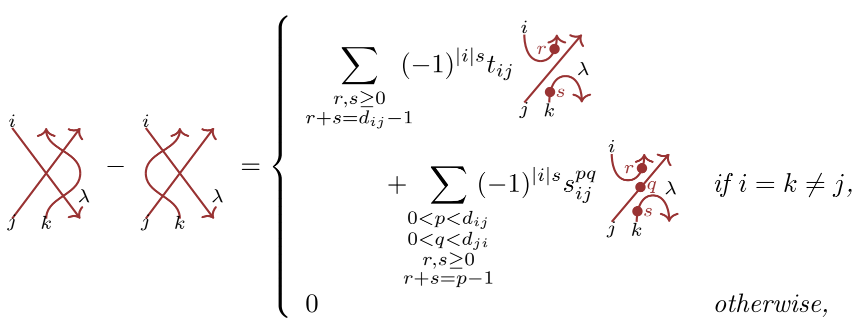
Lemma 3.2. *The following relations hold:*



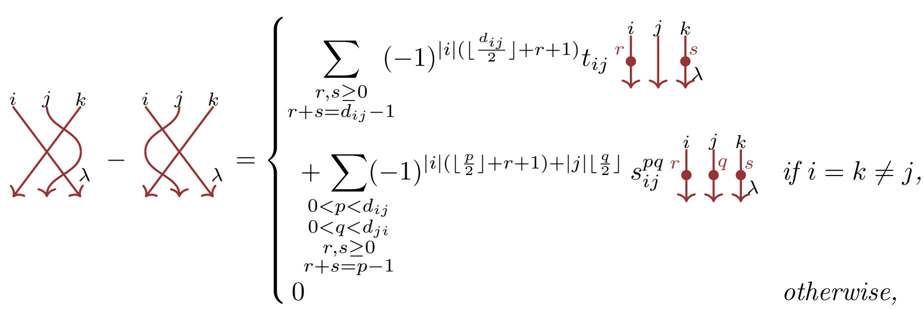
(3.7)

Proof. Rotate (1.7) clockwise as explained in the proof of Lemma 3.1. ☐

Lemma 3.3. *The following relations hold:*



(3.8)



(3.9)

Proof. Rotate (1.9) clockwise. ☐

Definition 3.4. For a supercategory $\mathcal{A}$, we write $\mathcal{A}^{\mathrm{sop}}$ for the supercategory with the same objects, morphisms

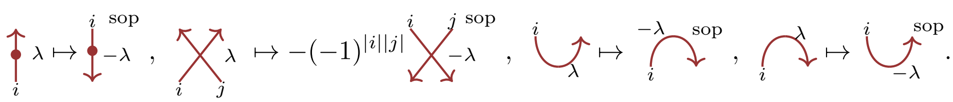
$$\mathrm{Hom}_{\mathcal{A}^{\mathrm{sop}}}(\lambda, \mu) := \mathrm{Hom}_{\mathcal{A}}(\mu, \lambda),$$

and new composition law defined from $f^{\mathrm{sop}} \circ g^{\mathrm{sop}} := (-1)^{|f||g|}(g \circ f)^{\mathrm{sop}}$, where we denote a morphism $f : \lambda \to \mu$ in $\mathcal{A}$ viewed as a morphism in $\mathcal{A}^{\mathrm{sop}}$ by $f^{\mathrm{sop}} : \mu \to \lambda$. For a 2-supercategory $\mathfrak{A}$, we write $\mathfrak{A}^{\mathrm{sop}}$ for the 2-supercategory with the same objects as $\mathfrak{A}$, and morphism categories defined from $\mathcal{H}om_{\mathfrak{A}^{\mathrm{sop}}}(\lambda, \mu) := \mathcal{H}om_{\mathfrak{A}}(\lambda, \mu)^{\mathrm{sop}}$. Horizontal composition in $\mathfrak{A}^{\mathrm{sop}}$ is the same as in $\mathfrak{A}$. Here is the check of the super interchange law in $\mathfrak{A}^{\mathrm{sop}}$:

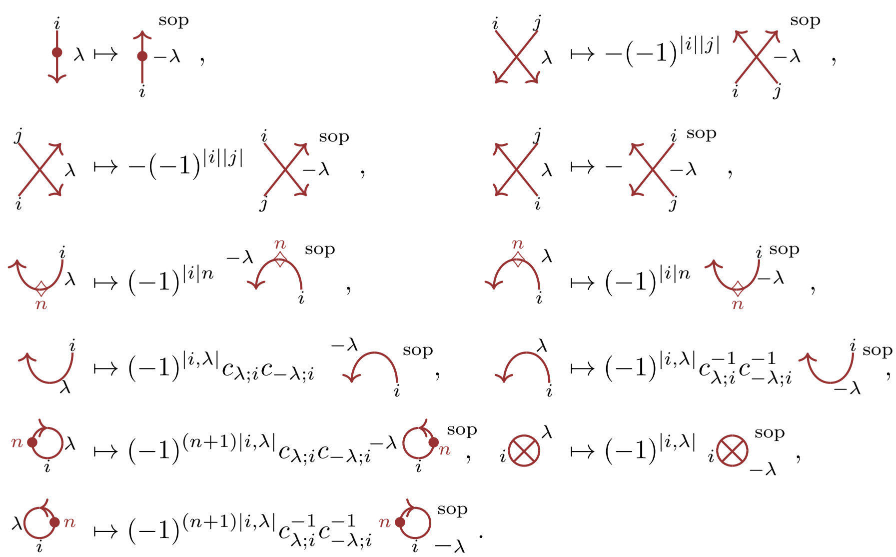
$$
\begin{aligned}
(x^{\mathrm{sop}}y^{\mathrm{sop}}) \circ (u^{\mathrm{sop}}v^{\mathrm{sop}}) &= (xy)^{\mathrm{sop}} \circ (uv)^{\mathrm{sop}} = (-1)^{(|x|+|y|)(|u|+|v|)}((uv) \circ (xy))^{\mathrm{sop}} \\
&= (-1)^{|x||u|+|y||u|+|y||v|}((u \circ x)(v \circ y))^{\mathrm{sop}} \\
&= (-1)^{|x||u|+|y||u|+|y||v|}(u \circ x)^{\mathrm{sop}}(v \circ y)^{\mathrm{sop}} \\
&= (-1)^{|y||u|}(x^{\mathrm{sop}} \circ u^{\mathrm{sop}})(y^{\mathrm{sop}} \circ v^{\mathrm{sop}}).
\end{aligned}
$$

We will often appeal to the following proposition to establish mirror images of relations in a horizontal axis. (This formulation is more convenient than the version in [B, Theorem 2.3], since ω really is an involution of $\mathfrak{U}(\mathfrak{g})$ rather than a map to another Kac-Moody 2-category.)

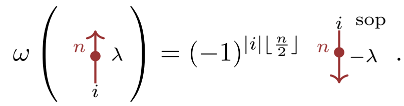
Proposition 3.5. *There is a 2-supercategory isomorphism* $\omega : \mathfrak{U}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{U}(\mathfrak{g})^{\mathrm{sop}}$ *defined by the strict 2-superfunctor* ω *given on objects by* $\omega(\lambda) := -\lambda$*, on generating 1-morphisms by* $\omega(E_i 1_\lambda) := F_i 1_{-\lambda}$ *and* $\omega(F_i 1_\lambda) := E_i 1_{-\lambda}$*, and on generating 2-morphisms by*



Moreover we have that $\omega^2 = \mathrm{id}$*, as follows from the following describing the effect of* ω *on the other named 2-morphisms in* $\mathfrak{U}(\mathfrak{g})$*:*

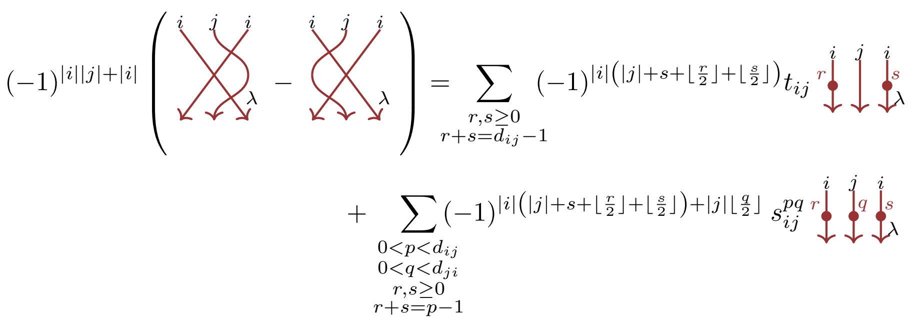


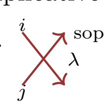
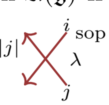
Proof. This is very similar to the proof of [B, Theorem 2.3] but the signs are considerably more subtle, so we include a few remarks. Note to start with that ω should send $x^{\circ n}$ (vertical composition computed in $\mathfrak{U}(\mathfrak{g})$) to $\omega(x)^{\circ n}$ (vertical composition computed in $\mathfrak{U}(\mathfrak{g})^{\mathrm{sop}}$), so that



(3.10)

It is important that the sign here matches the signs in (2.3). The proof of the existence of ω amounts to checking relations. For example, to verify (1.9) in the case $i = k \neq j$, one needs to show in view of (3.10) that

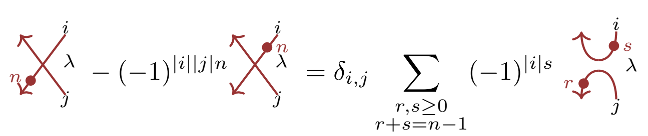


in $\mathfrak{U}(\mathfrak{g})$. This follows from (3.9). The other relations follow similarly using (1.10), (1.12)–(1.14), (3.7) and (3.5)–(3.6). The computation of the effect of ω on the other 2-morphisms is a mostly routine application of the definitions, but care is needed to distinguish multiplication (hence, multiplicative inverses) in $\mathfrak{U}(\mathfrak{g})$ from in $\mathfrak{U}(\mathfrak{g})^{\mathrm{sop}}$. For example, when $i \neq j$, the inverse of  is $(-1)^{|i||j|}$ . ∎

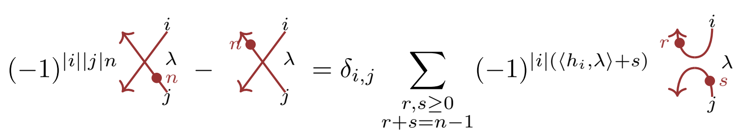
4. Leftward dot slides

We proceed to prove analogs of the relations (2.3) and (3.3)–(3.4) for leftward cups, caps and crossings.

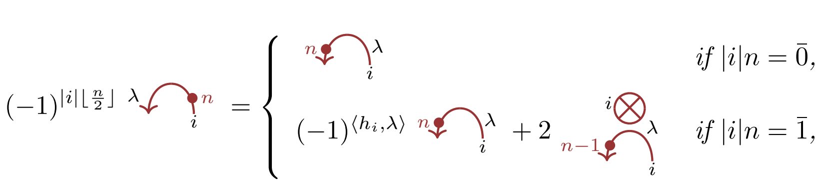
Proposition 4.1. *The following relations hold for all $n \geq 0$:*



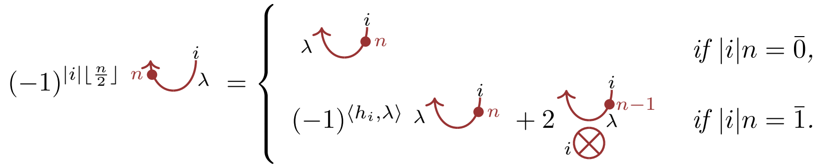
(4.1)



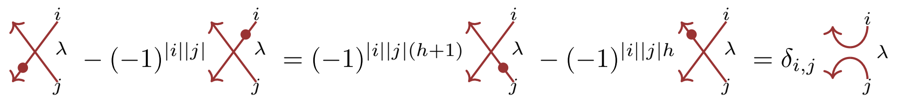
(4.2)



(4.3)

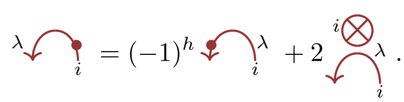
 (4.4)

Proof. Let $h := \langle h_i, \lambda \rangle$. The relations (4.1)–(4.2) follow easily by induction starting from the case $n = 1$, which asserts:

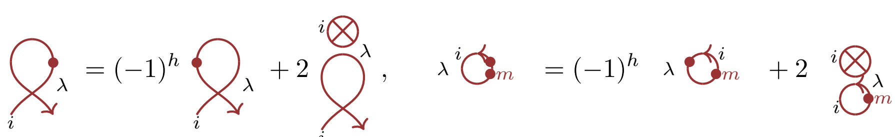


It suffices to prove this under the assumption that $h \geq 0$; the case $h < 0$ then follows by applying the Chevalley involution from Proposition 3.5. Under this assumption, one vertically composes the $n = 1$ case of (3.3)–(3.4) on top and bottom with a leftward crossing, then simplifies using (2.7) in case $i \neq j$ or (2.3) and (2.10)–(2.14) in case $i = j$.

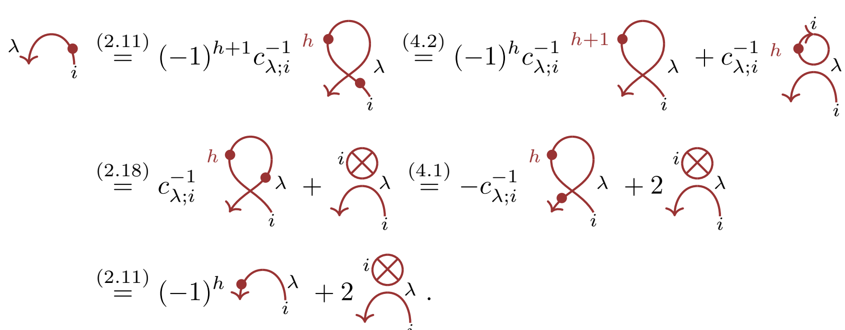
For (4.3)–(4.4), we just need to prove the former, since the latter then follows on applying ω. When i is even, (4.3) was already established in [B, Theorem 5.6], so let us assume for brevity that i is odd (though the argument here can easily be adapted to even i too). When $n = 1$ we must prove:

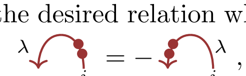


If $h < 0$ we vertically compose this on the bottom with the isomorphism $E_i F_i 1_\lambda \oplus 1_\lambda^{\oplus -h} \xrightarrow{\sim} F_i E_i 1_\lambda$ from (1.14) to reduce to checking



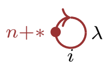
for all $0 \leq m < -h$. The first identity here is easily deduced from (3.3) and (2.14), while the second follows using (2.14) and the definition (2.18). Now assume that $h \geq 0$. Then we have:



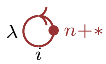
Thus we have proved the desired relation when $n = 1$. Applying it twice and using (1.24), we deduce that , which is the desired relation for $n = 2$. The general case follows easily from the two special cases established so far. $\square$

5. Infinite Grassmannian relations

Recall the shorthand for dotted bubbles from (2.17), and that the odd bubble  squares to zero. Our next proposition implies that the homomorphisms $\beta_{\lambda;i}$ from (1.19)–(1.20) in the introduction are well defined. In terms of these maps, it shows moreover that



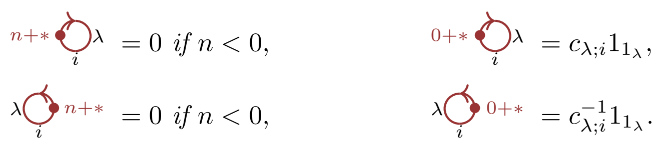
$$= \begin{cases} c_{\lambda;i}\beta_{\lambda;i}\left(\mathrm{e}_n\right) & \text{if } |i| = \bar{0}, \\ c_{\lambda;i}\beta_{\lambda;i}\left(\mathrm{e}_{\lfloor \frac{n}{2} \rfloor}\right) & \text{if } |i| = \bar{1} \text{ and } n \text{ is even}, \\ c_{\lambda;i}\beta_{\lambda;i}\left(\mathtt{d}\mathrm{e}_{\lfloor \frac{n}{2} \rfloor}\right) & \text{if } |i| = \bar{1} \text{ and } n \text{ is odd}, \end{cases} \tag{5.1}$$



$$= \begin{cases} c_{\lambda;i}^{-1}\beta_{\lambda;i}\left((-1)^n\, \mathrm{h}_n\right) & \text{if } |i| = \bar{0}, \\ c_{\lambda;i}^{-1}\beta_{\lambda;i}\left((-1)^{\lfloor \frac{n}{2} \rfloor}\, \mathrm{h}_{\lfloor \frac{n}{2} \rfloor}\right) & \text{if } |i| = \bar{1} \text{ and } n \text{ is even}, \\ c_{\lambda;i}^{-1}\beta_{\lambda;i}\left((-1)^{\lfloor \frac{n}{2} \rfloor}\, \mathtt{d}\mathrm{h}_{\lfloor \frac{n}{2} \rfloor}\right) & \text{if } |i| = \bar{1} \text{ and } n \text{ is odd}, \end{cases} \tag{5.2}$$

for all $n \geq 0$. This extends the infinite Grassmannian relation first introduced in [L]; see also [EL, Proposition 3.5] for a related result in the odd case.

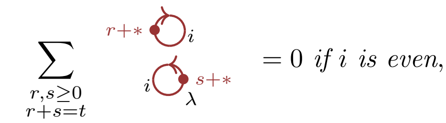
Proposition 5.1. *The following relations hold:*



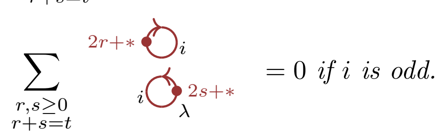
(5.3)

(5.4)

Also the following hold for all $t > 0$:

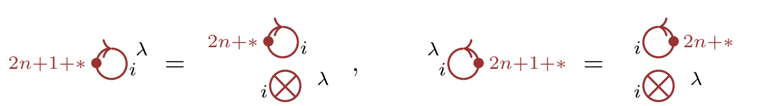


(5.5)



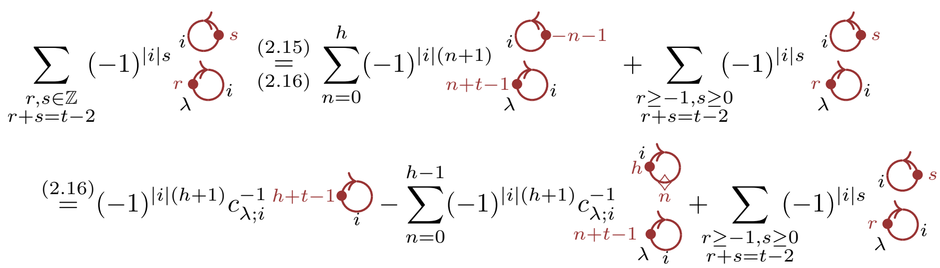
(5.6)

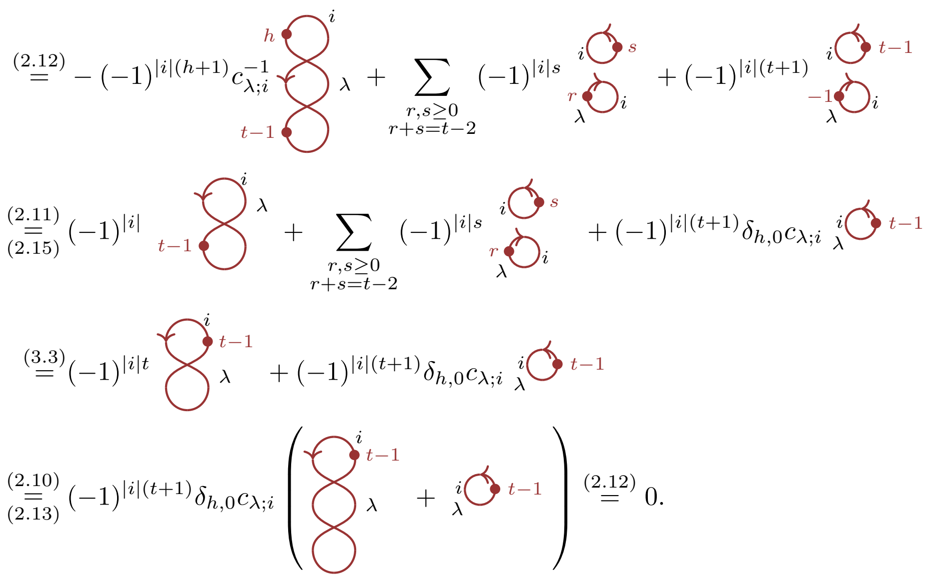
Finally if i is odd, the following hold for all $n \in \mathbb{Z}$:



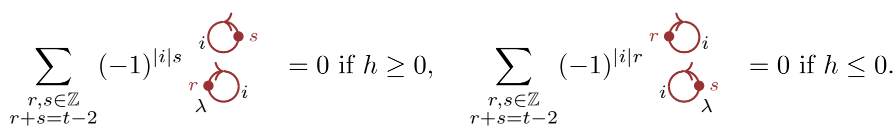
(5.7)

Proof. Let $h := \langle h_i, \lambda \rangle$. The equations (5.3)–(5.4) are implied by (2.13)–(2.16). For the rest, we first assume that $h \geq 0$ and calculate:





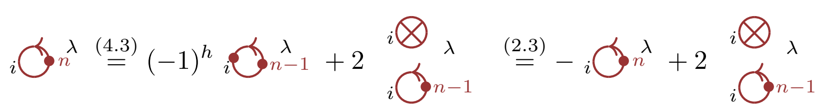
This establishes the first of the following identities, and the second follows from that on supercommuting the bubbles then applying the Chevalley involution from Proposition 3.5: for all $t > 0$ we have that



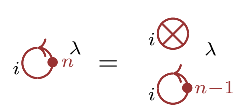
(5.8)

If i is even, (5.8) implies (5.5), and there is nothing more to be done.

For the remainder of the proof we assume that i is odd. Take $n > 0$ such that $n + h + 1$ is odd. We have that

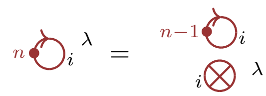


This shows that



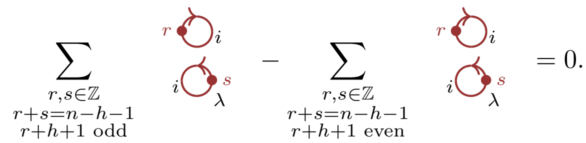
(5.9)

assuming $n > 0$ and $n + h + 1$ is odd. A similar argument for clockwise bubbles shows that

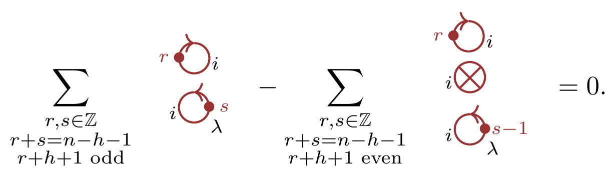


(5.10)

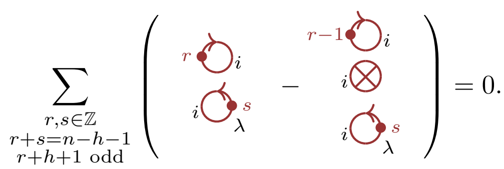
assuming that $n > 0$ and $n + h + 1$ is odd. Now we proceed to show by ascending induction on n that (5.10) also holds when $n \leq 0$ and $n + h + 1$ is odd. This statement is vacuous if $n < h$, and it is also clear in case $n = h$ thanks to the definition (2.18). So assume that $h < n \leq 0$, $n + h + 1$ is odd, and that (5.10) has been proved for all smaller n with $n+h+1$ odd. From (5.8), we get that

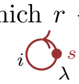


The terms in the first summation here are zero unless $s \geq -h-1$, hence, $r \leq n$. In the second summation we always have that $s+h+1$ is odd, hence, terms here are zero unless $s > -h-1 \geq 0$. Applying (5.9) to each non-zero term in the second summation, we deduce that

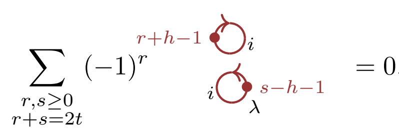


Now we reindex the second summation, replacing r by $r-1$ and s by $s+1$, to deduce that



In view of the induction hypothesis, all of the terms here in which $r < n$ vanish. This just leaves us with the term $r = n$, when $s = -h-1$ so  $= c_{\lambda;i}^{-1} 1_{1_\lambda}$, which we can cancel to establish the desired instance of (5.10). This completes the induction. Hence, we have established the first equation from (5.7); the second follows from that using Proposition 3.5.

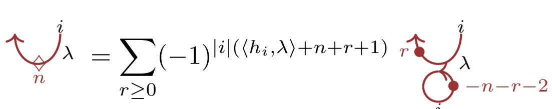
It just remains to prove (5.6). We explain this assuming that $h \leq 0$; then one can apply the Chevalley involution to get the other case. From (5.8) we get for any $t > 0$ that

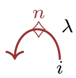
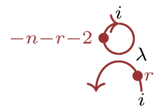


In all the terms of this summation we have that $r \equiv s \pmod 2$. If both r and s are odd, we can apply (5.7) twice to pull out two odd bubbles, hence, these terms vanish thanks to (1.24). This leaves just the terms in which both r and s are even, which is exactly what is needed to establish the identity (5.6). $\square$

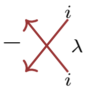
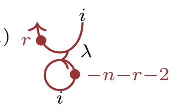
Once we have proved the next two corollaries, we will not need to refer to the decorated leftward cups and caps again.

Corollary 5.2. *The following relations hold:*

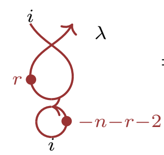
 *if* $0 \leq n < \langle h_i, \lambda \rangle$, (5.11)

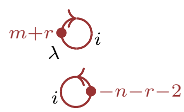

$$= \sum_{r\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle+n+r+1)}$$

$$\text{if } 0 \leq n < -\langle h_i, \lambda\rangle. \qquad (5.12)$$

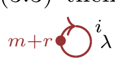
Proof. We explain the proof of (5.11); the proof of (5.12) is entirely similar or it may be deduced by applying ω using also (4.4). Let $h := \langle h_i, \lambda\rangle > 0$. Remembering the definition (2.8), it suffices to show that the vertical composition consisting of (1.13) on top of

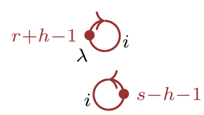
$$-$$

$$\oplus \bigoplus_{n=0}^{h-1} \sum_{r\geq 0} (-1)^{|i|(h+n+r+1)}$$


is equal to the identity. Using (2.12)–(2.13), this reduces to checking that

$$\sum_{r\geq 0} (-1)^{|i|(h+n+r+1)}$$

$$= 0 \qquad \text{if } 0 \leq n < h, \qquad (5.13)$$

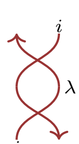
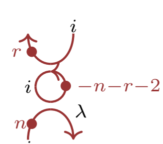
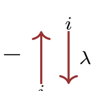
$$\sum_{r\geq 0} (-1)^{|i|(h+n+r+1)}$$

$$= \delta_{m,n} 1_{1_\lambda} \qquad \text{if } 0 \leq m, n < h. \qquad (5.14)$$

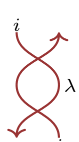
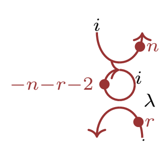
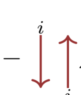
For (5.13), each term in the summation is zero: if $r \geq h$ the counterclockwise dotted bubble is zero by (5.4); if $0 \leq r < h$ one commutes the dots past the crossing using (3.3) then applies (2.13). To prove (5.14), note by (5.3)–(5.4) that in order for  to be non-zero we must have that $r \geq h - m - 1$, while for  to be non-zero we must have $r \leq h - n - 1$. Hence, we may assume that $m \geq n$, and are done for the same reasons in case $m = n$. If $m > n$ the left hand side of (5.14) is equal to

$$\sum_{\substack{r,s\geq 0\\ r+s=m-n}} (-1)^{|i|(m+n+r)}$$

.

Now one shows that this is zero using (5.5)–(5.7) and (1.24); when i is odd it is convenient when checking this to treat the cases $m \equiv n \pmod 2$ and $m \not\equiv n \pmod 2$ separately. □

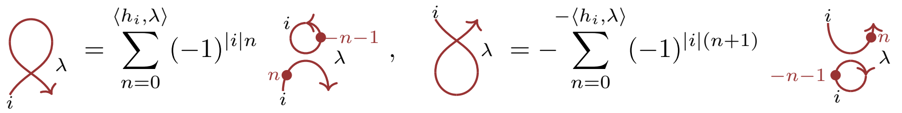
Corollary 5.3. *The following relations hold:*


$$= \sum_{n=0}^{\langle h_i,\lambda\rangle-1} \sum_{r\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle+n+r+1)}$$

$$-$$

$$, \qquad (5.15)$$


$$= \sum_{n=0}^{-\langle h_i,\lambda\rangle-1} \sum_{r\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle+n+r+1)}$$

$$-$$

$$. \qquad (5.16)$$

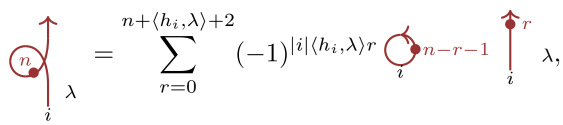
Proof. Substitute (5.11)–(5.12) into (2.12). □

Corollary 5.4. *The following relations hold:*

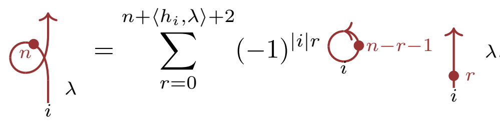


(5.17)

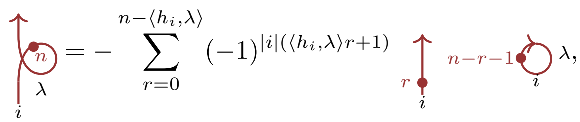
Hence, for $n \geq 0$ we have:



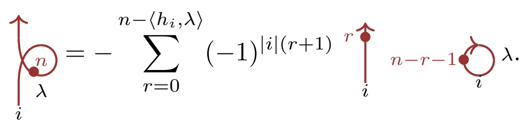
(5.18)



(5.19)

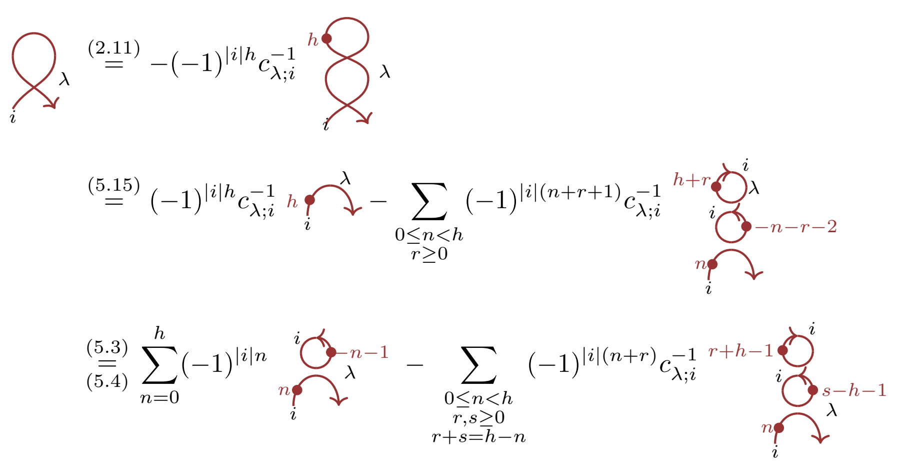


(5.20)



(5.21)

Proof. We first prove (5.17). By our usual argument with the Chevalley involution, it suffices to prove the left hand relation. We are done already by (2.14) if $h := \langle h_i, \lambda\rangle < 0$. If $h \geq 0$ then:



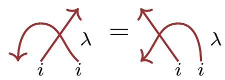
It remains to observe just like at the end of the proof of Corollary 5.2 that the second summation on the right hand side vanishes.

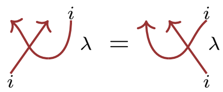
Finally to deduce (5.18)–(5.21), use (3.1)–(3.2) to commute the dots past the upward crossing, then convert the crossing to a rightward one using (1.10)–(1.11) and apply (5.17). □

6. Left adjunction

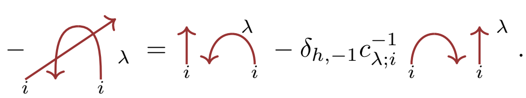
The leftward cups and caps form the unit and counit of another adjunction.

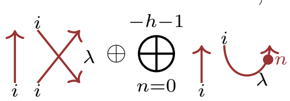
Lemma 6.1. *The following relations hold:*

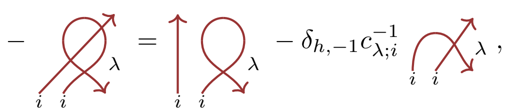

if $\langle h_i, \lambda\rangle \leq -1$, (6.1)

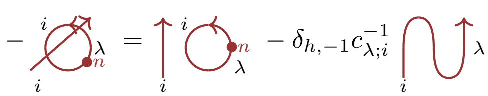

if $\langle h_i, \lambda\rangle \geq -1$. (6.2)

Proof. Let $h := \langle h_i, \lambda\rangle$ for short. First we prove (6.1), so $h \leq -1$. We claim that

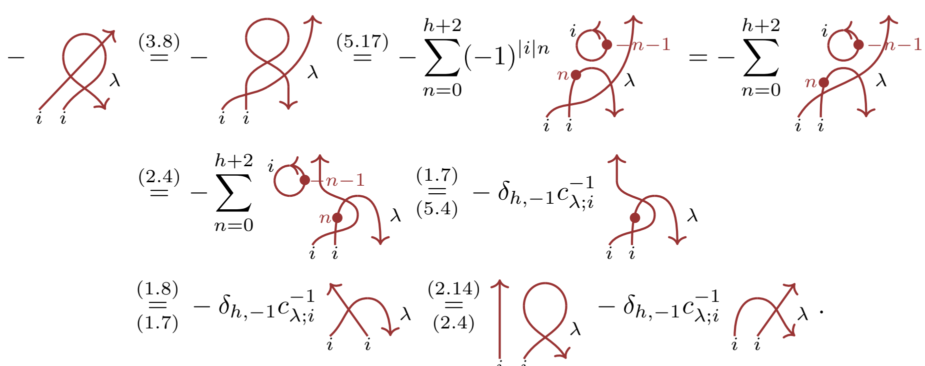

(6.3)

To establish the claim, we vertically compose on the bottom with the isomorphism  arising from (1.14) to reduce to showing equivalently that

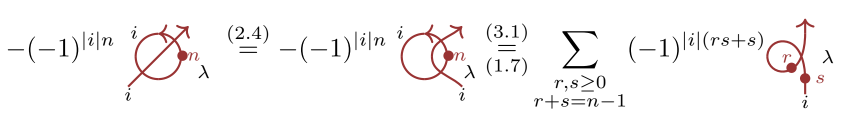

(6.4)


for $0 \leq n \leq -h-1$. (6.5)

Here is the verification of (6.4):

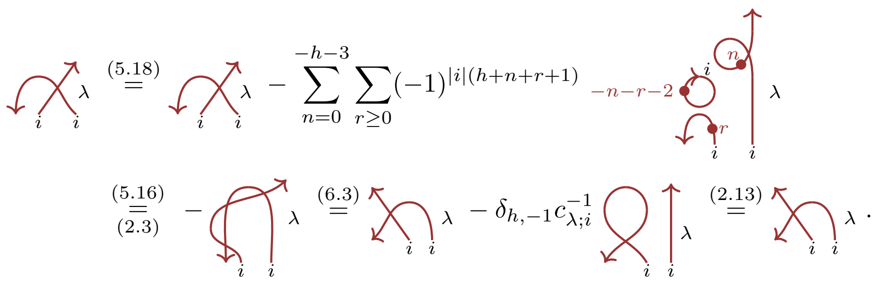


For (6.5), by (5.4) and (1.10), the right hand side is $c_{\lambda;i}^{-1}\uparrow_i\lambda$ if $n = -h-1 > 0$, and it is zero otherwise. On the other hand, the left hand side equals

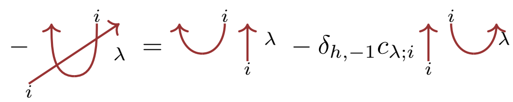


This is obviously zero if $n = 0$. Assuming $n > 0$, we apply (5.18) to see that it is zero unless $n = -h-1$, when the term with $r = -h-2, s = 0$ contributes $c_{\lambda;i}^{-1}\uparrow_i\lambda$.

This completes the proof of the claim. Now we can establish (6.1):

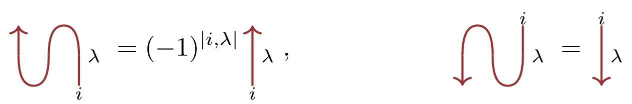


The proof of (6.2) follows by a very similar argument; one first checks that



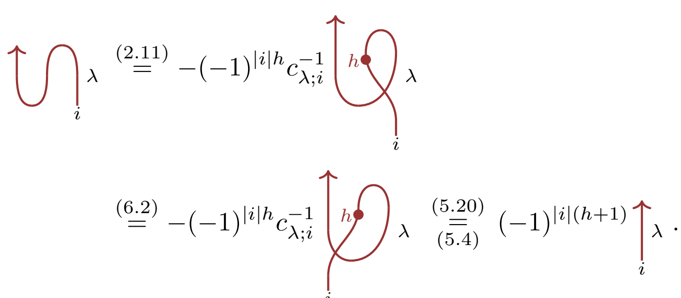
when $h \geq -1$. $\square$

Proposition 6.2. *The following relations hold:*

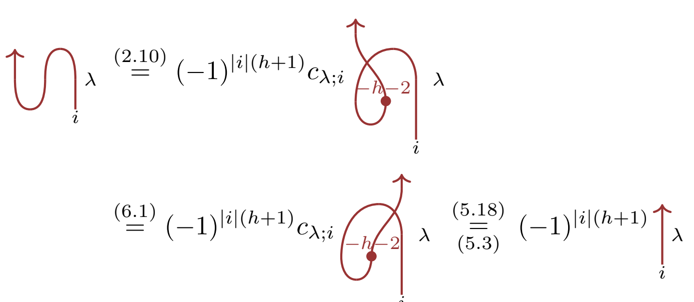


(6.6)

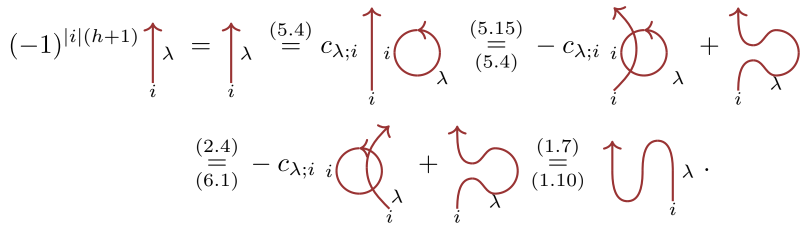
Proof. It suffices to prove the first equality; the second one then follows using Proposition 3.5. Let $h := \langle h_i, \lambda \rangle$ for short, and recall that $|i, \lambda| = |i|(h+1)$. If $h \geq 0$ then



If $h \leq -2$ then



Finally if $h = -1$ then

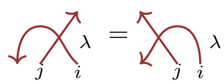


This completes the proof. □

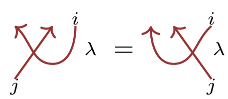
7. Final relations

There are just a few more important relations to be derived.

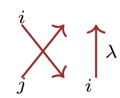
Lemma 7.1. *The following hold for all $i \neq j$:*

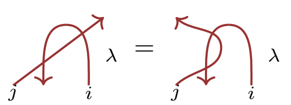


$$\text{if } \langle h_i, \lambda\rangle \leq \max(d_{ij} - 1, 0), \tag{7.1}$$



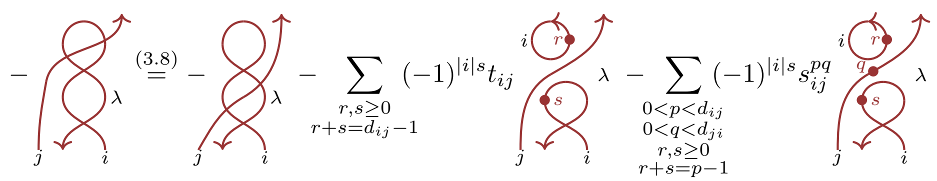
$$\text{if } \langle h_i, \lambda\rangle \geq d_{ij}. \tag{7.2}$$

Proof. Let $h := \langle h_i, \lambda\rangle$. First we prove (7.1) assuming that $0 < h \leq d_{ij} - 1$. Vertically composing on the bottom with the isomorphism  , we reduce to proving that

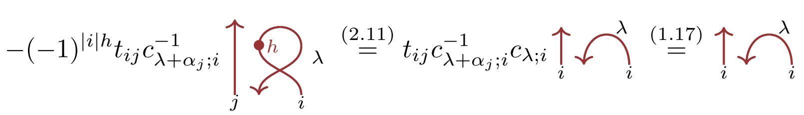


(7.3)

Then to check this, we apply (5.16) to transform the left hand side into

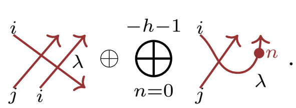


The first term on the right hand side here vanishes by (2.14). Also the terms in the summations are zero unless $r \geq d_{ij} - h - 1$ and $s \geq h$ by (2.13) and (5.4), hence, we are left just with the $r = d_{ij} - h - 1, s = h$ term, which equals

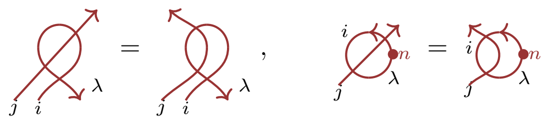


This is equal to the right hand side of (7.3) thanks to (2.7).

To complete the proof of (7.1), we need to show that it holds when $h \leq 0$. By (1.12) and (1.14), the following 2-morphism is invertible:

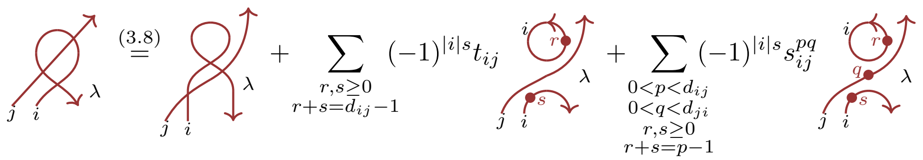


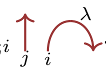
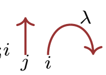
Vertically composing with this on the bottom, we deduce that the relation we are trying to prove is equivalent to the following relations:

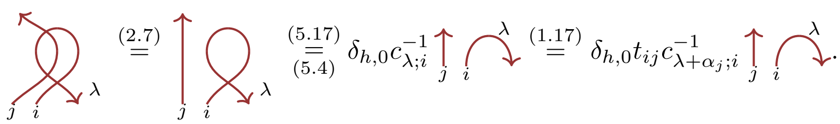


for $0 \le n < -h$. (7.4)

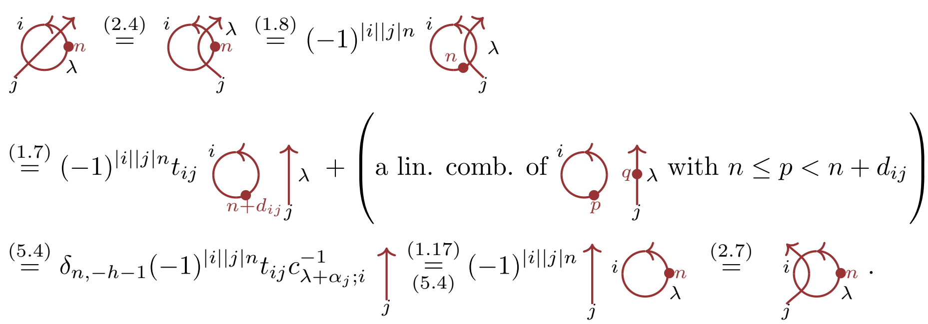
To establish the first of these, we pull the j-string past the ii-crossing:



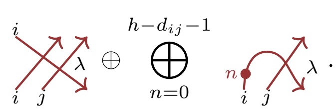
If $h < 0$ then all the terms on the right hand side vanish thanks to (2.14) and (5.4). If $h = 0$ and $d_{ij} > 0$ everything except for the $r = d_{ij} - 1$ term from the first sum vanishes, and we get $t_{ij} c^{-1}_{\lambda+\alpha_j;i}$ . Finally if $h = d_{ij} = 0$, we only have the first term on the right hand side, which contributes $t_{ij} c^{-1}_{\lambda+\alpha_j;i}$  again thanks to (5.17), (5.4), (2.4) and (1.7). This is what we want because:



We are just left with the right hand relations from (7.4) involving bubbles:



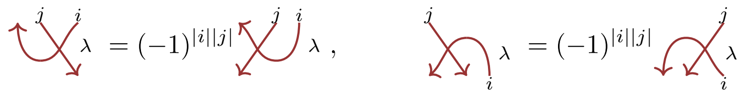
The relation (7.2) follows by very similar arguments to the previous paragraph; the first step is to vertically compose on the top with the isomorphism



□

Proposition 7.2. *The following relations hold for all i, j:*

(7.5)



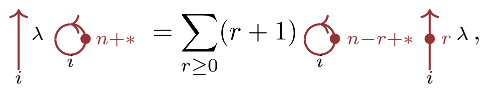
(7.6)

Proof. We get (7.5) in half of the cases from Lemmas 6.1 and 7.1. To deduce the other half of the cases, attach leftward cups (resp. caps) to the two strands at the bottom (resp. the top) of the relations established in these two lemma, then simplify using (6.6). Finally (7.6) follows from (7.5) using Proposition 3.5 as usual. □

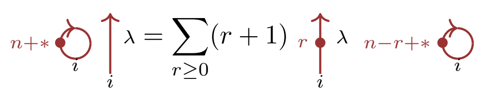
The final two propositions of the section extend [KL3, Propositions 3.3–3.5].

Proposition 7.3. *The following hold for all $n \geq 0$ and $\lambda \in P$.*

(i) *If i is even then*

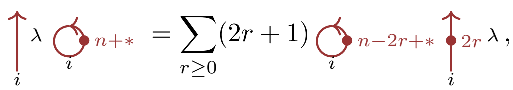


(7.7)

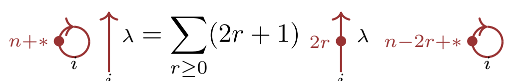


(7.8)

(ii) *If i is odd then*

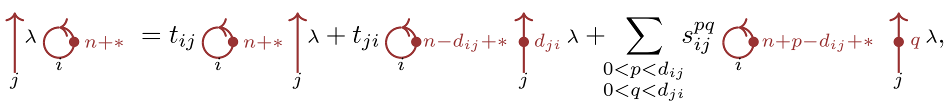


(7.9)

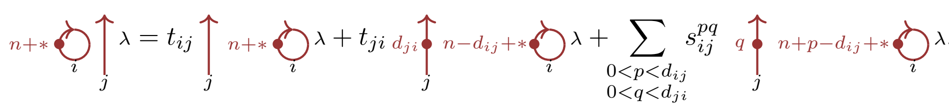


(7.10)

(iii) *For $i \neq j$ with $d_{ij} > 0$ we have that*

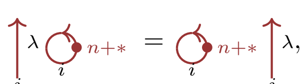


(7.11)

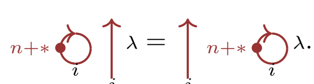


(7.12)

(iv) *For $i \neq j$ with $d_{ij} = 0$ we have that*



(7.13)



(7.14)

Proof. Let $h := \langle h_i, \lambda \rangle$ throughout the proof.

(i)–(ii) When i is even, this was already established in [L]. So we just need to prove (ii), assuming i is odd. We observe to start with that the identities (7.9) and (7.10) (for fixed λ and all $n \geq 0$) are equivalent. To see this, let us rephrase them in terms of power series. We make $\mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(E_i 1_\lambda)$ into a $\Bbbk[x]$-module so that x acts as by vertically composing on top with a dot. Let t be an indeterminate and $\mathrm{e}(t) := \sum_{n\geq 0} \mathrm{e}_n t^n$, $\mathrm{h}(t) := \sum_{n \geq 0} \mathrm{h}_n t^n$, which are power series in $\mathrm{Sym}[[t]]$.

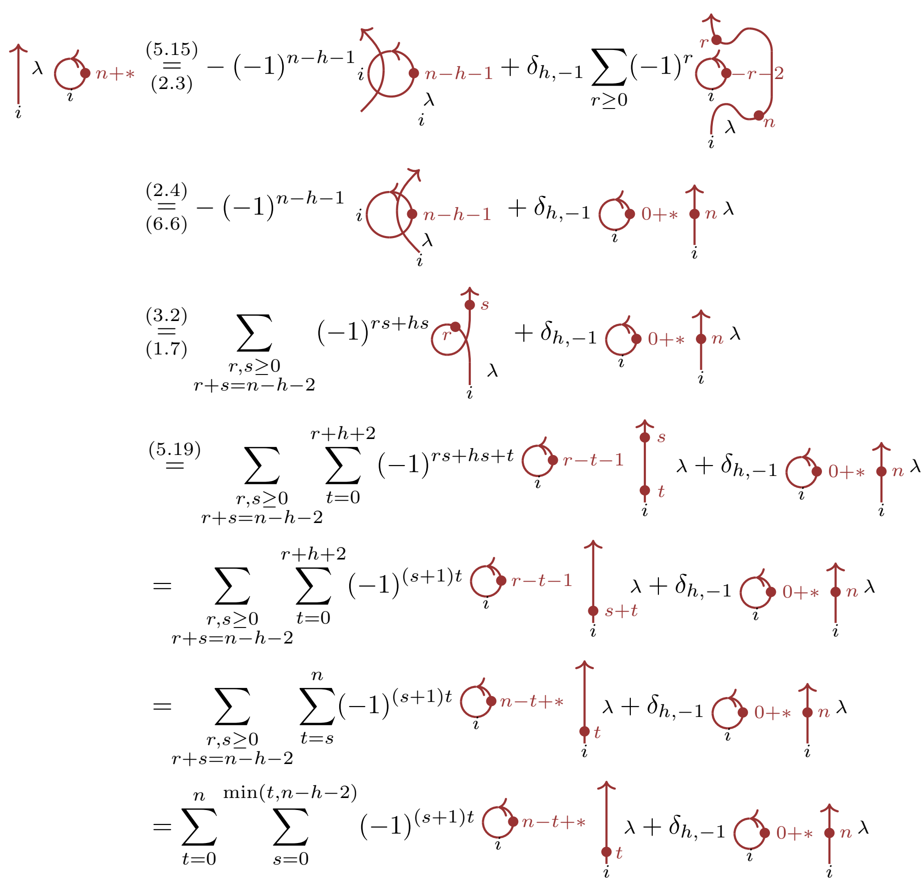
Recalling (5.1)–(5.2), the identities (7.9) and (7.10) for all $n \geq 0$ are equivalent to the generating function identities

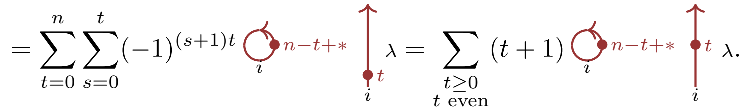
$$\uparrow_i \lambda\ \beta_{\lambda;i}((1-\mathtt{d}t)\mathrm{h}(-t^2)) = \left(\sum_{r\geq 0}(2r+1)x^{2r}t^{2r}\right)\beta_{\lambda+\alpha_i;i}((1-\mathtt{d}t)\mathrm{h}(-t^2))\uparrow_i \lambda,$$

$$\beta_{\lambda+\alpha_i;i}((1+\mathtt{d}t)\mathrm{e}(t^2))\uparrow_i \lambda = \left(\sum_{r\geq 0}(2r+1)x^{2r}t^{2r}\right)\uparrow_i \lambda\ \beta_{\lambda;i}((1+\mathtt{d}t)\mathrm{e}(t^2)),$$

respectively, as follows by equating coefficients of t. Since $\mathrm{e}(t)\mathrm{h}(-t) = 1$ in Sym and $\mathtt{d}^2 = 0$, we deduce that $(1+\mathtt{d}t)\mathrm{e}(t^2)$ and $(1-\mathtt{d}t)\mathrm{h}(-t^2)$ are two-sided inverses. Using this, it is easy to see that the two generating function identities are indeed equivalent, e.g. multiplying the first on the right by $\beta_{\lambda;i}((1+\mathtt{d}t)\mathrm{e}(t^2))$ and on the left by $\beta_{\lambda+\alpha_i;i}((1-\mathtt{d}t)\mathrm{e}(t^2))$ transforms it into the second.

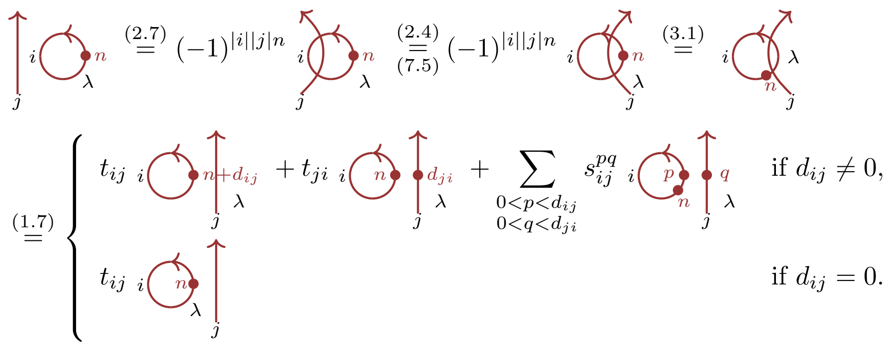
To complete the proof of (ii), we need to show that one of (7.9) or (7.10) holds for each fixed h. We proceed to verify (7.9) in case $h \leq -1$; a similar argument establishes (7.10) in case $h \geq -1$. So assume that $h \leq -1$. The identity to be proved is trivial in case $n = 0$ so suppose moreover that $n > 0$, so that $n-h-1 \geq 1$. Then we have that





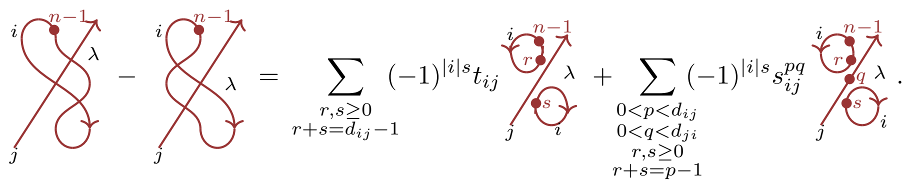
This is what we wanted.

(iii)–(iv) By an argument with generating functions similar to the one explained in the proof of (ii) above, the identities (7.11) and (7.12) are equivalent, as are (7.13) and (7.14). Therefore it suffices just to prove one of them for each fixed h and all $n \geq 0$. For any $n \geq 0$, we have that

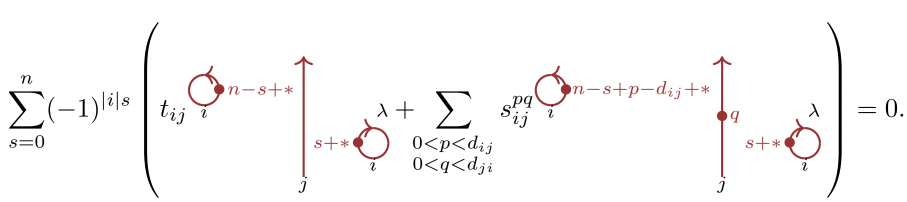


This proves both (7.11) and (7.13) for $n \geq h+1$. Also, the case $n = 0$ follows from (1.17), hence, we are completely done if $h \leq 0$. A similar argument establishes (7.12) and (7.14) for $n \geq d_{ij} - h + 1$, hence, we are completely done if $h \geq d_{ij}$.

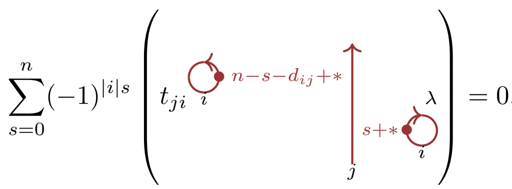
We are left with proving (7.11)–(7.12) when $1 \leq h \leq d_{ij} - 1$. We claim that (7.11) holds for all $n \leq d_{ij} - h$. The claim implies that (7.12) holds for all $n \leq d_{ij} - h$ too, and we have already established (7.12) for $n \geq d_{ij} - h + 1$, so the claim is enough to finish the proof. For the claim, we proceed by induction on $n = 0, 1, \dots, d_{ij} - h$. The base case $n = 0$ is trivial. For the induction step, take $1 \leq n \leq d_{ij} - h$. By (3.8), we have that



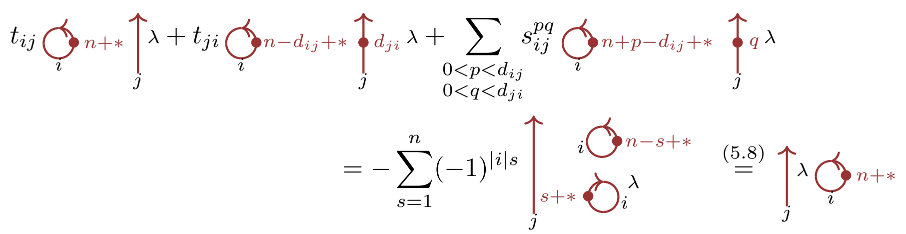
Both terms on the left hand side are zero: for the first this follows immediately from (2.13); for the second one this follows from (2.14) and (5.4) on applying (3.3) to pull the dots past the crossing. Replacing s by $s + h - 1$, we have proved that



Also because $n < d_{ij}$ we have that

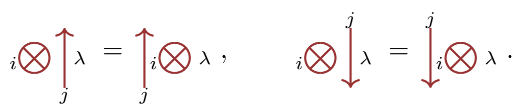


Now consider the identity obtained by adding these two expressions together. We use the induction hypothesis (7.11) to simplify all of the terms with $s \geq 1$, keeping the $s = 0$ terms on the left hand side, to obtain



This completes the proof of the claim. □

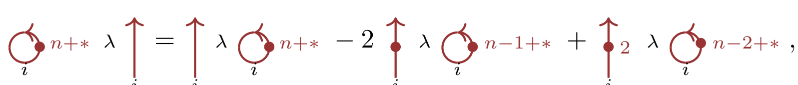
Corollary 7.4. *For $i, j \in I$ with i odd, we have that*



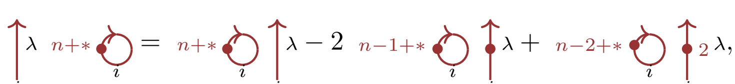
(7.15)

Proof. Remembering the definition (2.18), the first relation follows from the $n = 1$ cases of (7.9), (7.11) and (7.13); to see that the lower terms in (7.11) vanish, recall that d_{ij} is even. Hence, it satisfies $d_{ij} \geq 2$, and $s_{ij}^{pq} = 0$ if $p = d_{ij} - 1$. The second relation follows from the first by applying ω. □

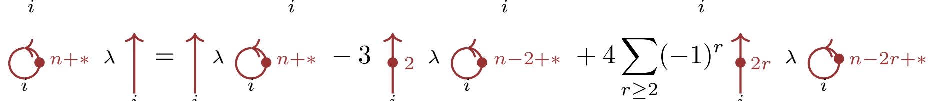
Remark 7.5. One can invert the formulae in Proposition 7.3 to obtain also various bubble slides in the other direction. For example, inverting (7.7)–(7.10) produces the following, for i even, i even, i odd and i odd, respectively:



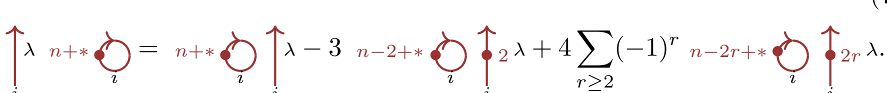
(7.16)



(7.17)

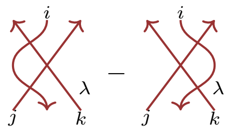


(7.18)

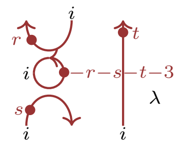


(7.19)

Proposition 7.6. *The following relation holds:*

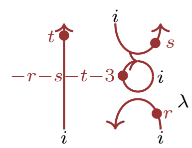


$$= \begin{cases} \displaystyle\sum_{r,s,t\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle + r+s+1)} \end{cases}$$



if $i = j = k$,

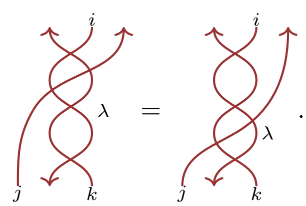
$$+ \sum_{r,s,t\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle + r+s+t)}$$



(7.20)

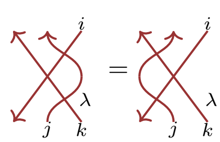
0 *otherwise.*

Proof. Assuming either $i = j = k$ or $i \neq k$, we attach crossings to the top left and bottom right pairs of strands of (3.8) to deduce that

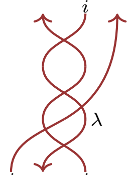


(7.21)

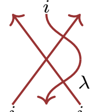
When $i \neq k$, the lemma follows easily from this on simplifying using (2.7). A similar argument treats the case $i \neq j$, attaching crossings to the top right and bottom left pairs of strands in the relation



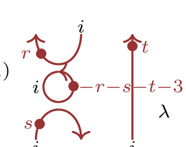
which may be deduced by attaching a leftward cap to the top left and a leftward cup to the bottom right of (1.9) and using (6.6) and (7.5). We are just left with the case that $i = j = k$. For this we use (7.21) again to reduce to checking:



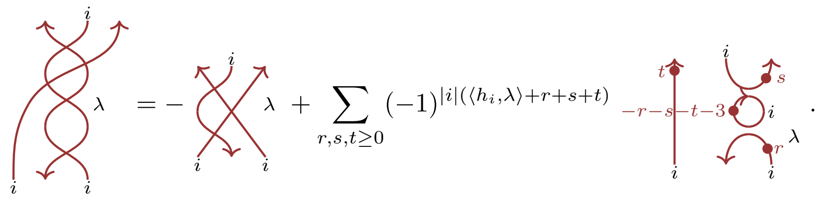
$= -$



$$- \sum_{r,s,t\geq 0} (-1)^{|i|(\langle h_i,\lambda\rangle + r+s+1)}$$



,



These two identities are proved in similar ways. One first uses (5.15)–(5.16) to reduce the double crossings, then (3.1)–(3.2) to pull the dots to the boundary, remembering also (2.13)–(2.14), (2.4) and (1.7). By now we can safely leave the details to the reader! □

8. The nondegeneracy conjecture

The main result of this section is a generalization of [KL3, Proposition 3.11]. We need some further notation which is adapted from [KL3]. Let Seq be the set of all words in the alphabet $\{\uparrow_i, \downarrow_i \mid i \in I\}$; our words correspond to the *signed sequences* of [KL3]. For $\mathtt{a} = \mathtt{a}_m \cdots \mathtt{a}_1 \in \mathrm{Seq}$, let

$$\mathrm{wt}(\mathtt{a}) := \sum_{i\in I} \Big(\#\{n = 1,\dots,m | \mathtt{a}_n = \uparrow_i\} - \#\{n = 1,\dots,m | \mathtt{a}_n = \downarrow_i\}\Big)\alpha_i \in Q. \quad (8.1)$$

To $\lambda \in P$ and $\mathtt{a} = \mathtt{a}_m \cdots \mathtt{a}_1 \in \mathrm{Seq}$, we associate the 1-morphism

$$E_{\mathtt{a}} 1_\lambda := E_{\mathtt{a}_m} \cdots E_{\mathtt{a}_1} 1_\lambda : \lambda \to \lambda + \mathrm{wt}(\mathtt{a}) \quad (8.2)$$

in $\mathfrak{U}(\mathfrak{g})$, with the convention that $E_{\uparrow_i} = E_i$ and $E_{\downarrow_i} = F_i$. As λ and $\mathtt{a}$ vary, these give all of the 1-morphisms in $\mathfrak{U}(\mathfrak{g})$.

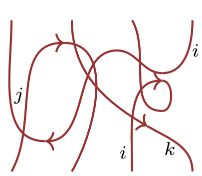
Suppose that we are given $\mathtt{a} = \mathtt{a}_m \cdots \mathtt{a}_1$ and $\mathtt{b} = \mathtt{b}_n \cdots \mathtt{b}_1 \in \mathrm{Seq}$. An $\mathtt{ab}$-*matching* is a planar diagram with

- m distinct vertices on a horizontal axis at the bottom labeled from right to left by the letters $\mathtt{a}_1, \dots, \mathtt{a}_m$;
- n distinct vertices on a horizontal axis at the top labeled from right to left by the letters $\mathtt{b}_1, \dots, \mathtt{b}_n$;
- $(m + n)/2$ smoothly immersed directed I-colored strands drawn between the horizontal axes whose endpoints are the given $(m + n)$ vertices.

We require moreover that:

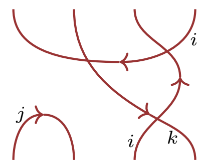
- the strands have only finitely many intersections and critical points (= points of slope zero);
- there are no intersections at critical points, no triple intersections, and no tangencies;
- the colors and directions of the strands are consistent with the letters at their endpoints.

Note at least one $\mathtt{ab}$-matching exists if and only if $\mathrm{wt}(\mathtt{a}) = \mathrm{wt}(\mathtt{b})$. Here is an example with $\mathtt{a} = \uparrow_j \downarrow_j \uparrow_i \downarrow_k$ and $\mathtt{b} = \uparrow_i \downarrow_k \uparrow_i \downarrow_i$:



A matching is *reduced* if each strand has at most one critical point which should either be a minimum or a maximum, there are no self-intersections of strands, and distinct strands intersect at most once.

Any $\mathtt{ab}$-matching defines a pairing between the letters of the words $\mathtt{a}$ and $\mathtt{b}$, two letters being paired if they are endpoints of the same strand. We say that two matchings are *equivalent* if they define the same pairing. Every matching is equivalent to at least one reduced matching. For example, here is a reduced matching equivalent to the matching displayed above:



A *decorated* $\mathtt{ab}$-*matching* is an $\mathtt{ab}$-matching whose strands have been decorated by finitely many dots located away from intersections and critical points, each of which is labeled by a non-negative integer. Given any decorated $\mathtt{ab}$-matching σ and $\lambda \in P$, there is a unique way to label the regions of σ by elements of P so that it becomes the diagrammatic representation of a 2-morphism $f(\sigma, \lambda) \in \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}}1_\lambda, E_{\mathtt{b}}1_\lambda)$ as above.

For each $\mathtt{a}, \mathtt{b} \in \mathrm{Seq}$, we choose a set $M(\mathtt{a}, \mathtt{b})$ of representatives for the equivalence classes of reduced $\mathtt{ab}$-matchings. For each element of $M(\mathtt{a}, \mathtt{b})$, we also choose a distinguished point on each of its strands located away from intersections and critical points. Then let $\widehat{M}(\mathtt{a}, \mathtt{b})$ be the set of decorated $\mathtt{ab}$-matchings obtained by taking each of the matchings in $M(\mathtt{a}, \mathtt{b})$ and putting a dot labeled with a non-negative integer at each of its distinguished points. Finally recall the homorphism $\beta_\lambda : \mathrm{SYM} \to \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda)$ from (1.22).

Theorem 8.1. *Take* $\mathtt{a}, \mathtt{b} \in \mathrm{Seq}$ *with* $\mathrm{wt}(\mathtt{a}) = \mathrm{wt}(\mathtt{b})$ *and any* $\lambda \in P$. *Viewing* $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}}1_\lambda, E_{\mathtt{b}}1_\lambda)$ *as a right* SYM*-module so that* $p \in \mathrm{SYM}$ *acts by horizontally composing on the right with* $\beta_\lambda(p)$*, the 2-morphisms* $\left\{ f(\sigma, \lambda) \middle| \sigma \in \widehat{M}(\mathtt{a}, \mathtt{b}) \right\}$ *generate* $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}}1_\lambda, E_{\mathtt{b}}1_\lambda)$ *as a right* SYM*-module.*

Proof. By the definitions, any 2-morphism in $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}}1_\lambda, E_{\mathtt{b}}1_\lambda)$ is a linear combination of diagrams obtained by horizontally and vertically composing the generators $x, \tau, \eta, \varepsilon, \eta'$ and ε'. Now the point is that we have derived enough relations above to be able to algorithmically rewrite any 2-morphism represented by such a diagram as a linear combination of the 2-morphisms $f(\sigma, \lambda)\beta_\lambda(p)$ for $\sigma \in \widehat{M}(\mathtt{a}, \mathtt{b})$ and $p \in \mathrm{SYM}$. This proceeds by induction on the total number of crossings in the diagram. We omit the details since it is essentially the same argument as used to prove [KL3, Proposition 3.11]. □

Now we can properly state the Nondegeneracy Conjecture from the introduction:

Nondegeneracy Conjecture. *For all* $\mathtt{a}, \mathtt{b} \in \mathrm{Seq}$ *with* $\mathrm{wt}(\mathtt{a}) = \mathrm{wt}(\mathtt{b})$ *and any* $\lambda \in P$*, the superspace* $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}}1_\lambda, E_{\mathtt{b}}1_\lambda)$ *is a free right* SYM*-module with basis* $\left\{ f(\sigma, \lambda) \,\middle|\, \sigma \in \widehat{M}(\mathtt{a}, \mathtt{b}) \right\}$.

9. The covering quantum group

Henceforth, we assume that the Cartan matrix is symmetrized by positive integers $(d_i)_{i \in I}$, and that the parameters are chosen to satisfy the homogeneity condition (1.31). Let $(-,-)$ be the symmetric bilinear form on the root lattice Q defined from $(\alpha_i, \alpha_j) := -d_i d_{ij}$. In this section, we recall the definition of the covering quantum group $\dot{U}_{q,\pi}(\mathfrak{g})$ of Clark, Hill and Wang [CHW1, CHW2]. Our exposition is based mostly on [CFLW] and [C]. Note that our q is the parameter denoted q^{-1} in [CHW1, CHW2, CFLW], which is v^{-1} in [C]. We write $e_i 1_\lambda$ and $f_i 1_\lambda$ in place of $E_i 1_\lambda$ and $F_i 1_\lambda$; we would also write k_i for the generator K_i^{-1} although we won't actually need this here. In [CHW2, CFLW, C], an additional assumption of "bar-consistency" is made on the super Cartan datum; we do not insist on this until later.

Let $\mathbb{L}$ be the ring $\mathbb{Q}(q)[\pi]/(\pi^2-1)$, and $\mathcal{L} := \mathbb{Z}[q, q^{-1}, \pi]/(\pi^2-1)$ as in the introduction. For $n \in \mathbb{Z}$, we let

$$[n]_{q,\pi} := \frac{q^n - (\pi q)^{-n}}{q - (\pi q)^{-1}} = \begin{cases} q^{n-1} + \pi q^{n-3} + \cdots + \pi^{n-1} q^{1-n} & \text{if } n \geq 0, \\ -\pi^n (q^{-n-1} + \pi q^{-n-3} + \cdots + \pi^{-n-1} q^{1+n}) & \text{if } n \leq 0. \end{cases}$$

There are corresponding quantum factorials and binomial coefficients:

$$[n]^!_{q,\pi} := [n]_{q,\pi}[n-1]_{q,\pi} \cdots [1]_{q,\pi}, \qquad \begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi} := \frac{[n]^!_{q,\pi}}{[r]^!_{q,\pi}[n-r]^!_{q,\pi}}.$$

We let $-$ be the involution of $\mathbb{L}$ (or $\mathcal{L}$) with $\overline{q} = q^{-1}$ and $\overline{\pi} = \pi$. Note this is different from the bar involution used in [CFLW, C]; in particular, our quantum integers are *not* bar invariant, but satisfy

$$\overline{[n]}_{q,\pi} = \pi^{n-1}[n]_{q,\pi} = -\pi[-n]_{q,\pi}. \tag{9.1}$$

We have that $\overline{\begin{bmatrix} n \\ r \end{bmatrix}}_{q,\pi} = \pi^{r(n-r)} \begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi}$, so that the quantum binomial coefficient is bar invariant if n is odd. For $i \in I$, we set

$$q_i := q^{d_i}, \qquad \pi_i := \pi^{|i|}.$$

Let $\dot{U}_{q,\pi}(\mathfrak{g})$ be the locally unital $\mathbb{L}$-algebra with mutually orthogonal idempotents $\{1_\lambda \mid \lambda \in P\}$, and generators $e_i 1_\lambda = 1_{\lambda+\alpha_i} e_i$ and $f_i 1_\lambda = 1_{\lambda-\alpha_i} f_i$ for all $i \in I$ and $\lambda \in P$, subject to the following relations:

$$(e_i f_j - \pi^{|i||j|} f_j e_i) 1_\lambda = \delta_{i,j} [\langle h_i, \lambda \rangle]_{q_i,\pi_i} 1_\lambda, \tag{9.2}$$

$$\sum_{r=0}^{d_{ij}+1} (-1)^r \pi_i^{r|j|+r(r-1)/2} \begin{bmatrix} d_{ij}+1 \\ r \end{bmatrix}_{q_i,\pi_i} e_i^{d_{ij}+1-r} e_j e_i^r 1_\lambda = 0 \qquad (i \neq j), \tag{9.3}$$

$$\sum_{r=0}^{d_{ij}+1} (-1)^r \pi_i^{r|j|+r(r-1)/2} \begin{bmatrix} d_{ij}+1 \\ r \end{bmatrix}_{q_i,\pi_i} f_i^{d_{ij}+1-r} f_j f_i^r 1_\lambda = 0 \qquad (i \neq j). \tag{9.4}$$

Also let $\dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$ be the $\mathcal{L}$-subalgebra of $\dot{U}_{q,\pi}(\mathfrak{g})$ generated by the divided powers

$$e_i^{(n)} 1_\lambda := e_i^n 1_\lambda / [n]^!_{q_i,\pi_i}, \qquad f_i^{(n)} 1_\lambda := f_i^n 1_\lambda / [n]^!_{q_i,\pi_i} \tag{9.5}$$

for all $i \in I, \lambda \in P$ and $n \geq 1$; see also [C, Lemma 3.5].

We also need the antilinear (with respect to the bar involution of the ground ring) algebra automorphisms $\psi, \omega : \dot{U}_{q,\pi}(\mathfrak{g}) \to \dot{U}_{q,\pi}(\mathfrak{g})$ and the linear algebra anti-automorphism $\rho : \dot{U}_{q,\pi}(\mathfrak{g}) \to \dot{U}_{q,\pi}(\mathfrak{g})$, which are defined on generators by

$$\omega(1_\lambda) = 1_{-\lambda}, \qquad \omega(e_i 1_\lambda) = f_i 1_{-\lambda}, \qquad \omega(f_i 1_\lambda) = e_i 1_{-\lambda}, \tag{9.6}$$

$$\psi(1_\lambda) = 1_\lambda, \qquad \psi(e_i 1_\lambda) = e_i 1_\lambda, \qquad \psi(f_i 1_\lambda) = \pi^{|i,\lambda|} f_i 1_\lambda, \tag{9.7}$$

$$\rho(1_\lambda) = 1_\lambda, \qquad \rho(e_i 1_\lambda) = q_i^{-\langle h_i,\lambda\rangle -1} 1_\lambda f_i, \qquad \rho(f_i 1_\lambda) = q_i^{\langle h_i,\lambda\rangle -1} 1_\lambda e_i. \tag{9.8}$$

Note all of these are involutions. Let $* := \rho \circ \psi$ and $! := \psi \circ \rho$. These are mutually inverse antilinear antiautomorphisms with

$$1_\lambda^* = 1_\lambda, \quad (e_i 1_\lambda)^* = q_i^{-\langle h_i,\lambda\rangle -1} 1_\lambda f_i, \qquad (f_i 1_\lambda)^* = q_i^{\langle h_i,\lambda\rangle -1} \pi^{|i,\lambda|} 1_\lambda e_i, \tag{9.9}$$

$$1_\lambda^! = 1_\lambda, \quad (e_i 1_\lambda)^! = \pi^{|i,\lambda|} q_i^{1+\langle h_i,\lambda\rangle} 1_\lambda f_i, \quad (f_i 1_\lambda)^! = q_i^{1-\langle h_i,\lambda\rangle} 1_\lambda e_i. \tag{9.10}$$

The notation here varies somewhat across the literature, e.g. the counterparts of our ω, ψ and ρ in the purely even setting are denoted by $\omega \circ \psi, \psi$ and $\bar\rho$ in [KL3]. In the remainder of the section, we are going to explain how to lift ω, ψ and ρ to the Kac-Moody 2-supercategory.

First, we must explain how to deal with antilinearity at the level of 2-categories. Let $\mathcal{A}$ be a graded supercategory. The supercategory $\mathcal{A}^{\mathrm{sop}}$ from Definition 3.4 is actually a graded supercategory with the same grading as $\mathcal{A}$, i.e. $\deg(f^{\mathrm{sop}}) = \deg(f)$. Similarly, if $\mathfrak{A}$ is a graded 2-supercategory then $\mathfrak{A}^{\mathrm{sop}}$ is a graded 2-supercategory. If $\mathfrak{A}$ is a graded (Q,Π)-2-supercategory in the sense of [BE, Definition 6.5], with structure maps $\sigma_\lambda : q_\lambda \xrightarrow{\sim} 1_\lambda, \bar\sigma_\lambda : q_\lambda^{-1} \xrightarrow{\sim} 1_\lambda$ and $\zeta_\lambda : \pi_\lambda \xrightarrow{\sim} 1_\lambda$, we can regard $\mathfrak{A}^{\mathrm{sop}}$ as a graded (Q,Π)-2-supercategory by declaring that its structure maps are $(\bar\sigma_\lambda^{-1})^{\mathrm{sop}} : q_\lambda^{-1} \xrightarrow{\sim} 1_\lambda$, $(\sigma_\lambda^{-1})^{\mathrm{sop}} : q_\lambda \xrightarrow{\sim} 1_\lambda$ and $(\zeta_\lambda^{-1})^{\mathrm{sop}} : \pi_\lambda \xrightarrow{\sim} 1_\lambda$. The key point here is that we have interchanged the roles of q and q^{-1}.

Lemma 9.1. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are graded 2-supercategories, and recall the definition of their (Q,Π)-envelopes $\mathfrak{A}_{q,\pi}$ and $\mathfrak{B}_{q,\pi}$ from Definition 1.6. Given a graded 2-superfunctor $\phi : \mathfrak{A} \to (\mathfrak{B}_{q,\pi})^{\mathrm{sop}}$, there is a canonical induced graded 2-superfunctor $\tilde\phi : \mathfrak{A}_{q,\pi} \to (\mathfrak{B}_{q,\pi})^{\mathrm{sop}}$.*

Proof. View $(\mathfrak{B}_{q,\pi})^{\mathrm{sop}}$ as a graded (Q,Π)-2-supercategory as explained above. Then apply the universal property of (Q,Π)-envelopes from [BE, Lemma 6.11(i)]. □

Remark 9.2. In the setup of Lemma 9.1, the construction from the proof of [BE, Lemma 6.11(i)] implies the following explicit description for $\tilde\phi$. It is equal to ϕ on objects. On a 1-morphism F in $\mathfrak{A}$ with $\phi(F) = Q^{m'}\Pi^{a'}F'$ for a 1-morphism F' in $\mathfrak{B}$, we have that $\tilde\phi(Q^m\Pi^a F) = Q^{m'-m}\Pi^{a+a'}F'$. Given another 1-morphism G in $\mathfrak{A}$ with $\phi(G) = Q^{n'}\Pi^{b'}G'$ and $x \in \mathrm{Hom}_{\mathfrak{A}}(F,G)$ with $\phi(x) = \left((x')_{n',b'}^{m',a'}\right)^{\mathrm{sop}}$ for $x' \in \mathrm{Hom}_{\mathfrak{B}}(G',F')$, we have that

$$\tilde\phi\left(x_{m,a}^{n,b}\right) = (-1)^{a|x|+b|x|+ab+b}\left((x')_{n'-n,b+b'}^{m'-m,a+a'}\right)^{\mathrm{sop}}.$$

Note also that $\tilde\phi$ is not strict (even if ϕ itself is strict). Its coherence map

$$\tilde c_{Q^n\Pi^b G, Q^m\Pi^a F} : \tilde\phi(Q^n\Pi^b G)\tilde\phi(Q^m\Pi^a F) \xrightarrow{\sim} \tilde\phi(Q^{m+n}\Pi^{a+b}GF)$$

is $(-1)^{ab}\left(f_{k'-m-n,a+b+c'}^{m'+n'-m-n,a+b+a'+b'}\right)^{\mathrm{sop}}$, where $\left(f_{k',c'}^{m'+n',a'+b'}\right)^{\mathrm{sop}}$ denotes the coherence map $c_{G,F} : \phi(G)\phi(F) \xrightarrow{\sim} \phi(GF)$ for ϕ, for H' defined so that $\phi(GF) = Q^{k'}\Pi^{c'}H'$ and $f \in \mathrm{Hom}_{\mathfrak{B}}(H', G'F')$.

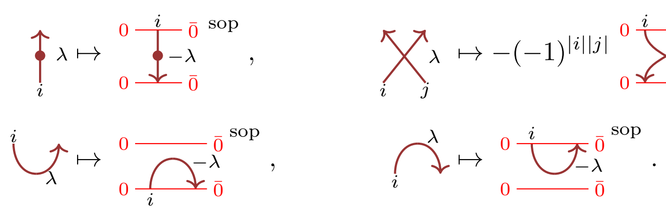
Since we are assuming now that the parameters satisfy (1.31), the Kac-Moody 2-supercategory $\mathfrak{U}(\mathfrak{g})$ is a graded 2-supercategory with $\mathbb{Z}$-grading defined as in the introduction. Let $\mathfrak{U}_{q,\pi}(\mathfrak{g})$ be its (Q,Π)-envelope from Definition 1.6. We now proceed to define the categorical counterparts of the antilinear automorphisms (9.6)–(9.7). Actually, the first was already defined in Proposition 3.5, but we need to extend this to the envelope.

Proposition 9.3. *There is an isomorphism of graded 2-supercategories*

$$\tilde{\omega} : \mathfrak{U}_{q,\pi}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$$

defined on objects by $\lambda \mapsto -\lambda$ and 1-morphisms by $Q^m\Pi^a E_i 1_\lambda \mapsto Q^{-m}\Pi^a F_i 1_{-\lambda}$, $Q^m\Pi^a F_i 1_\lambda \mapsto Q^{-m}\Pi^a E_i 1_{-\lambda}$.

Proof. If we compose the strict 2-superfunctor from Proposition 3.5 with the canonical inclusion $\mathfrak{U}(\mathfrak{g})^{\mathrm{sop}} \to \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$, we obtain a strict graded 2-superfunctor $\omega : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$. This is defined on objects by $\lambda \mapsto \lambda$, on 1-morphisms by $E_i 1_\lambda \mapsto Q^0\Pi^{\bar{0}} F_i 1_{-\lambda}$, $F_i 1_\lambda \mapsto Q^0\Pi^{\bar{0}} E_i 1_{-\lambda}$, and on 2-morphisms by the following:



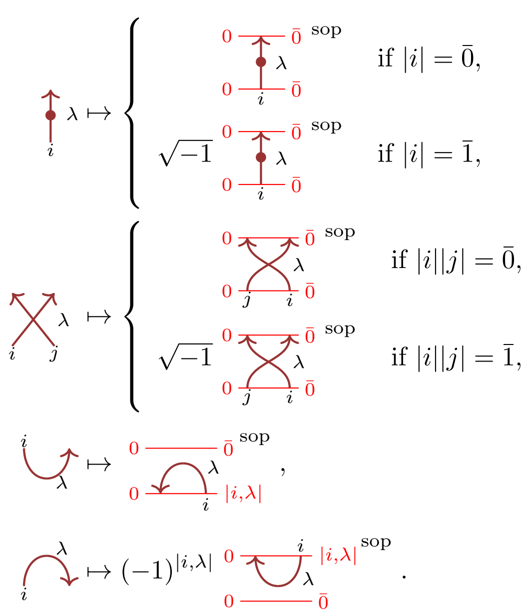
It remains to apply Lemma 9.1 to get the desired graded 2-superfunctor $\tilde{\omega}$ (which is no longer strict). ☐

Proposition 9.4. *Assume that there is a given element $\sqrt{-1} \in \Bbbk_{\bar{0}}$ which squares to -1. Then there is an isomorphism of graded 2-supercategories*

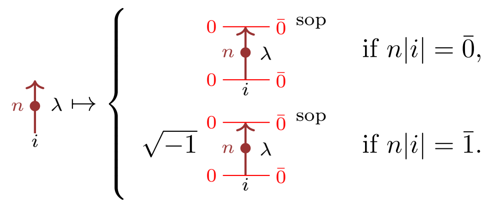
$$\tilde{\psi} : \mathfrak{U}_{q,\pi}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$$

defined on objects by $\lambda \mapsto \lambda$ and 1-morphisms by $Q^m\Pi^a E_i 1_\lambda \mapsto Q^{-m}\Pi^a E_i 1_\lambda$, $Q^m\Pi^a F_i 1_\lambda \mapsto Q^{-m}\Pi^{a+|i,\lambda|} F_i 1_\lambda$.

Proof. We claim that there is a strict graded 2-superfunctor $\psi : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$ which is defined on objects by $\lambda \mapsto \lambda$, 1-morphisms by $E_i 1_\lambda \mapsto Q^0\Pi^{\bar{0}} E_i 1_\lambda$, $F_i 1_\lambda \mapsto Q^0\Pi^{|i,\lambda|} F_i 1_\lambda$, and 2-morphisms by the following:



To prove the claim, one needs to verify the relations. Note to start with that



Using this, the quiver Hecke superalgebra relations (1.7)–(1.9) are straightforward. The inversion relations (1.12)–(1.14) are also fine. The adjunction relations (1.10) need a little more care since the signs coming from (1.30) play a role. Then apply Lemma 9.1 to get the desired graded 2-superfunctor $\tilde{\psi}$ (which is no longer strict). □

Definition 9.5. Let $\mathfrak{A}$ be a graded 2-supercategory. Define $\mathfrak{A}^{\text{srev}}$ to be the graded 2-supercategory with the same objects as $\mathfrak{A}$, and morphism categories

$$\mathrm{Hom}_{\mathfrak{A}^{\text{srev}}}(\mu, \lambda) := \mathrm{Hom}_{\mathfrak{A}}(\lambda, \mu).$$

We write $F^{\text{srev}} : \mu \to \lambda$ (resp. $x^{\text{srev}} : F^{\text{srev}} \to G^{\text{srev}}$) for the 1-morphism (resp. 2-morphism) in $\mathfrak{A}^{\text{srev}}$ defined by the 1-morphism $F : \lambda \to \mu$ (resp. $x : F \to G$) in $\mathfrak{A}$. Then, horizontal composition in $\mathfrak{A}^{\text{srev}}$ is defined on 1-morphisms by $(F^{\text{srev}})(G^{\text{srev}}) := (GF)^{\text{srev}}$ and on homogeneous 2-morphisms by $(x^{\text{srev}})(y^{\text{srev}}) := (-1)^{|x||y|}(yx)^{\text{srev}}$. Vertical composition of 2-morphisms in $\mathfrak{A}^{\text{srev}}$ is the same as in $\mathfrak{A}$. Here is the check of the super interchange law in $\mathfrak{A}^{\text{srev}}$:

$$\begin{aligned}(x^{\text{srev}}y^{\text{srev}}) \circ (u^{\text{srev}}v^{\text{srev}}) &= (-1)^{|x||y|+|u||v|}(yx)^{\text{srev}} \circ (vu)^{\text{srev}} \\ &= (-1)^{|x||y|+|u||v|}((yx) \circ (vu))^{\text{srev}} \\ &= (-1)^{|x||y|+|u||v|+|x||v|}((y \circ v)(x \circ u))^{\text{srev}} \\ &= (-1)^{|y||u|}(x \circ u)^{\text{srev}}(y \circ v)^{\text{srev}} \\ &= (-1)^{|y||u|}(x^{\text{srev}} \circ u^{\text{srev}})(y^{\text{srev}} \circ v^{\text{srev}}).\end{aligned}$$

If $\mathfrak{A}$ is a graded (Q, Π)-2-supercategory with structure maps $\sigma_\lambda : q_\lambda \xrightarrow{\sim} 1_\lambda, \bar{\sigma}_\lambda : q_\lambda^{-1} \xrightarrow{\sim} 1_\lambda$ and $\zeta_\lambda : \pi_\lambda \xrightarrow{\sim} 1_\lambda$, we can regard $\mathfrak{A}^{\text{srev}}$ as a graded (Q, Π)-2-supercategory by declaring that its structure maps are $(\sigma_\lambda)^{\text{srev}} : (q_\lambda)^{\text{srev}} \xrightarrow{\sim} (1_\lambda)^{\text{srev}}$, $(\bar{\sigma}_\lambda)^{\text{srev}} : (q_\lambda^{-1})^{\text{srev}} \xrightarrow{\sim} (1_\lambda)^{\text{srev}}$ and $(\zeta_\lambda)^{\text{srev}} : (\pi_\lambda)^{\text{srev}} \xrightarrow{\sim} (1_\lambda)^{\text{srev}}$.

Lemma 9.6. *Suppose that $\mathfrak{A}$ and $\mathfrak{B}$ are graded 2-supercategories. Given a graded 2-superfunctor $\phi : \mathfrak{A} \to (\mathfrak{B}_{q,\pi})^{\text{srev}}$, there is a canonical induced graded 2-superfunctor $\tilde{\phi} : \mathfrak{A}_{q,\pi} \to (\mathfrak{B}_{q,\pi})^{\text{srev}}$.*

Proof. View $(\mathfrak{B}_{q,\pi})^{\text{srev}}$ as a graded (Q, Π)-2-supercategory as explained above. Then apply [BE, Lemma 6.11(i)]. □

Remark 9.7. In the setup of Lemma 9.6, the construction from the proof of [BE, Lemma 6.11(i)] implies the following explicit description for $\tilde{\phi}$. It is equal to ϕ on objects. On a 1-morphism F in $\mathfrak{A}$ with $\phi(F) = (Q^{m'}\Pi^{a'}F')^{\text{srev}}$ for a 1-morphism F' in $\mathfrak{B}$, we have that $\tilde{\phi}(Q^m\Pi^a F) = (Q^{m+m'}\Pi^{a+a'}F')^{\text{srev}}$. Given another 1-morphism G in $\mathfrak{A}$ with $\phi(G) = (Q^{n'}\Pi^{b'}G')^{\text{srev}}$ and $x \in \mathrm{Hom}_{\mathfrak{A}}(F, G)$ with $\phi(x) = \left((x')^{n',b'}_{m',a'}\right)^{\text{srev}}$ for $x' \in \mathrm{Hom}_{\mathfrak{B}}(F', G')$, we have that

$$\tilde{\phi}\left(x^{n,b}_{m,a}\right) = (-1)^{aa'+bb'}\left((x')^{n+n',b+b'}_{m+m',a+a'}\right)^{\text{srev}}.$$

The coherence map

$$\tilde{c}_{Q^n\Pi^b G, Q^m\Pi^a F} : \tilde{\phi}(Q^n\Pi^b G)\tilde{\phi}(Q^m\Pi^a F) \xrightarrow{\sim} \tilde{\phi}(Q^{m+n}\Pi^{a+b}GF)$$

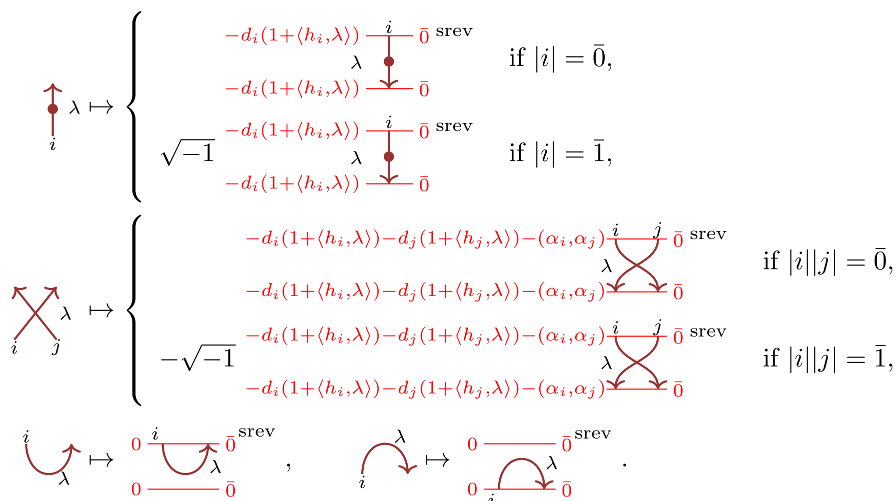
is $(-1)^{a(b+b')+(a+b)(a'+b'+c')}\left(f^{m+n+k',a+b+c'}_{m+n+m'+n',a+b+a'+b'}\right)^{\mathrm{srev}}$, where $\left(f^{k',c'}_{m'+n',a'+b'}\right)^{\mathrm{srev}}$ denotes the coherence map $c_{G,F} : \phi(G)\phi(F) \xrightarrow{\sim} \phi(GF)$ for ϕ, for H' defined so that $\phi(GF) = Q^{k'}\Pi^{c'}H'$ and $f \in \mathrm{Hom}_{\mathfrak{B}}(F'G', H')$.

Proposition 9.8. *Assume that there is a given element $\sqrt{-1} \in \Bbbk_{\bar{0}}$ which squares to -1. Then there is an isomorphism of graded 2-supercategories*

$$\tilde{\rho} : \mathfrak{U}_{q,\pi}(\mathfrak{g}) \xrightarrow{\sim} \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{srev}}$$

such that $\lambda \mapsto \lambda$ and $Q^m\Pi^a E_i 1_\lambda \mapsto (Q^{m-d_i(1+\langle h_i,\lambda\rangle)}\Pi^a 1_\lambda F_i)^{\mathrm{srev}}$, $Q^m\Pi^a F_i 1_\lambda \mapsto (Q^{m-d_i(1-\langle h_i,\lambda\rangle)}\Pi^a 1_\lambda E_i)^{\mathrm{srev}}$.

Proof. We claim that there is a strict graded 2-superfunctor $\rho : \mathfrak{U}(\mathfrak{g}) \to \mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{srev}}$ defined on objects by $\lambda \mapsto \lambda$, 1-morphisms by $E_i 1_\lambda \mapsto (Q^{-d_i(1+\langle h_i,\lambda\rangle)}\Pi^{\bar{0}} 1_\lambda F_i)^{\mathrm{srev}}$, $F_i 1_\lambda \mapsto (Q^{-d_i(1-\langle h_i,\lambda\rangle)}\Pi^{|i,\lambda|} 1_\lambda E_i)^{\mathrm{srev}}$, and 2-morphisms by the following:



To prove the claim, one needs to verify the relations. The quiver Hecke relations are the most complicated; for these, use (3.5)–(3.6), (3.7) and (3.9). (Note the degree shifts actually play no role in this argument; they are included to match (9.8).) Finally, apply Lemma 9.6 to get $\tilde{\rho}$. □

Suppose in this paragraph that $\Bbbk = \Bbbk_{\bar{0}}$ is a field. Then the underlying 2-category $\underline{\mathfrak{U}}_{q,\pi}(\mathfrak{g})$ is a (Q,Π)-2-category in the sense of [BE, Definition 6.14], as is its additive Karoubi envelope $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$. The Grothendieck ring $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ is a locally unital $\mathcal{L}$-algebra with distinguished idempotents $\{1_\lambda \mid \lambda \in P\}$. The analogous Grothendieck ring arising from $\mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{sop}}$ may be identified with $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ as a ring, but now q acts as q^{-1}. This means that the isomorphisms $\tilde{\omega}$ and $\tilde{\psi}$ from Propositions 9.3–9.4 induce *antilinear* locally unital algebra automorphisms

$$[\tilde{\omega}], [\tilde{\psi}] : K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})) \to K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})).$$

Also, the Grothendieck ring arising from $\mathfrak{U}_{q,\pi}(\mathfrak{g})^{\mathrm{srev}}$ may be identified with the opposite $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))^{\mathrm{op}}$, so that the isomorphism $\tilde{\rho}$ from Proposition 9.8 induces a linear algebra antiautomorphism

$$[\tilde\rho] : K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})) \to K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})).$$

The epimorphism $\gamma : \dot U_{q,\pi}(\mathfrak{g})_{\mathcal{L}} \twoheadrightarrow K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ to be constructed in Theorem 11.7 below intertwines the maps ω, ψ and ρ from (9.6)–(9.8) with $[\tilde\omega], [\tilde\psi]$ and $[\tilde\rho]$.

Remark 9.9. One can also consider the compositions $\tilde\rho \circ \tilde\psi$ and $\tilde\psi \circ \tilde\rho$. Both of these maps can be defined directly on generators, revealing that they actually do not require the existence of $\sqrt{-1} \in \Bbbk$, unlike $\tilde\rho$ and $\tilde\psi$ themselves. Just as discussed in [KL3, (3.46)–(3.47)], these maps may also be interpreted as taking right duals/mates and left duals/mates, respectively. They decategorify to the maps $*$ and ! from (9.9)–(9.10).

10. The sesquilinear form

Continue with the assumptions from section 9. Let $\mathbf{f}$ be the $\mathbb{L}$-superalgebra on generators $\{\theta_i \mid i \in I\}$ with $|\theta_i| := |i|$, subject to relations

$$\sum_{r=0}^{d_{ij}+1} (-1)^r \pi_i^{r|j|+r(r-1)/2} \begin{bmatrix} d_{ij}+1 \\ r \end{bmatrix}_{q_i,\pi_i} \theta_i^{d_{ij}+1-r} \theta_j \theta_i^r = 0 \tag{10.1}$$

for all $i \neq j$. There is a Q-grading $\mathbf{f} = \bigoplus_{\alpha \in Q} \mathbf{f}_\alpha$ on $\mathbf{f}$ compatible with the $\mathbb{Z}/2$-grading defined by declaring that each θ_i is of degree α_i. Viewing $\mathbf{f} \otimes \mathbf{f}$ as an algebra with the twisted multiplication $(x \otimes y)(x' \otimes y') := \pi^{|y||x'|} q^{-(\beta,\alpha')} xx' \otimes yy'$ for homogeneous $x \in \mathbf{f}_\alpha, y \in \mathbf{f}_\beta, x' \in \mathbf{f}_{\alpha'}, y' \in \mathbf{f}_{\beta'}$, we let $r : \mathbf{f} \to \mathbf{f} \otimes \mathbf{f}$ be the superalgebra homomorphism defined from $r(\theta_i) = \theta_i \otimes 1 + 1 \otimes \theta_i$ for each $i \in I$. By [CHW1, Proposition 3.4.1], there is a (non-degenerate) symmetric bilinear form $(-,-)$ on $\mathbf{f}$ defined by the following properties:

- $(\theta_i, \theta_j) = \delta_{i,j}/(1 - \pi_i q_i^2)$;
- $(xy, z) = (x \otimes y, r(z))$;
- $(x, yz) = (r(x), y \otimes z)$.

Here, the form on $\mathbf{f} \otimes \mathbf{f}$ is defined from $(x \otimes y, x' \otimes y') := (x, x')(y, y')$. Note that $\mathbf{f}_\alpha$ and $\mathbf{f}_\beta$ are orthogonal for $\alpha \neq \beta$.

Theorem 10.1 (Lusztig, Clark)**.** *There is a unique sesquilinear form (= antilinear in the first argument, linear in the second) $\langle -,- \rangle : \dot U_{q,\pi}(\mathfrak{g}) \times \dot U_{q,\pi}(\mathfrak{g}) \to \mathbb{L}$ such that the following hold:*

(1) $\langle 1_\mu x 1_\lambda, 1_{\mu'} x' 1_{\lambda'} \rangle = 0$ if $\lambda \neq \lambda'$ or $\mu \neq \mu'$;
(2) $\langle xy, z \rangle = \langle y, x^* z \rangle$;
(3) $\langle e_{i_d} \cdots e_{i_1} 1_\lambda, e_{j_d} \cdots e_{j_1} 1_\lambda \rangle = (\theta_{i_1} \cdots \theta_{i_d}, \theta_{j_1} \cdots \theta_{j_d})$.

Moreover:

(4) $\langle x, y \rangle = \langle \psi(y), \psi(x) \rangle$;
(5) $\langle x, yz \rangle = \langle y^! x, z \rangle$.

Assuming in addition that the bar-consistency assumption of [C, Definition 2.1(d)] *holds, i.e.*

$$d_i \equiv |i| \pmod 2 \qquad \text{for each } i \in I, \tag{10.2}$$

the form $\langle -,- \rangle$ is non-degenerate.

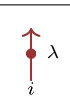
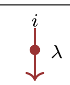
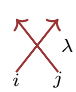
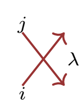
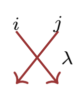
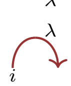
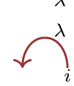
Proof. There is clearly at most one sesquilinear form on $\dot U_{q,\pi}(\mathfrak{g})$ satisfying properties (1)–(3). To see that there is indeed such a form, we appeal to [C, Proposition 5.8], which defines a bilinear form $(-,-)'$ on $\dot{U}_{q,\pi}(\mathfrak{g})$ satisfying four properties. Our form $\langle -,-\rangle$ is obtained from Clark's form by setting

$$\langle x,y\rangle := (\sigma(\psi(u)),\sigma(v))', \tag{10.3}$$

where ψ is the antilinear automorphism from (9.7) and σ is the linear antiautomorphism defined by declaring that $\sigma(1_\lambda)=1_\lambda$, $\sigma(e_i1_\lambda)=1_\lambda f_i$ and $\sigma(f_i1_\lambda)=1_\lambda e_i$. We leave it as an exercise to the reader to check that Clark's four properties translate into our properties (1)–(4); actually, one needs the opposite formulation of Clark's second property which may be derived from [C, Proposition 5.3], noting that Clark's τ_1 is our $\sigma\circ *\circ\psi\circ\sigma$. The property (5) is immediate from (2) and (4) plus the definition of (9.10). Finally, assuming bar-consistency, the non-degeneracy follows from [C, Theorem 5.12]. □

Remark 10.2. One could also define a bilinear (rather than sesquilinear) form $(-,-)$ on $\dot{U}_{q,\pi}(\mathfrak{g})$ by setting $(x,y):=\langle\psi(x),y\rangle$. This is a generalization of Lusztig's form from [Lu, Theorem 26.1.2] which is slightly different from the one introduced in [C]. Theorem 10.1 implies that $(-,-)$ is symmetric and it satisfies $(xy,z)=(y,\rho(x)z)$.

The next theorem gives a graphical description of the form $\langle -,-\rangle$ in the spirit of [KL3, Theorem 2.7]. Recall the notation Seq from section 8. For $\mathtt{a},\mathtt{b}\in\mathrm{Seq}$, let $\widehat{M}(\mathtt{a},\mathtt{b})$ be chosen as in Theorem 8.1. For $\sigma\in\widehat{M}(\mathtt{a},\mathtt{b})$ and $\lambda\in P$, define the *degree* $\deg(\sigma,\lambda)$ and the *parity* $|\sigma,\lambda|$ to be the degree and parity of the homogeneous 2-morphism $f(\sigma,\lambda)$, i.e. we sum the degrees and parities of all of the generating dots, cups, caps and crossings in the diagram for $f(\sigma,\lambda)$ as listed in the following table:

Generator	Degree	Parity	Generator	Degree	Parity
	$2d_i$	$\lvert i\rvert$		$2d_i$	$\lvert i\rvert$
	$-(\alpha_i,\alpha_j)$	$\lvert i\rvert\lvert j\rvert$		0	$\lvert i\rvert\lvert j\rvert$
	$-(\alpha_i,\alpha_j)$	$\lvert i\rvert\lvert j\rvert$		0	$\lvert i\rvert\lvert j\rvert$
	$d_i(1+\langle h_i,\lambda\rangle)$	$\bar{0}$		$d_i(1-\langle h_i,\lambda\rangle)$	$\lvert i,\lambda\rvert$
	$d_i(1-\langle h_i,\lambda\rangle)$	$\bar{0}$		$d_i(1+\langle h_i,\lambda\rangle)$	$\lvert i,\lambda\rvert$

Just as we did in (8.2), a word $\mathtt{a}=\mathtt{a}_m\cdots\mathtt{a}_1\in\mathrm{Seq}$ defines a monomial

$$e_{\mathtt{a}}1_\lambda := e_{\mathtt{a}_m}\cdots e_{\mathtt{a}_1}1_\lambda\in\dot{U}_{q,\pi}(\mathfrak{g}), \tag{10.4}$$

where $e_{\uparrow_i}:=e_i$ and $e_{\downarrow_i}:=f_i$. Clearly, these monomials taken over all $\mathtt{a}\in\mathrm{Seq}$ and all $\lambda\in P$ span $\dot{U}_{q,\pi}(\mathfrak{g})$.

Theorem 10.3. *The sesquilinear form $\langle -,-\rangle$ from Theorem 10.1 satisfies*

$$\langle e_{\mathtt{a}}1_\lambda, e_{\mathtt{b}}1_\mu\rangle = \delta_{\lambda,\mu}\sum_{\sigma\in\widehat{M}(\mathtt{a},\mathtt{b})} q^{\deg(\sigma,\lambda)}\pi^{|\sigma,\lambda|} \tag{10.5}$$

for each $\mathtt{a}, \mathtt{b} \in \mathrm{Seq}$ *and* $\lambda, \mu \in P$.

Proof. This argument parallels the proof of [KL3, Theorem 2.7] closely. We can clearly assume $\mu = \lambda$. Let $\langle \mathtt{a}, \mathtt{b}\rangle_\lambda$ denote the expression on the right hand side of (10.5). Note to start with that $\langle \mathtt{a}, \mathtt{b}\rangle_\lambda$ does not depend on the particular choice made for $\widehat{M}(\mathtt{a}, \mathtt{b})$. This follows because one can pass between any two choices of decorated reduced matchings by a sequence of isotopies which do not change degrees or parities of diagrams. (This is similar to the proof of Theorem 8.1, which applied more complicated relations which are the same as these isotopies plus terms with fewer crossings.) To complete the proof of the theorem, we must show:

$$\langle e_{\mathtt{a}} 1_\lambda, e_{\mathtt{b}} 1_\lambda\rangle = \langle \mathtt{a}, \mathtt{b}\rangle_\lambda. \tag{10.6}$$

We proceed with a series of claims, which mimic [KL3, Lemmas 2.8–2.12].

Claim 1. The identity (10.6) is true in case $\mathtt{a}$ and $\mathtt{b}$ are positive, i.e. they only involve upward arrows.

To see this, if $\mathtt{a} = \uparrow_{i_c} \cdots \uparrow_{i_1}$ and $\mathtt{b} = \uparrow_{j_d} \cdots \uparrow_{j_1}$, then $M(\mathtt{a}, \mathtt{b})$ is empty unless $c = d$, in which case its elements are in bijection with permutations $w \in S_d$ such that $i_{w(r)} = j_r$ for each $r = 1, \dots, d$, and we have that

$$\langle \mathtt{a}, \mathtt{b}\rangle_\lambda = \delta_{c,d} \sum_{w \in S_d} \left(\prod_{r=1}^d \frac{\delta_{i_{w(r)}, j_r}}{1 - \pi_{i_r} q_{i_r}^2}\right)\Bigg(\prod_{\substack{1 \leq r < s \leq d\\ w(r) > w(s)}} \pi^{|i_r||i_s|} q^{-(\alpha_{i_r}, \alpha_{i_s})}\Bigg).$$

Using Theorem 10.1(iv), it remains to check that this equals $(\theta_{i_1} \cdots \theta_{i_c}, \theta_{j_1}, \dots, \theta_{j_d})$. This follows by the explicit definition of the latter form on $\mathbf{f}$.

Claim 2. $\langle e_i e_{\mathtt{a}} 1_\lambda, e_{\mathtt{b}} 1_\lambda\rangle = \langle \uparrow_i \mathtt{a}, \mathtt{b}\rangle_\lambda \Leftrightarrow \langle e_{\mathtt{a}} 1_\lambda, f_i e_{\mathtt{b}} 1_\lambda\rangle = \langle \mathtt{a}, \downarrow_i \mathtt{b}\rangle_\lambda$.
Claim 3. $\langle f_i e_{\mathtt{a}} 1_\lambda, e_{\mathtt{b}} 1_\lambda\rangle = \langle \downarrow_i \mathtt{a}, \mathtt{b}\rangle_\lambda \Leftrightarrow \langle e_{\mathtt{a}} 1_\lambda, e_i e_{\mathtt{b}} 1_\lambda\rangle = \langle \mathtt{a}, \uparrow_i \mathtt{b}\rangle_\lambda$.

The proofs of these are the same as for [KL3, Lemma 2.9]. For example, for Claim 2, one considers the bijection between $\widehat{M}(\uparrow_i \mathtt{a}, \mathtt{b})$ and $\widehat{M}(\mathtt{a}, \downarrow_i \mathtt{b})$ obtained by attaching a cup on the bottom left. On the algebraic side, one uses Theorem 10.1(2) and (9.9).

Claim 4. $\langle e_{\mathtt{a}} e_i f_j e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \langle \mathtt{a}\uparrow_i\downarrow_j \mathtt{b}, \varnothing\rangle_\lambda \Leftrightarrow \langle e_{\mathtt{a}} f_j e_i e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \langle \mathtt{a}\downarrow_j\uparrow_i \mathtt{b}, \varnothing\rangle_\lambda$, assuming $i \neq j$.

Since $\langle e_{\mathtt{a}} e_i f_j e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \pi^{|i||j|}\langle e_{\mathtt{a}} f_j e_i e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle$ by (9.2), we must show that

$$\langle \mathtt{a}\uparrow_i\downarrow_j \mathtt{b}, \varnothing\rangle_\lambda = \pi^{|i||j|}\langle \mathtt{a}\downarrow_j\uparrow_i \mathtt{b}, \varnothing\rangle_\lambda.$$

This follows by considering the bijection between $\widehat{M}(\mathtt{a}\uparrow_i\downarrow_j \mathtt{b}, \varnothing)$ and $\widehat{M}(\mathtt{a}\downarrow_j\uparrow_i \mathtt{b}, \varnothing)$ obtained attaching a rightward crossing under the $\uparrow_i\downarrow_j$ to convert it to $\downarrow_j\uparrow_i$; see the proof of [KL3, Lemma 2.11] for further explanations. The only difference for us is that the crossing is odd in case $|i||j| = \bar{1}$.

Claim 5. Assuming that $\langle e_{\mathtt{a}} e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \langle \mathtt{ab}, \varnothing\rangle_\lambda$, we have that $\langle e_{\mathtt{a}} e_i f_i e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \langle \mathtt{a}\uparrow_i\downarrow_i \mathtt{b}, \varnothing\rangle_\lambda \Leftrightarrow \langle e_{\mathtt{a}} f_i e_i e_{\mathtt{b}} 1_\lambda, 1_\lambda\rangle = \langle \mathtt{a}\downarrow_i\uparrow_i \mathtt{b}, \varnothing\rangle_\lambda$.

Define μ so that $e_{\mathtt{b}} 1_\lambda = 1_\mu e_{\mathtt{b}}$. In view of (9.2), we must show that

$$\langle \mathtt{a}\uparrow_i\downarrow_i \mathtt{b}, \varnothing\rangle_\lambda - \pi^{|i|}\langle \mathtt{a}\downarrow_i\uparrow_i \mathtt{b}, \varnothing\rangle_\lambda = [\langle h_i, \mu\rangle]_{q_i, \pi_i}\langle \mathtt{ab}, \varnothing\rangle_\lambda. \tag{10.7}$$

To see this, we divide the decorated matchings in $\widehat{M}(\mathtt{a}\uparrow_i\downarrow_i \mathtt{b}, \varnothing)$ and $\widehat{M}(\mathtt{a}\downarrow_i\uparrow_i \mathtt{b}, \varnothing)$ into three classes exactly as explained in the proof of [KL3, Lemma 2.12]. It is then easy to see that the contributions to the left hand side of (10.7) from the first two classes cancel. The third classes arise from decorated matchings in $\widehat{M}(\mathtt{ab})$ by inserting a cap (clockwise or counterclockwise in the two cases) between $\mathtt{a}$ and $\mathtt{b}$. Hence, like in the proof of [KL3, Lemma 2.12] remembering also the sesquilinearity of $\langle -,-\rangle$, we see that the left hand side of (10.7) expands to

$$\overline{\left[q_i^{1-\langle h_i,\mu\rangle}/(1-\pi_i q_i^2) - \pi_i \pi^{|i,\mu|} q_i^{1+\langle h_i,\mu\rangle}/(1-\pi_i q_i^2)\right]}\langle \mathtt{ab}, \varnothing\rangle_\lambda.$$

This simplifies to the right hand side of (10.7).

Now we can complete the proof of (10.6) in general. Using Claims 2 and 3, we reduce to checking (10.6) in the special case that $\mathtt{b} = \varnothing$. Under this assumption, we then proceed by induction on the length of $\mathtt{a}$. Using Claims 4 and 5 plus the induction hypothesis, we can rearrange $\mathtt{a}$ to assume that all $\downarrow$'s appear to the left of all $\uparrow$'s. Then we use Claims 3 and 1 to finish the proof. □

Example 10.4 (cf. [C, Example 5.7])**.**

$$\langle e_i^{(r)} 1_\lambda, e_i^{(r)} 1_\lambda\rangle = \langle f_i^{(r)} 1_\lambda, f_i^{(r)} 1_\lambda\rangle = \prod_{s=1}^{r} \frac{1}{1-(\pi_i q_i^2)^s},$$
$$\langle e_i f_i 1_\lambda, 1_\lambda\rangle = \pi^{|i,\lambda|}\langle 1_\lambda, e_i f_i 1_\lambda\rangle = \frac{q_i^{1-\langle h_i,\lambda\rangle}}{1-\pi_i q_i^2},$$
$$\langle e_i f_i 1_\lambda, f_i e_i 1_\lambda\rangle = \langle f_i e_i 1_\lambda, e_i f_i 1_\lambda\rangle = \frac{\pi_i + q_i^2}{(1-\pi_i q_i^2)^2}.$$

11. Surjectivity of γ

In this section, we continue with the assumptions of §9, and also assume that $\Bbbk = \Bbbk_{\bar 0}$ is a field. For a graded superalgebra A, we write A-$\mathcal{GSM}od$ for the Abelian category of graded left A-supermodules with morphisms that preserve degree and parity. Let Q and Π denote the grading and parity shift functors on A-$\mathcal{GSM}od$, so that $(QV)_n = V_{n-1}$ and $(\Pi V)_a = V_{a+\bar 1}$. Let A-$\mathcal{GSP}roj$ be the full subcategory of A-$\mathcal{GSM}od$ consisting of the finitely generated projective supermodules. Let $K_0(A)$ denote the split Grothendieck group of A-$\mathcal{GSP}roj$. It is naturally an $\mathcal{L}$-module with q and π acting by $[Q]$ and $[\Pi]$, respectively. For a detailed discussion of the following basic facts, we refer the reader to [KL3, §§3.8.1–3.8.2], all of which is easily extended to the case of supermodules.

- Assume the graded superalgebra A is *Laurentian*, i.e. its graded pieces are finite-dimensional and are zero in sufficiently negative degree. Then, the Krull-Schmidt property holds in A-$\mathcal{GSP}roj$. Moreover, $K_0(A)$ is free as an $\mathcal{L}$-module, with basis as a free $\mathbb{Z}$-module given by the isomorphism classes of indecomposable projectives in A-$\mathcal{GSP}roj$.
- If $\alpha : A \to B$ is a homomorphism of graded superalgebras, there is an induced $\mathcal{L}$-module homomorphism $[\alpha] : K_0(A) \to K_0(B)$. If A and B are finite-dimensional and α is surjective, then $[\alpha]$ is surjective.
- Assume A is Laurentian, and let I be a two-sided homogeneous ideal that is non-zero only in strictly positive degree. Then, the canonical quotient map $A \twoheadrightarrow A/I$ induces an isomorphism $K_0(A) \stackrel{\sim}{\to} K_0(A/I)$.
- If A and B are finite-dimensional graded superalgebras all of whose irreducible graded supermodules are absolutely irreducible of type $\mathtt{M}$, then there is an isomorphism $K_0(A) \otimes_{\mathcal{L}} K_0(B) \stackrel{\sim}{\to} K_0(A \otimes B), [P] \otimes [Q] \mapsto [P \otimes Q]$.

For more background about K_0 for supercategories, see [BE, §1.5].

We also need to review some basic facts about quiver Hecke superalgebras established in [KL1, KL2] in the even case, and in [HW] in general. Note in [HW] that the additional assumption (10.2) of bar-consistency is made throughout, but it is not needed for the proofs of the particular results from [HW] cited below.

The *quiver Hecke supercategory* $\mathcal{H}$ is the (strict) monoidal supercategory generated by objects I and morphisms  $: i \to i$ and  $: i \otimes j \to j \otimes i$ of parities $|i|$ and $|i||j|$, respectively, subject to the relations (1.7)–(1.9) (omitting the label λ from these diagrams). For objects $\boldsymbol{i} = i_n \otimes \cdots \otimes i_1 \in I^{\otimes n}$ and $\boldsymbol{j} = j_m \otimes \cdots \otimes j_1 \in I^{\otimes m}$, there are no non-zero morphisms $\boldsymbol{i} \to \boldsymbol{j}$ in $\mathcal{H}$ unless $m = n$. The graded endomorphism superalgebra

$$H_n := \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \mathrm{Hom}_{\mathcal{H}}(\boldsymbol{i}, \boldsymbol{j}) \tag{11.1}$$

is the *quiver Hecke superalgebra* from [KKT]. Let $\mathcal{H}_{q,\pi}$ be the (Q,Π)-envelope of the monoidal supercategory $\mathcal{H}$, which is defined like in Definition 1.6 remembering that monoidal supercategories are 2-supercategories with one object; see also [BE, Definition 1.16]. Let $\underline{\mathcal{H}}_{q,\pi}$ be the underlying monoidal category (same objects, even morphisms of degree zero). The idempotent completion of the additive envelope of $\underline{\mathcal{H}}_{q,\pi}$ is denoted $\dot{\underline{\mathcal{H}}}_{q,\pi}$ as usual. It is equivalent to the category $\bigoplus_{n \geq 0} H_n\text{-}\mathcal{GSP}roj$, hence, we may identify

$$K_0(\dot{\underline{\mathcal{H}}}_{q,\pi}) = \bigoplus_{n \geq 0} K_0(H_n). \tag{11.2}$$

In particular, this means that the $\mathcal{L}$-module on the right hand side of (11.2) is actually an $\mathcal{L}$-algebra; its multiplication comes from the usual induction product $- \circ -$ on graded H_n-supermodules.

Fix $i \in I$ and consider the idempotent $1_{i^n} := 1_{i \otimes i \otimes \cdots \otimes i} \in H_n$. The graded subalgebra $1_{i^n} H_n 1_{i^n}$ is a copy of the *nil-Hecke algebra* in case $|i| = \bar{0}$, or the *odd nil-Hecke algebra* in case $|i| = \bar{1}$. In either case, we write simply X_r for the dot on the rth strand and T_r for the crossing of the rth and $(r+1)$th strands (numbering strands by $1, \dots, n$ from right to left). The elements $D_r := -T_r X_r$ from [HW, (5.20)] are homogeneous idempotents which satisfy the braid relations of the symmetric group S_n. Hence, for each $w \in S_n$ there is an element D_w defined as usual from a reduced expression for w. Letting w_0 be the longest element of S_n, we define

$$1_{i^{(n)}} := D_{w_0} \in 1_{i^n} H_n 1_{i^n}. \tag{11.3}$$

This is known to be a primitive homogeneous idempotent, hence,

$$P(i^{(n)}) := Q^{-d_i n(n-1)/2}\, H_n 1_{i^{(n)}} \tag{11.4}$$

is an indecomposable projective graded H_n-supermodule.

Lemma 11.1. *There is a graded supermodule isomorphism $H_n 1_{i^n} \cong P(i^{(n)})^{\oplus [n]^!_{q_i,\pi_i}}$ (meaning the obvious direct sum of copies of $P(i^{(n)})$ with parity and degree shifts matching the expansion of $[n]^!_{q_i,\pi_i}$).*

Proof. This is well known in the even case, and is noted after [HW, (5.28)] in the odd case. A different convention for (q,π)-integers is adopted in [HW], which we have taken into account by changing the parity shift in (11.4) compared to [HW, (5.28)]. □

Next suppose that we are given two different elements $i, j \in I$. For $r, s \geq 0$, the tensor product in $\mathcal{H}$ gives a superalgebra embedding $H_r \otimes H_1 \otimes H_s \hookrightarrow H_{r+s+1}$. Let $1_{i^{(r)}ji^{(s)}}$ denote the image of $1_{i^{(r)}} \otimes 1_j \otimes 1_{i^{(s)}}$ under this map, then set

$$P(i^{(r)}ji^{(s)}) := Q^{-d_i r(r-1)/2 - d_i s(s-1)/2}\, H_{r+s+1} 1_{i^{(r)}ji^{(s)}}. \tag{11.5}$$

In other words, $P(i^{(r)}ji^{(s)}) = P(i^{(r)}) \circ P(j) \circ P(i^{(s)})$. This is a graded projective H_{r+s+1}-supermodule.

Proposition 11.2 (Khovanov-Lauda, Rouquier, Hill-Wang)**.** *For $i \neq j \in I$, let $n := d_{ij}+1$. Then there exists a split exact sequence of graded H_{r+s+1}-supermodules*

$$0 \longrightarrow P(i^{(n)}j) \longrightarrow \cdots \longrightarrow \Pi^{\frac{r(r-1)}{2}|i|+r|i||j|} P(i^{(n-r)}ji^{(r)}) \longrightarrow \cdots$$
$$\longrightarrow \Pi^{\frac{n(n-1)}{2}|i|+n|i||j|} P(ji^{(n)}) \longrightarrow 0.$$

In particular, there is an isomorphism

$$\bigoplus_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \Pi^{k|i|} P(i^{(n-2k)}ji^{(2k)}) \cong \bigoplus_{k=0}^{\lfloor \frac{n}{2} \rfloor} \Pi^{k|i|+|i||j|} P(i^{(n-2k-1)}ji^{(2k+1)}).$$

Proof. See [HW, Theorem 5.9]. □

Recall the $\mathbb{L}$-algebra $\mathbf{f}$ defined at the beginning of section 10. Let $\mathbf{f}_{\mathcal{L}}$ be the $\mathcal{L}$-subalgebra generated by the divided powers $\theta_i^{(n)} := \theta_i^n / [n]^!_{q_i,\pi_i}$ for all $i \in I$ and $n \geq 1$. Using Lemma 11.1 and Proposition 11.2, it follows that there is a unique $\mathcal{L}$-algebra homomorphism

$$\bar{\gamma} : \mathbf{f}_{\mathcal{L}} \to \bigoplus_{n \geq 0} K_0(H_n), \qquad \theta_i^{(n)} \mapsto [P(i^{(n)})]. \tag{11.6}$$

Theorem 11.3 (Khovanov-Lauda, Hill-Wang)**.** *The homomorphism $\bar{\gamma}$ from (11.6) is an isomorphism.*

Proof. See [HW, Theorem 6.14]. □

Corollary 11.4. *Every irreducible graded H_n-supermodule is absolutely irreducible of type* $\mathtt{M}$.

Proof. The absolute irreducibility follows from Theorem 11.3; see the proof of [KL1, Corollary 3.19]. They are all of type $\mathtt{M}$ by [HW, Proposition 6.15]. □

Now we are going upgrade some of these results to $\mathfrak{U}(\mathfrak{g})$. For each $\lambda \in P$, there is a graded superalgebra homomorphism

$$\alpha_{n,\lambda} : H_n \to \bigoplus_{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n}} \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\boldsymbol{i}} 1_\lambda, E_{\boldsymbol{j}} 1_\lambda), \tag{11.7}$$

where for $\boldsymbol{i} = i_n \otimes \cdots \otimes i_1$ we write $E_{\boldsymbol{i}} 1_\lambda$ for $E_{i_n} \cdots E_{i_1} 1_\lambda$. In diagrammatic terms, $\alpha_{n,\lambda}$ takes the string diagram for an element of H_n to the 2-morphism whose diagram is obtained by adding the label λ on the right hand edge. Applying this to $1_{i^{(n)}}$, we obtain the homogeneous idempotent $\alpha_{n,\lambda}(1_{i^{(n)}}) \in \mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(E_i^n 1_\lambda)$. Then define the *divided power* $E_i^{(n)} 1_\lambda$ to be the 1-morphism in the idempotent completion $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ associated to the idempotent $\left(\alpha_{n,\lambda}(1_{i^{(n)}})\right)^{0,\bar{0}}_{0,\bar{0}}$ in the (Q,Π)-envelope. Composing with the isomorphism ω from Proposition 3.5, we get also a graded superalgebra homomorphism

$$\alpha'_{n,\lambda} := \omega \circ \alpha_{n,\lambda} : H_n^{\mathrm{sop}} \to \bigoplus_{\boldsymbol{i},\boldsymbol{j}\in I^{\otimes n}} \mathrm{Hom}_{\mathcal{U}(\mathfrak{g})}(F_{\boldsymbol{i}}1_\lambda, F_{\boldsymbol{j}}1_\lambda), \tag{11.8}$$

where $F_{\boldsymbol{i}}1_\lambda := F_{i_n}\cdots F_{i_1}1_\lambda$. Let $F_i^{(n)}1_\lambda$ be the 1-morphism in $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ associated to the idempotent $\left(\alpha'_{n,\lambda}\left(1_{i^{(n)}}\right)\right)_{0,\bar{0}}^{0,\bar{0}}$.

Lemma 11.5. *In $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$, we have that $[Q^0\Pi^{\bar{0}}E_i^n1_\lambda] = [n]^!_{q_i,\pi_i}[E_i^{(n)}1_\lambda]$ and $[Q^0\Pi^{\bar{0}}F_i^n1_\lambda] = [n]^!_{q_i,\pi_i}[F_i^{(n)}1_\lambda]$.*

Proof. This follows from the definitions and Lemma 11.1. To give some more detail, Lemma 11.1 means that the idempotent $1_{i^n} \in H_n$ splits as a sum of $n!$ idempotents, each of which is conjugate via some unit in H_n to $1_{(i^n)}$. These units are homogeneous of various degrees and parities encoded in the (q,π)-factorial $[n]^!_{q_i,\pi_i}$. When we apply the homomorphism $\alpha_{n,\lambda}$ to this decomposition, we deduce that the 2-morphism $1_{E_i^n\lambda}$ splits as a sum of $n!$ idempotents, each of which is conjugate by some homogeneous unit in $\mathrm{End}_{\mathcal{U}(\mathfrak{g})}(E_i^n1_\lambda)$ to $\alpha_{n,\lambda}\left(1_{i^{(n)}}\right)$. Passing to $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$, we get from this an isomorphism $Q^0\Pi^{\bar{0}}E_i^n1_\lambda \xrightarrow{\sim} E_i^{(n)}1_\lambda^{\oplus[n]_{q_i,\pi_i}}$ by taking the direct sum of these units appropriately shifted so that they become even of degree zero. □

Lemma 11.6. *In $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$, we have that*

$$[Q^0\Pi^{\bar{0}}E_iF_j1_\lambda] - [Q^0\Pi^{|i||j|}F_jE_i1_\lambda] = \delta_{i,j}[\langle h_i,\lambda\rangle]_{q_i,\pi_i}[1_\lambda],$$

$$\sum_{r=0}^{d_{ij}+1}(-1)^r\pi_i^{r|j|+r(r-1)/2}[E_i^{(d_{ij}+1-r)}E_j^{(1)}E_i^{(r)}1_\lambda] = 0 \qquad (i\neq j),$$

$$\sum_{r=0}^{d_{ij}+1}(-1)^r\pi_i^{r|j|+r(r-1)/2}[F_i^{(d_{ij}+1-r)}F_j^{(1)}F_i^{(r)}1_\lambda] = 0 \qquad (i\neq j).$$

Proof. The first identity follows from the inversion relations (1.12)–(1.14). For example, to prove it in the case $i = j$ and $\langle h_i,\lambda\rangle \leq 0$, we use (1.14) to see that there is an isomorphism in $\dot{\underline{\mathcal{U}}}_{q,\pi}(\mathfrak{g})$

$$Q^0\Pi^{|i|}E_iF_i1_\lambda \oplus \bigoplus_{n=0}^{-\langle h_i,\lambda\rangle-1} Q^{d_i(-\langle h_i,\lambda\rangle-1-2n)}\Pi^{n|i|}1_\lambda \xrightarrow{\sim} Q^0\Pi^{\bar{0}}F_iE_i1_\lambda.$$

Since $[\langle h_i,\lambda\rangle]_{q_i,\pi_i} = -\pi_i\sum_{n=0}^{-\langle h_i,\lambda\rangle-1} q_i^{-\langle h_i,\lambda\rangle-1-2n}\pi_i^n$, this gives what we need on passing to the Grothendieck group.

The second two identities are consequences of Proposition 11.2. One needs to interpret the isomorphism there first in terms of idempotents, then apply the homomorphisms $\alpha_{n+1,\lambda}$ and $\alpha'_{n+1,\lambda}$. □

Theorem 11.7. *There is a unique surjective $\mathcal{L}$-algebra homomorphism*

$$\gamma : \dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}} \twoheadrightarrow K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$$

sending $1_\lambda, e_i^{(n)}1_\lambda$ and $f_i^{(n)}1_\lambda$ to $[1_\lambda]$, $[E_i^{(n)}1_\lambda]$ and $[F_i^{(n)}1_\lambda]$, respectively.

Proof. To establish the existence of the homomorphism γ, note to start with that there is an $\mathbb{L}$-algebra homomorphism $\dot{U}_{q,\pi}(\mathfrak{g}) \to \mathbb{L}\otimes_{\mathcal{L}} K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ sending $1_\lambda, e_i^{(r)}1_\lambda$

and $f_i^{(r)}1_\lambda$ to $[1_\lambda]$, $[E_i^{(r)}1_\lambda]$ and $[F_i^{(r)}1_\lambda]$, respectively. To see this, we just have to check the defining relations of $\dot{U}_{q,\pi}(\mathfrak{g})$ from (9.2)–(9.4), which follow by Lemma 11.6. Then we restrict this homomorphism to $\dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$, observing that the image of the restriction lies in $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ thanks to Lemma 11.5.

It remains to prove that γ is surjective. The proof of this is essentially the same as the proof in the purely even case given in [KL3, §3.8], so we will try to be brief. For $n, n' \geq 0$ and $\lambda \in P$, we let

$$H_{n,n',\lambda} := \bigoplus_{\substack{\boldsymbol{i},\boldsymbol{j} \in I^{\otimes n} \\ \boldsymbol{i}',\boldsymbol{j}' \in I^{\otimes n'}}} \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\boldsymbol{i}}F_{\boldsymbol{i}'}1_\lambda, E_{\boldsymbol{j}}F_{\boldsymbol{j}'}1_\lambda).$$

Idempotents in this algebra are idempotent 2-morphisms in $\mathfrak{U}(\mathfrak{g})$, hence, there is a canonical homomorphism

$$\delta_{n,n',\lambda} : K_0(H_{n,n',\lambda}) \to K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})).$$

Moreover, there is an $\mathcal{L}$-algebra homomorphism

$$\alpha_{n,n',\lambda} : H_n \otimes H_{n'}^{\mathrm{sop}} \otimes \mathrm{SYM} \to H_{n,n',\lambda}$$

sending $a \otimes a' \otimes p$ to $\alpha_{n,\mu}(a)\alpha'_{n',\lambda}(a')\beta_\lambda(p)$, where μ is the weight labeling the left hand edge of the diagram $\alpha'_{n',\lambda}(a')1_\lambda$. Let $I_{n,n',\lambda}$ be the two-sided ideal of $H_{n,n',\lambda}$ spanned by all string diagrams which involve a U-turn, i.e. they involve at least one arc whose endpoints are both on the top edge; cf. [KL3, Proposition 3.17]. Let

$$\beta_{n,n',\lambda} : H_{n,n',\lambda} \twoheadrightarrow H_{n,n',\lambda}/I_{n,n',\lambda}$$

be the canonical quotient map. The composition $\gamma_{n,n',\lambda} := \beta_{n,n',\lambda} \circ \alpha_{n,n',\lambda}$ is surjective. We get induced a commutative diagram at the level of Grothendieck groups:

$$\begin{array}{ccc} K_0(H_n \otimes H_{n'}^{\mathrm{sop}} \otimes \mathrm{SYM}) & \xrightarrow{[\gamma_{n,n',\lambda}]} & K_0\left(H_{n,n',\lambda}/I_{n,n',\lambda}\right) \\ & {\scriptstyle [\alpha_{n,n',\lambda}]} \searrow \quad \nearrow {\scriptstyle [\beta_{n,n',\lambda}]} & \\ & K_0(H_{n,n',\lambda}) & \end{array}.$$

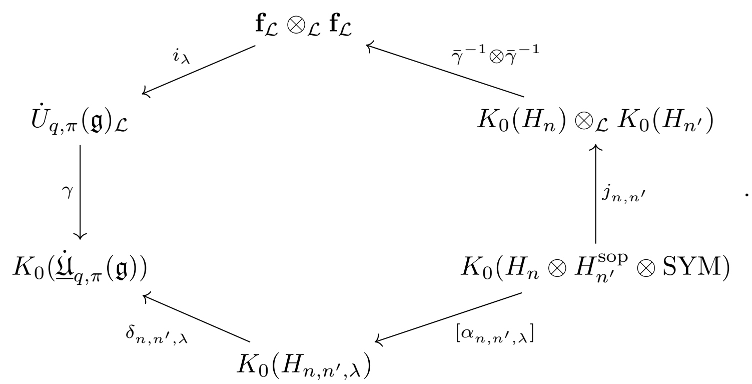
Following the proof of [KL3, Proposition 3.36], using the facts summarized at the start of this section plus the fact that H_n is finite as a module over its center, one shows that $[\gamma_{n,n',\lambda}]$ is onto, hence, so too is $[\beta_{n,n',\lambda}]$.

Now let X be an indecomposable object in $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$. Define its *width* to be the smallest $N \geq 0$ such that X is isomorphic to a summand of $Q^m\Pi^bE_{\mathtt{a}}1_\lambda$ for some $\mathtt{a} \in \mathrm{Seq}$ of length N and some $m \in \mathbb{Z}, b \in \mathbb{Z}/2$ and $\lambda \in P$. We are going to show by induction on width that each $[X]$ is in the image of γ. For the base case, if X is of width zero, we claim that it is isomorphic to some $Q^m\Pi^b1_\lambda$. To see this, recall that $\mathrm{End}_{\mathfrak{U}(\mathfrak{g})}(1_\lambda)$ is a quotient of SYM, which is strictly postively graded with $\Bbbk$ in degree zero. Hence, 1_λ is either indecomposable or zero, which implies our claim. Since $[1_\lambda]$ is in the image of γ, the base of the induction is now established.

For the induction step, take X of width $N > 0$. We can find some $n, n' \geq 0$ with $n + n' = N$ and $\boldsymbol{i} \in I^{\otimes n}, \boldsymbol{i}' \in I^{\otimes n'}$ such that X is isomorphic to a summand of $Q^m\Pi^bE_{\boldsymbol{i}}F_{\boldsymbol{i}'}1_\lambda$. This is a consequence of the relations (1.12)–(1.14); cf. the proof of [KL3, Lemma 3.38]. It follows that $[X]$ is in the image of $\delta_{n,n',\lambda}$, i.e. there is some $Y \in H_{n,n',\lambda}$-$\mathcal{GSP}roj$ such that $\delta_{n,n',\lambda}([Y]) = [X]$. The minimality in the definition of width ensures that $\beta_{n,n',\lambda}([Y]) \neq 0$. Pick $Z \in H_n \otimes H_{n'}^{\mathrm{sop}} \otimes \mathrm{SYM}$-$\mathcal{GSP}roj$ such that $[\gamma_{n,n',\lambda}]([Z]) = [\beta_{n,n',\lambda}]([Y])$. Then one argues explicitly with idempotents as in [KL3, §3.8.4] to see that

$$[\alpha_{n,n',\lambda}]([Z]) = [Y] + [Y']$$

for $Y' \in H_{n,n',\lambda}\text{-}\mathcal{GSP}roj$ with $[\beta_{n,n',\lambda}]([Y']) = 0$. By induction, $\delta_{n,n',\lambda}([Y'])$ is in the image of γ. Hence, to show that $[X] = \delta_{n,n',\lambda}([Y])$ is so, we are reduced to showing that $\delta_{n,n',\lambda}([\alpha_{n,n',\lambda}]([Z]))$ is in the image of γ. This follows using the following commutative diagram:



Here, $\bar{\gamma}$ is the isomorphism from Theorem 11.3. the isomorphism $j_{n,n'}$ exists because of Corollary 11.4, and i_λ sends $\theta_{i_1} \cdots \theta_{i_n} \otimes \theta_{j_1} \cdots \theta_{j_m} \mapsto e_{i_1} \cdots e_{i_n} f_{j_1} \cdots f_{j_m} 1_\lambda$ □

12. The decategorification conjecture

We continue to assume the homogeneity condition (1.31) holds and that $\Bbbk = \Bbbk_{\bar{0}}$ is a field. Let us restate the Decategorification Conjecture from the introduction:

Decategorification Conjecture. *The surjective homomorphism γ from Theorem 11.7 is an isomorphism.*

The proof of the following theorem mimics [KL3, §3.9].

Theorem 12.1. *Assume that the Nondegeneracy Conjecture holds and moreover that the Cartan datum is bar-consistent, i.e. (10.2) holds. Then the Decategorification Conjecture holds as well.*

Proof. For a graded superspace V, we let $\dim_{q,\pi} V := \sum_{n\in\mathbb{Z}} \sum_{a\in\mathbb{Z}/2} (\dim V_{n,a}) q^n \pi^a$. For example, viewing the algebra SYM from (1.21) as a graded superalgebra so that the isomorphism (1.22) preserves degrees and parities, we have that

$$S := \dim_{q,\pi} \mathrm{SYM} = \prod_{i\in I} \prod_{r\geq 1} \frac{1}{1-(\pi_i q_i^2)^r} \in \mathbb{Z}[[q]][\pi]/(\pi^2-1).$$

The Nondegeneracy Conjecture implies (indeed, is equivalent to) the assertion that

$$\langle e_{\mathtt{a}} 1_\lambda, e_{\mathtt{b}} 1_\lambda \rangle = S^{-1} \dim_{q,\pi} \mathrm{Hom}_{\mathfrak{U}(\mathfrak{g})}(E_{\mathtt{a}} 1_\lambda, E_{\mathtt{b}} 1_\lambda) \tag{12.1}$$

for $\mathtt{a}, \mathtt{b} \in \mathrm{Seq}$ with $\mathrm{wt}(\mathtt{a}) = \mathrm{wt}(\mathtt{b})$ and $\lambda \in P$.

Now consider the sesquilinear form on $K_0(\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ defined by letting $\langle [X], [Y] \rangle$ be zero if X, Y are 1-morphisms in $\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ whose domains or codomains are different, and setting

$$\langle [X], [Y] \rangle := S^{-1} \sum_{n\in\mathbb{Z}} \sum_{a\in\mathbb{Z}/2} \dim \mathrm{Hom}_{\dot{\underline{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})}(Q^n \Pi^a X, Y) q^n \pi^a$$

if X and Y have the same domain and codomain. Equivalently, for 1-morphisms $X, Y : \lambda \to \mu$ in $\mathfrak{U}_{q,\pi}(\mathfrak{g})$, we have that

$$\langle [X], [Y] \rangle = S^{-1} \dim_{q,\pi} \mathrm{Hom}_{\mathfrak{U}_{q,\pi}(\mathfrak{g})}(X, Y).$$

Comparing with (12.1), using also Theorem 10.3, we deduce that the forms $\langle -, - \rangle$ on $\dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$ and $K_0(\underline{\dot{\mathfrak{U}}}_{q,\pi}(\mathfrak{g}))$ are intertwined by the homomorphism γ in the sense that $\langle x, y \rangle = \langle \gamma(x), \gamma(y) \rangle$.

Finally, suppose that $x \in \dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$ is in the kernel of γ. By the previous paragraph, we have that $\langle x, y \rangle = 0$ for all $y \in \dot{U}_{q,\pi}(\mathfrak{g})_{\mathcal{L}}$. In view of the non-degeneracy of the form $\langle -, - \rangle$ from Theorem 10.1, this implies that $x = 0$. □

Remark 12.2. The assumption of bar-consistency made in both of Theorems 10.1 and 12.1 is probably unnecessary. We have included it because we have appealed to [C, Theorem 5.12], where it is assumed from the outset. Providing one allows that the canonical basis should be bar-invariant only up to multiplication by π, we expect that the arguments of [C] should still be valid without bar-consistency, but we have not checked this assertion in detail.

Example 12.3. Take $\mathfrak{g}$ to be odd $\mathfrak{b}_1$ and identify P with $\mathbb{Z}$ as in the introduction. Then, [EL, Proposition 8.3] implies that the indecomposable 1-morphisms in $\underline{\dot{\mathfrak{U}}}_{q,\pi}(\mathfrak{g})$ (up to degree and parity shift) are

$$\{E^{(a)} F^{(b)} 1_\lambda \mid a, b \geq 0, \lambda \in \mathbb{Z}, \lambda \leq b - a\} \cup \{F^{(b)} E^{(a)} 1_\lambda \mid a, b \geq 0, \lambda \in \mathbb{Z}, \lambda \geq b - a\}.$$

Also by [EL, Theorem 8.4], the Decategorification Conjecture holds in this case, i.e. γ is an isomorphism. As has already been noted in [C, Example 4.16], γ maps the classes of the indecomposable 1-morphisms listed above to the canonical basis for $\dot{U}_{q,\pi}(\mathfrak{g})$ from [CW, Theorem 6.2] (up to multiplication by π).

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