SYMMETRIC FUNCTIONS, PARABOLIC CATEGORY $\mathcal{O}$ AND THE SPRINGER FIBER

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ABSTRACT. We prove that the center of a regular block of parabolic category $\mathcal{O}$ for the general linear Lie algebra is isomorphic to the cohomology algebra of a corresponding Springer fiber. This was conjectured by Khovanov. We also find presentations for the centers of singular blocks, which are cohomology algebras of Spaltenstein varieties.

1. Introduction

Fix a natural number n. By a *composition*, respectively, a *partition* of n we mean a tuple $\lambda = (\lambda_i)_{i\in\mathbb{Z}}$, respectively, a sequence $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$ of non-negative integers summing to n. For a composition ν of n, we write

$$S_\nu = \cdots \times S_{\nu_1} \times S_{\nu_2} \times \cdots$$

for the usual parabolic subgroup of the symmetric group S_n parametrized by ν, and call ν *regular* if $S_\nu = \{1\}$. Let $P := \mathbb{C}[x_1, \dots, x_n]$, viewed as a graded commutative algebra with each x_i in degree two. The symmetric group S_n acts as usual on P by permuting the generators. Let P_ν be the subalgebra $\mathbb{C}[x_1, \dots, x_n]^{S_\nu}$ of all S_ν-invariants in P. Given distinct integers $i_1, \dots, i_m$, we write $e_r(\nu; i_1, \dots, i_m)$ and $h_r(\nu; i_1, \dots, i_m)$ for the rth elementary and complete symmetric polynomials in variables $X_{i_1} \cup \cdots \cup X_{i_m}$, where

$$X_i := \left\{ x_j \;\middle|\; \sum_{h<i} \nu_h < j \leq \sum_{h\leq i} \nu_h \right\}.$$

Note P_ν is itself a free polynomial algebra of rank n generated by the elements $\{e_r(\nu; i) \mid i \in \mathbb{Z}, 1 \leq r \leq \nu_i\}$ and also by $\{h_r(\nu; i) \mid i \in \mathbb{Z}, 1 \leq r \leq \nu_i\}$. Moreover, we have that $e_r(\nu; i_1, \dots, i_m) = 0$ for $r > \nu_{i_1} + \cdots + \nu_{i_m}$.

Now fix a composition μ of n and let λ denote the *transpose partition*, that is, λ_j counts the number of $i \in \mathbb{Z}$ such that $\mu_i \geq j$. Let I_ν^μ denote the homogeneous ideal of P_ν generated by

$$\left\{ h_r(\nu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\ i_1, \dots, i_m \text{ distinct integers,} \\ r > \lambda_1 + \cdots + \lambda_m - \nu_{i_1} - \cdots - \nu_{i_m} \end{array} \right\}.$$

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Equivalently (see Lemma 2.2), I_ν^μ is the ideal generated by

$$\left\{ e_r(\nu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\;\; i_1, \dots, i_m \text{ distinct integers,} \\ r > \nu_{i_1} + \cdots + \nu_{i_m} - \lambda_{l+1} - \lambda_{l+2} - \cdots \\ \text{where } l := \#\{i \in \mathbb{Z} \mid \nu_i > 0, i \neq i_1, \dots, i_m\} \end{array} \right\}.$$

Let C_ν^μ denote the graded quotient P_ν / I_ν^μ. These algebras have natural geometric realizations, as follows.

- If both μ and ν are regular then I_ν^μ is simply the ideal of P generated by all homogeneous symmetric polynomials of positive degree. In this case, we write simply I and C for I_ν^μ and C_ν^μ. The algebra C is the *coinvariant algebra*, which by a classical theorem of Borel is isomorphic to the cohomology algebra (with complex coefficients) of the *flag manifold* F of complete flags in $\mathbb{C}^n$.
- If just μ is regular, we denote I_ν^μ and C_ν^μ simply by I_ν and C_ν. The algebra C_ν is isomorphic to the subalgebra C^{S_ν} of all S_ν-invariants in C, which is the cohomology algebra of the *partial flag manifold* F_ν of flags $\cdots \subseteq V_1 \subseteq V_2 \subseteq \cdots$ with $\dim V_j = \sum_{i \leq j} \nu_i$ for each $j \in \mathbb{Z}$.
- If just ν is regular, we denote I_ν^μ and C_ν^μ simply by I^μ and C^μ. The ideal I^μ is generated by the elementary symmetric functions $e_r(x_{i_1}, \dots, x_{i_m})$ for every $m \geq 1$, $1 \leq i_1 < \cdots < i_m \leq n$ and $r > m - \lambda_{n-m+1} - \lambda_{n-m+2} - \cdots$. This is Tanisaki's presentation [T] (simplifying De Concini and Procesi's original work [DP]) for the cohomology algebra of the *Springer fiber* F^μ of all flags in F stabilized by the nilpotent matrix x_μ of Jordan type μ.
- Generalizing these special cases, we will prove in a subsequent article [BO] (see also Remark 4.6 and Example 4.7 below) that the algebra C_ν^μ for arbitrary μ and ν is isomorphic to the cohomology algebra of the *Spaltenstein variety* F_ν^μ introduced in [Sp], that is, the subvariety of F_ν consisting of all partial flags $\cdots \subseteq V_1 \subseteq V_2 \subseteq \cdots$ such that $x_\mu V_j \subseteq V_{j-1}$ for each j.

The main goal of this article is to explain how the algebras C_ν^μ arise in representation theory. Consider the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$. Let $\mathfrak{h}$ be the Cartan subalgebra of diagonal matrices and $\mathfrak{b}$ be the Borel subalgebra of upper triangular matrices. Let $\varepsilon_1, \dots, \varepsilon_n$ be the basis for the vector space $\mathfrak{h}^*$ that is dual to the standard basis $e_{1,1}, \dots, e_{n,n}$ of $\mathfrak{h}$ consisting of matrix units. The Bernstein-Gelfand-Gelfand category $\mathcal{O}$ introduced originally in [BGG] is the category of all finitely generated $\mathfrak{g}$-modules that are locally finite over $\mathfrak{b}$ and semisimple over $\mathfrak{h}$. For $\alpha \in \mathfrak{h}^*$, write $L(\alpha)$ for the irreducible highest weight module of highest weight $(\alpha - \rho)$ where $\rho := -\varepsilon_2 - 2\varepsilon_3 - \cdots - (n-1)\varepsilon_n$[1]. These are the irreducible objects in $\mathcal{O}$.

For ν as above, let $\mathcal{O}_\nu$ be the Serre subcategory of $\mathcal{O}$ generated by the irreducible modules $L(\alpha)$ for all $\alpha = \sum_{i=1}^n a_i \varepsilon_i \in \mathfrak{h}^*$ such that exactly ν_i of the coefficients $a_1, \dots, a_n$ are equal to i for each $i \in \mathbb{Z}$. This is an *integral*

[1]This choice of ρ is congruent to the usual choice (half the sum of the positive roots) modulo $\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n$. The choice here is more convenient when working with $\mathfrak{gl}_n(\mathbb{C})$ rather than $\mathfrak{sl}_n(\mathbb{C})$ because it is itself an integral weight.

block of $\mathcal{O}$. For μ as above, let $\mathfrak{p}$ be the standard parabolic subalgebra of block upper triangular matrices with blocks of size $\dots, \mu_1, \mu_2, \dots$ down the diagonal. Let $\mathcal{O}^\mu$ be the full subcategory of $\mathcal{O}$ consisting of all modules that are locally finite over $\mathfrak{p}$. This is *parabolic category* $\mathcal{O}$. Finally we set $\mathcal{O}^\mu_\nu := \mathcal{O}^\mu \cap \mathcal{O}_\nu$, an integral block[2] of parabolic category $\mathcal{O}$. Recall the center of an additive category $\mathcal{C}$ is the ring $Z(\mathcal{C})$ of all natural transformations from the identity functor to itself.

Main Theorem. $Z(\mathcal{O}^\mu_\nu) \cong C^\mu_\nu$.

To make this isomorphism explicit, let $z_1, \dots, z_n$ be the generators for the center $Z(\mathfrak{g})$ of the universal enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ determined by the property that, for $\alpha = \sum_{i=1}^n a_i \varepsilon_i \in \mathfrak{h}^*$, the element z_r acts on the irreducible module $L(\alpha)$ by the scalar $e_r(a_1, \dots, a_n)$, the rth elementary symmetric polynomial evaluated at $a_1, \dots, a_n$. Let $p_\nu : Z(\mathfrak{g}) \to Z(\mathcal{O}_\nu)$ denote the canonical homomorphism sending z to the natural transformation defined by left multiplication by z. There is also a homomorphism $q_\nu : Z(\mathfrak{g}) \to C_\nu$ with $q_\nu(z_r) = e_r(x_1 + a_1, \dots, x_n + a_n)$ for each $r = 1, \dots, n$, where $a_1, \dots, a_n$ here are defined so that $a_1 \leq \dots \leq a_n$ and as before exactly ν_i of the integers $a_1, \dots, a_n$ are equal to i for each $i \in \mathbb{Z}$. By a famous result of Soergel [S] (see §6 below for precise references) both of the maps p_ν and q_ν are surjective, and there is a unique isomorphism c_ν making the following diagram commute:

$$\begin{array}{ccc} & Z(\mathfrak{g}) & \\ {\scriptstyle p_\nu}\swarrow & & \searrow{\scriptstyle q_\nu} \\ Z(\mathcal{O}_\nu) & \xrightarrow[c_\nu]{\sim} & C_\nu. \end{array}$$

We actually prove that there exists a unique isomorphism c^μ_ν making the following diagram commute:

$$\begin{array}{ccc} Z(\mathcal{O}_\nu) & \xrightarrow[c_\nu]{\sim} & C_\nu \\ {\scriptstyle r^\mu_\nu}\downarrow & & \downarrow{\scriptstyle s^\mu_\nu} \\ Z(\mathcal{O}^\mu_\nu) & \xrightarrow[c^\mu_\nu]{\sim} & C^\mu_\nu, \end{array}$$

where $r^\mu_\nu : Z(\mathcal{O}_\nu) \to Z(\mathcal{O}^\mu_\nu)$ is the restriction map arising from the embedding of $\mathcal{O}^\mu_\nu$ into $\mathcal{O}_\nu$ and $s^\mu_\nu : C_\nu \twoheadrightarrow C^\mu_\nu$ is the canonical quotient map, which exists because $I_\nu \subseteq I^\mu_\nu$. Note the surjectivity of the map r^μ_ν, an essential step in the proof, was already established in [B, Theorem 2].

In the special case that ν is regular, this proves that the center of a regular block of $\mathcal{O}^\mu$ is isomorphic to the cohomology algebra of the Springer fiber F^μ, exactly as was conjectured by Khovanov in [Kh, Conjecture 3][3]. Using the surjectivity of the map r^μ_ν established in [B], Stroppel [S2, Theorem 1]

[2]The fact that $\mathcal{O}^\mu_\nu$ really is an indecomposable subcategory of $\mathcal{O}^\mu$, so is a block of $\mathcal{O}^\mu$ in the usual sense, is explained in the discussion immediately following the statement of Theorem 2 in the introduction of [B].

[3]Of the other conjectures from Khovanov's paper, [Kh, Conjecture 4] was already proved by Mazorchuk and Stroppel [MS, Theorem 5.2], and [Kh, Conjecture 2] then

has also independently found a proof of Khovanov's conjecture. Stroppel's approach based on deformation is quite different from the strategy followed here. The approach here has the advantage of yielding at the same time the explicit description of the centers of all singular blocks. By [B, Theorem 5.11], these can also be reinterpreted in terms of the centers of blocks of degenerate cyclotomic Hecke algebras.

The key idea in the proof is the construction of an action of the general linear Lie algebra $\hat{\mathfrak{g}} := \mathfrak{gl}_\infty(\mathbb{C})$ on the direct sum $\bigoplus_\nu Z(\mathcal{O}_\nu) \cong \bigoplus_\nu C_\nu$ of the centers of all integral blocks of $\mathcal{O}$. In this construction, the Chevalley generators of $\hat{\mathfrak{g}}$ act as the trace maps in the sense of [Be] associated to some canonical adjunctions between the special translation functors that arise by tensoring with the natural $\mathfrak{g}$-module and its dual. In a similar way, there is an action of $\hat{\mathfrak{g}}$ on the direct sum $\bigoplus_\nu Z(\mathcal{O}^\mu_\nu) \cong \bigoplus_\nu C^\mu_\nu$ of the centers of all integral blocks of $\mathcal{O}^\mu$ such that the canonical map $\oplus_\nu r^\mu_\nu$ is a $\hat{\mathfrak{g}}$-module homomorphism (see Theorem 4.3). As explained in more detail in [BO], this action is closely related to Ginzburg's geometric construction of representations of the general linear group [G1]; see also [BG] and [G2, §7].

The remainder of the article is organized as follows. In §2, we begin with some preliminaries on symmetric functions, then deduce some elementary properties of the algebras C^μ_ν. In §3, we construct an action of $\hat{\mathfrak{g}}$ on the direct sum $\bigoplus_\nu C_\nu$ of all partial coinvariant algebras by exploiting Schur-Weyl duality, paralleling the idea of Braverman and Gaitsgory [BG] on the geometric side. In §4, we use this action to deduce the dimension of C^μ_ν from known properties of C^μ. Then in §5 we reinterpret the actions of the Chevalley generators of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu C_\nu$ as certain trace maps. Only in §6 do we finally relate things back to category $\mathcal{O}$, using the full strength of Soergel's theory from [S] to complete the proof of the Main Theorem.

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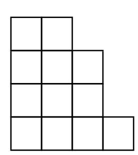
2. Preliminaries

Let λ be a partition of n. Recall that the *Young diagram* of λ consists of λ_i boxes in its ith row; for example, the Young diagram of $\lambda = (4\,3^2\,2)$ is



Given in addition a composition ν of n, a λ-*tableau* of *type* ν means some filling of the boxes of this diagram by integers so that there are exactly ν_i entries equal to i for each $i \in \mathbb{Z}$. A λ-tableau is *column strict* if its entries are strictly increasing in each column from bottom to top, and it is *standard* if it is column strict and in addition its entries are weakly increasing in each

follows by [S2, Theorem 2]. This just leaves the geometric [Kh, Conjecture 1], which appears still to be unresolved.

row from left to right. The *Kostka number* $K_{\lambda,\nu}$ is the number of standard λ-tableaux of type ν. It is well known that $K_{\lambda,\nu}$ is non-zero if and only if $\lambda \geq \nu^+$, where $\geq$ denotes the usual dominance ordering on partitions and ν^+ is the unique partition of n whose non-zero parts have the same multiplicities as in the composition ν; see for example [Mac, (I.6.5)].

Let $e_r(x_1, \dots, x_n)$ and $h_r(x_1, \dots, x_n)$ denote the rth elementary and complete symmetric polynomials in commuting variables $x_1, \dots, x_n$, adopting the convention that $e_r(x_1, \dots, x_n) = h_r(x_1, \dots, x_n) = 0$ for $r < 0$ and $e_0(x_1, \dots, x_n) = h_0(x_1, \dots, x_n) = 1$. The following basic identity [Mac, (I.2.6′)] will be used repeatedly:

$$\sum_{s=0}^{r}(-1)^s e_s(x_1, \dots, x_n)h_{r-s}(x_1, \dots, x_n) = 0 \tag{2.1}$$

for all $r \geq 1$. For $m, n \geq 0$, we obviously have that

$$h_r(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{s=0}^{r} h_s(x_1, \dots, x_m)h_{r-s}(y_1, \dots, y_n), \tag{2.2}$$

$$e_r(x_1, \dots, x_m, y_1, \dots, y_n) = \sum_{s=0}^{r} e_s(x_1, \dots, x_m)e_{r-s}(y_1, \dots, y_n). \tag{2.3}$$

Moreover, for all $r \geq 0$, we have that

$$h_r(y_1, \dots, y_n) = \sum_{s=0}^{r}(-1)^s e_s(x_1, \dots, x_m)h_{r-s}(x_1, \dots, x_m, y_1, \dots, y_n), \tag{2.4}$$

$$e_r(y_1, \dots, y_n) = \sum_{s=0}^{r}(-1)^s h_s(x_1, \dots, x_m)e_{r-s}(x_1, \dots, x_m, y_1, \dots, y_n). \tag{2.5}$$

These identities follow by expanding the right hand sides using the identities (2.2)–(2.3) and then simplifying using (2.1).

Let P denote the polynomial algebra $\mathbb{C}[x_1, \dots, x_n]$, graded so that each x_i is in degree 2. The algebra of *symmetric polynomials* is the subalgebra P^{S_n} of all S_n-invariants in P. It is classical that P is a free P^{S_n}-module of rank $n!$ with basis

$$\{x_1^{r_1}x_2^{r_2}\cdots x_n^{r_n} \mid 0 \leq r_i < i\}. \tag{2.6}$$

An equivalent statement is that $P^{S_{n-1}}$ is a free P^{S_n}-module with basis $1, x_n, x_n^2, \dots, x_n^{n-1}$. The expansion of higher powers of x_n in terms of this basis can be obtained by setting $u = x_n$ in the following identity: for any $r \geq 0$ and any u such that $(u - x_1)\cdots(u - x_n) = 0$ we have that

$$u^{n+r} = \sum_{s=1}^{n}\sum_{t=0}^{r}(-1)^{s+t-1}e_{s+t}(x_1, \dots, x_n)h_{r-t}(x_1, \dots, x_n)u^{n-s}. \tag{2.7}$$

This follows by induction on r; the base case $r = 0$ is exactly the assumption $(u - x_1)\cdots(u - x_n) = 0$, and then the induction step is obtained by multiplying both sides by u then making some easy manipulations using (2.1).

Applying (2.1) once more, an equivalent way of writing the same identity is as

$$u^{n+r} = \sum_{s=1}^{n} \sum_{t=1}^{s} (-1)^{s-t} e_{s-t}(x_1, \dots, x_n) h_{r+t}(x_1, \dots, x_n) u^{n-s}, \tag{2.8}$$

again valid for all $r \geq 0$ and u satisfying $(u - x_1) \cdots (u - x_n) = 0$

Next fix a composition ν of n and let P_ν denote the subalgebra P^{S_ν} of S_ν-invariants in P. For $i \in \mathbb{Z}$ we have the elements

$$h_r(\nu; i) := h_r(x_{j+1}, x_{j+2}, \dots, x_{j+\nu_i}), \tag{2.9}$$

$$e_r(\nu; i) := e_r(x_{j+1}, x_{j+2}, \dots, x_{j+\nu_i}), \tag{2.10}$$

where $j := \sum_{h<i} \nu_h$. The algebra P_ν is freely generated by either the $h_r(\nu; i)$'s or the $e_r(\nu; i)$'s for all $i \in \mathbb{Z}$ and $1 \leq r \leq \nu_i$. Also, $e_r(\nu; i) = 0$ for $r > \nu_i$. More generally, given distinct integers $i_1, \dots, i_m$, let

$$h_r(\nu; i_1, \dots, i_m) := \sum_{r_1+\cdots+r_m=r} h_{r_1}(\nu; i_1) h_{r_2}(\nu; i_2) \cdots h_{r_m}(\nu; i_m), \tag{2.11}$$

$$e_r(\nu; i_1, \dots, i_m) := \sum_{r_1+\cdots+r_m=r} e_{r_1}(\nu; i_1) e_{r_2}(\nu; i_2) \cdots e_{r_m}(\nu; i_m). \tag{2.12}$$

Because of (2.2)–(2.3), these are the same elements as defined in the introduction. If $i_1, \dots, i_m, j_1, \dots, j_l$ are distinct integers such that $\nu_{i_1} + \cdots + \nu_{i_m} + \nu_{j_1} + \cdots + \nu_{j_l} = n$, then we have by (2.4) that

$$h_r(\nu; i_1, \dots, i_m) = \sum_{s=0}^{r} (-1)^s e_s(\nu; j_1, \dots, j_l) h_{r-s}(x_1, \dots, x_n). \tag{2.13}$$

The *coinvariant algebra* C is the quotient P/I, where I denotes the ideal of P generated by all homogeneous symmetric functions of positive degree. The images of the monomials (2.6) give a basis for C as a vector space, so $\dim C = n!$. In fact, by a theorem of Chevalley [C], the algebra C viewed as a module over the symmetric group is isomorphic to the left regular module $\mathbb{C}S_n$, where the action of S_n on C is the action induced by the natural permutation action on P. More generally, let I_ν be the ideal of P_ν generated by all homogeneous symmetric polynomials of positive degree and define the *partial coinvariant algebra* C_ν to be the quotient P_ν/I_ν. The first lemma is well known.

Lemma 2.1. *The map $P_\nu \to C$ obtained by restricting the canonical quotient map $P \twoheadrightarrow C$ to the subalgebra P_ν has kernel I_ν and image C^{S_ν}. Hence it induces a canonical isomorphism between $C_\nu = P_\nu/I_\nu$ and C^{S_ν}. In particular,* $\dim C_\nu = |S_n/S_\nu|$.

Proof. Since we are over a field of characteristic zero, taking S_ν-fixed points is an exact functor. Applying it to $0 \to I \to P \to C \to 0$ gives a short exact sequence $0 \to I^{S_\nu} \to P_\nu \to C^{S_\nu} \to 0$. So to prove the first statement of the lemma, we just need to show that $I^{S_\nu} = I_\nu$. Let $\gamma : P \twoheadrightarrow P_\nu$ denote the projection defined by $\gamma(f) := \frac{1}{|S_\nu|} \sum_{w \in S_\nu} wf$. Any element of I is a linear combination of terms of the form fg for $f \in P$ and $g \in P^{S_n}$

homogeneous of positive degree. Applying γ, we deduce that any element of $\gamma(I)$ is a linear combination of terms of the form $\gamma(fg) = \gamma(f)g$ for $f \in P$ and $g \in P^{S_n}$ homogeneous of positive degree. Since $\gamma(f) \in P_\nu$, all such terms belong to I_ν. So we get that $I^{S_\nu} = \gamma(I^{S_\nu}) \subseteq \gamma(I) \subseteq I_\nu$. On the other hand, we obviously have that $I_\nu \subseteq I^{S_\nu}$. Hence, $I_\nu = I^{S_\nu}$ and $C_\nu \cong C^{S_\nu}$. The final statement about dimension now follows easily using Chevalley's theorem, since $\dim C^{S_\nu} = \dim(\mathbb{C}S_n)^{S_\nu} = |S_n/S_\nu|$. $\square$

In view of the above lemma, we will always identify C_ν with the subalgebra C^{S_ν} of C. Moreover, we will use the same notation for elements of P_ν and for their canonical images in C_ν; since we will usually be working in C_ν from now on this should not cause any confusion. By (2.13), we have in C_ν that

$$h_r(\nu; i_1, \dots, i_m) = (-1)^r e_r(\nu; j_1, \dots, j_l) \tag{2.14}$$

for $m, l \geq 0$ and distinct integers $i_1, \dots, i_m, j_1, \dots, j_l$ with the property that $\nu_{i_1} + \dots + \nu_{i_m} + \nu_{j_1} + \dots + \nu_{j_l} = n$. Hence by (2.2) we get that

$$\sum_{s=0}^{r} (-1)^s e_s(\nu; i_1, \dots, i_m) h_{r-s}(\nu; i_1, \dots, i_m) = 0 \tag{2.15}$$

for all $r \geq 1$, equality again written in C_ν.

Now we come to the crucial definition. Fix another composition μ of n and let λ denote the transpose partition. Let I_ν^μ be the ideal of P_ν generated by the elements

$$\left\{ h_r(\nu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\ \ i_1, \dots, i_m \text{ distinct integers,} \\ r > \lambda_1 + \dots + \lambda_m - \nu_{i_1} - \dots - \nu_{i_m} \end{array} \right\} \tag{2.16}$$

and set

$$C_\nu^\mu := P_\nu / I_\nu^\mu \tag{2.17}$$

exactly as in the introduction. If we choose m and $i_1, \dots, i_m$ so that $\lambda_1 + \dots + \lambda_m = \nu_{i_1} + \dots + \nu_{i_m} = n$, then $h_r(\nu; i_1, \dots, i_m)$ belongs to I_ν^μ for all $r > 0$, and it equals $h_r(x_1, \dots, x_n)$. Since I_ν is generated by the elements $h_r(x_1, \dots, x_n)$ for all $r > 0$, this shows that $I_\nu \subseteq I_\nu^\mu$. So it is natural to regard C_ν^μ also as a quotient of C_ν. Using Lemma 2.2 below, it is easy to see that if μ is regular, i.e. $\lambda_1 = n$, then $I_\nu = I_\nu^\mu$. So in this special case we have simply that $C_\nu^\mu = C_\nu$.

Lemma 2.2. *The ideal I_ν^μ is also generated by*

$$\left\{ e_r(\nu; i_1, \dots, i_m) \;\middle|\; \begin{array}{l} m \geq 1,\ \ i_1, \dots, i_m \textit{ distinct integers,} \\ r > \nu_{i_1} + \dots + \nu_{i_m} - \lambda_{l+1} - \lambda_{l+2} - \cdots \\ \textit{where } l := \#\{i \in \mathbb{Z} \mid \nu_i > 0, i \neq i_1, \dots, i_m\} \end{array} \right\}.$$

Proof. Let J_ν^μ be the ideal generated by the given set of elementary symmetric functions. We just explain how to prove that each generator of J_ν^μ belongs to I_ν^μ, hence $J_\nu^\mu \subseteq I_\nu^\mu$. Then a similar argument in the other direction shows that each generator of I_ν^μ belongs to J_ν^μ, hence $I_\nu^\mu \subseteq J_\nu^\mu$, to complete the proof.

So take some element $e_r(\nu; i_1, \dots, i_m)$ for $m \geq 1$, distinct integers $i_1, \dots, i_m$ and $r > \nu_{i_1} + \cdots + \nu_{i_m} - \lambda_{l+1} - \lambda_{l+2} - \cdots$ where

$$l := \#\{i \in \mathbb{Z} \mid \nu_i > 0, i \neq i_1, \dots, i_m\}.$$

The definition of l means we can find distinct integers $j_1, \dots, j_l \notin \{i_1, \dots, i_m\}$ such that $\nu_{i_1} + \cdots + \nu_{i_m} + \nu_{j_1} + \cdots + \nu_{j_l} = n$. It is then the case that

$$\nu_{i_1} + \cdots + \nu_{i_m} - \lambda_{l+1} - \lambda_{l+2} - \cdots = \lambda_1 + \cdots + \lambda_l - \nu_{j_1} - \cdots - \nu_{j_l},$$

hence we have that $r > \lambda_1 + \cdots + \lambda_l - \nu_{j_1} - \cdots - \nu_{j_l}$. If $l > 0$ this shows that $h_r(\nu; j_1, \dots, j_l)$ is an element of the set (2.16), while if $l = 0$ then $r > 0$ so $h_r(\nu; j_1, \dots, j_l) = 0$. Either way, this means that $h_r(\nu; j_1, \dots, j_l) \in I_\nu^\mu$. We know by (2.14) that $e_r(\nu; i_1, \dots, i_m)$ is equal to $(-1)^r h_r(\nu; j_1, \dots, j_l)$ plus some element of I_ν. We observed already above that $I_\nu \subseteq I_\nu^\mu$, so we have now proved that $e_r(\nu; i_1, \dots, i_m) \in I_\nu^\mu$. □

Lemma 2.3. *$C_\nu^\mu \neq 0$ if and only if $\lambda \geq \nu^+$.*

Proof. Since everything is graded, we have that $C_\nu^\mu \neq 0$ if and only if all the generators of I_ν^μ are of positive degree, i.e. $\lambda_1 + \cdots + \lambda_m \geq \nu_{i_1} + \cdots + \nu_{i_m}$ for all $m \geq 1$ and distinct integers $i_1, \dots, i_m$. By the definition of the dominance ordering on partitions, this is the statement that $\lambda \geq \nu^+$. □

When ν is regular, we write I^μ for I_ν^μ and C^μ for C_ν^μ. As we said already in the introduction, Lemma 2.2 is all that is needed to see that our definition of C^μ is equivalent to Tanisaki's presentation [T] for the cohomology algebra $H^*(F^\mu, \mathbb{C})$ of the Springer fiber F^μ. In the following theorem, we record some known facts about this algebra.

Theorem 2.4. *Let μ be a composition of n with transpose partition λ.*

(i) *As a $\mathbb{C}S_n$-module, C^μ is isomorphic to the permutation representation M^μ of S_n on the cosets of the parabolic subgroup S_μ.*

(ii) *The top graded component of C^μ is in degree*

$$d^\mu := \sum_{i \geq 1} \lambda_i(\lambda_i - 1) \tag{2.18}$$

(which is twice the dimension of the Springer fiber F^μ).

(iii) *As a $\mathbb{C}S_n$-module, the top graded component $C^\mu(d^\mu)$ is isomorphic to the irreducible Specht module parametrized by the partition μ^+.*

(iv) *For any non-zero vector $z \in C^\mu$, there exists $y \in C^\mu$ such that yz is a non-zero vector in the top graded component $C^\mu(d^\mu)$.*

Proof. Parts (i)–(iii) are proved in [T]. Part (iv) is noted in the proof of [Go, Theorem 6.6(vi)], where it is proved using the monomial basis for C^μ constructed in [GP]. □

Remark 2.5. Recall from [Mac, (I.6.6(vi))] that the composition multiplicity $[M^\mu : S^\tau]$ of the irreducible Specht module S^τ parametrized by a partition τ in the permutation module M^μ is equal to the Kostka number $K_{\tau,\mu}$. In view of Theorem 2.4(i), the polynomial

$$K_{\tau,\mu}(t) := \sum_{r\geq 0}[C^\mu(d^\mu - 2r) : S^\tau]t^r \qquad (2.19)$$

arising from graded composition multiplicities in C^μ satisfies $K_{\tau,\mu}(1) = K_{\tau,\mu}$. In fact, it is known by [GP] that $K_{\tau,\mu}(t)$ is equal to the *Kostka-Foulkes polynomial* as defined in [Mac, section III.6].

3. Partial coinvariant algebras

In the remainder of the article, the notation $\bigoplus_\nu$ always denotes the direct sum over the set of all compositions ν of n. Let $\hat{\mathfrak{g}} := \mathfrak{gl}_\infty(\mathbb{C})$ be the Lie algebra of matrices over $\mathbb{C}$ with rows and columns parametrized by index set $\mathbb{Z}$, all but finitely many of whose entries are zero. For each $i \in \mathbb{Z}$, let D_i, E_i and F_i denote the (i,i)-matrix unit, the $(i,i{+}1)$-matrix unit and the $(i{+}1,i)$-matrix unit in $\hat{\mathfrak{g}}$, respectively. These elements generate $\hat{\mathfrak{g}}$.

We are going to exploit some basic facts about *polynomial representations* of $\hat{\mathfrak{g}}$. All of these facts are standard results about polynomial representations finite dimensional general linear Lie algebras (see e.g. [Gr]) extended to $\hat{\mathfrak{g}}$ by taking direct limits. To start with, given a $\hat{\mathfrak{g}}$-module M and a composition ν of n, the *ν-weight space* of M is

$$M_\nu := \{v \in M \mid D_i v = \nu_i v \text{ for all } i \in \mathbb{Z}\}. \qquad (3.1)$$

We call M a *polynomial representation* of $\hat{\mathfrak{g}}$ of *degree* n if $M = \bigoplus_\nu M_\nu$ and all M_ν are finite dimensional. For example, the nth tensor power $\widehat{V}^{\otimes n}$ of the natural $\hat{\mathfrak{g}}$-module $\widehat{V}$ is a polynomial representation of degree n, as is the module

$$\textstyle\bigwedge^\mu(\widehat{V}) := \cdots \otimes \bigwedge^{\mu_1}(\widehat{V}) \otimes \bigwedge^{\mu_2}(\widehat{V}) \otimes \cdots \qquad (3.2)$$

for a composition μ of n. For each partition λ of n, there is an irreducible polynomial representation $P^\lambda(\widehat{V})$ characterized uniquely up to isomorphism by the property that the dimension of the ν-weight space of $P^\lambda(\widehat{V})$ is equal to the Kostka number $K_{\lambda,\nu}$ for every ν. These modules give all of the irreducible polynomial representations of degree n.

The symmetric group S_n acts on the right on $\widehat{V}^{\otimes n}$ by permuting tensors. Consider the functor $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} ?$ from the category of finite dimensional left $\mathbb{C}S_n$-modules to the category of polynomial representations of $\widehat{\mathfrak{g}}$ of degree n. By Schur's classical theory, this functor is known to be an equivalence of categories. Recall that M^μ is the permutation module parametrized by a composition μ of n and S^λ is the irreducible Specht module parametrized by a partition λ of n. It is well known that

$$\textstyle\bigwedge^\mu(\widehat{V}) \cong \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{M}^\mu, \qquad (3.3)$$

$$P^\lambda(\widehat{V}) \cong \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} S^\lambda, \qquad (3.4)$$

where $\widetilde{M}^\mu$ denotes the $\mathbb{C}S_n$-module obtained from M^μ by twisting the action by sign.

Lemma 3.1. *For any left $\mathbb{C}S_n$-module M, there is a natural vector space isomorphism $\kappa_M : \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} M \xrightarrow{\sim} \bigoplus_\nu M^{S_\nu}$ defined as follows. Suppose ν is a composition of n and $a_1 \leq \cdots \leq a_n$ are integers exactly ν_i of which are equal to i for each $i \in \mathbb{Z}$. Then*

$$\kappa_M(v_{a_1} \otimes \cdots \otimes v_{a_n} \otimes m) = \frac{1}{|S_\nu|} \sum_{w \in S_\nu} wm \in M^{S_\nu}$$

for any $m \in M$. The inverse map κ_M^{-1} satisfies

$$\kappa_M^{-1}(m) = v_{a_1} \otimes \cdots \otimes v_{a_n} \otimes m$$

for any $m \in M^{S_\nu}$.

Proof. Fix a composition ν of n and let $a_1 \leq \cdots \leq a_n$ be the unique integers such that exactly ν_i of them are equal to i for each $i \in \mathbb{Z}$. Let $\widehat{V}_\nu^{\otimes n}$ denote the ν-weight space of $\widehat{V}^{\otimes n}$. Writing $\mathbb{C}$ for the trivial module, it is well known that the right $\mathbb{C}S_\nu$-module homomorphism $\mathbb{C} \to \widehat{V}_\nu^{\otimes n}$ under which $1 \mapsto v_{a_1} \otimes \cdots \otimes v_{a_n}$ extends by Frobenius reciprocity to a right $\mathbb{C}S_n$-module isomorphism $\mathbb{C} \otimes_{\mathbb{C}S_\nu} \mathbb{C}S_n \xrightarrow{\sim} \widehat{V}_\nu^{\otimes n}$. Also there is a familiar vector space isomorphism $\mathbb{C} \otimes_{\mathbb{C}S_\nu} M \xrightarrow{\sim} M^{S_\nu}, 1 \otimes m \mapsto \frac{1}{|S_\nu|} \sum_{w \in S_\nu} wm$. Composing these maps, we obtain an isomorphism

$$\widehat{V}_\nu^{\otimes n} \otimes_{\mathbb{C}S_n} M \xrightarrow{\sim} \mathbb{C} \otimes_{\mathbb{C}S_\nu} \mathbb{C}S_n \otimes_{\mathbb{C}S_n} M = \mathbb{C} \otimes_{\mathbb{C}S_\nu} M \xrightarrow{\sim} M^{S_\nu}$$

such that $v_{a_1} \otimes \cdots \otimes v_{a_n} \otimes m \mapsto \frac{1}{|S_\nu|} \sum_{w \in S_\nu} wm$. The isomorphism κ_M is the direct sum of these maps over all compositions ν of n. □

As in the introduction, let F be the flag manifold of (complex) dimension $\frac{1}{2}n(n-1)$, and let F_ν denote the partial flag manifold associated to a composition ν of n. So elements of F_ν are nested chains $(V_j)_{j \in \mathbb{Z}}$ of subspaces of $\mathbb{C}^n$ such that $\dim V_j = \sum_{i \leq j} \nu_i$ for each j. The (complex) dimension of F_ν is equal to $\frac{1}{2}d_\nu$ where

$$d_\nu := n(n-1) - \sum_{i \in \mathbb{Z}} \nu_i(\nu_i - 1). \tag{3.5}$$

Let $\pi : F \twoheadrightarrow F_\nu$ denote the natural projection; informally, π is the map forgetting all subspaces of a full flag of the wrong dimension. Identifying the cohomology algebra $H^*(F, \mathbb{C})$ with the coinvariant algebra C as in [F, 10.2(3)], the fundamental class of a point of F (regarded as an element of $H^*(F, \mathbb{C})$ via Poincaré duality) is the canonical image of the polynomial

$$\varepsilon := \frac{1}{n!} \prod_{1 \leq i < j \leq n} (x_i - x_j) \tag{3.6}$$

in C. More generally, the cohomology algebra $H^*(F_\nu, \mathbb{C})$ can be identified with the partial coinvariant algebra $C_\nu = C^{S_\nu}$ so that the pull-back homomorphism $\pi^* : H^*(F_\nu, \mathbb{C}) \to H^*(F, \mathbb{C})$ coincides with the natural inclusion

$C_\nu \hookrightarrow C$. Let

$$\varepsilon_\nu := \frac{1}{|S_\nu|} \prod_{\substack{1 \le i < j \le n \\ i \overset{\nu}{\sim} j}} (x_i - x_j), \tag{3.7}$$

where we write $i \overset{\nu}{\sim} j$ if i and j lie in the same S_ν-orbit. Note ε is divisible by ε_ν and the quotient $\varepsilon/\varepsilon_\nu$ belongs to P_ν. The fundamental class of a point of F_ν is the canonical image of $\varepsilon/\varepsilon_\nu$ in C_ν.

Let $\widetilde{C}$ denote the $\mathbb{C}S_n$-module obtained from C by twisting the natural permutation action by sign, i.e. $w \in S_n$ acts on $\widetilde{C}$ as the map $f \mapsto \operatorname{sgn}(w) wf$. Since the regular $\mathbb{C}S_n$-module twisted by sign is isomorphic to the regular module, it follows from Chevalley's theorem that $\widetilde{C}$ is isomorphic to C as (ungraded) $\mathbb{C}S_n$-modules. Let $\widetilde{C}_\nu$ denote the space $\widetilde{C}^{S_\nu}$ of all S_ν-invariants in $\widetilde{C}$. In other words, $\widetilde{C}_\nu$ is the space of all S_ν-*anti-invariants* in C. Note ε_ν belongs to $\widetilde{C}_\nu$.

In the next lemma, we use Poincaré duality to regard the usual push-forward π_* in homology as a homogeneous map $\pi_* : H^*(F, \mathbb{C}) \to H^*(F_\nu, \mathbb{C})$ of degree $-\sum_{i \in \mathbb{Z}} \nu_i(\nu_i - 1)$.

Lemma 3.2. *The restriction of the push-forward* $\pi_* : H^*(F, \mathbb{C}) \to H^*(F_\nu, \mathbb{C})$ *to the subspace* $\widetilde{C}_\nu$ *defines a* C_ν*-module isomorphism* $\pi_* : \widetilde{C}_\nu \xrightarrow{\sim} C_\nu$ *with* $\pi_*(\varepsilon_\nu) = 1$. *In particular,* $\widetilde{C}_\nu = \varepsilon_\nu C_\nu$ *and* $\dim \widetilde{C}_\nu = |S_n/S_\nu|$.

Proof. By the projection formula, π_* is a C_ν-module homomorphism, hence so is its restriction to the C_ν-submodule $\widetilde{C}_\nu$ of C. By degree considerations, we have that $\pi_*(\varepsilon_\nu) = c \cdot 1$ for some scalar c. To compute the scalar, π_* sends the fundamental class of a point to the fundamental class of a point, so $\pi_*(\varepsilon) = \varepsilon/\varepsilon_\nu$. Hence in C_ν we have that

$$\varepsilon/\varepsilon_\nu = \pi_*(\varepsilon) = \pi_*(\varepsilon_\nu \cdot \varepsilon/\varepsilon_\nu) = \pi_*(\varepsilon_\nu)\varepsilon/\varepsilon_\nu = c\varepsilon/\varepsilon_\nu.$$

So $c = 1$ and $\pi_*(\varepsilon_\nu) = 1$. It follows at once from this that $\pi_* : \widetilde{C}_\nu \to C_\nu$ is surjective. It is an isomorphism because $\dim \widetilde{C}_\nu = \dim C_\nu = |S_n/S_\nu|$. □

Now let us consider the $\hat{\mathfrak{g}}$-module $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}$. Since $\widetilde{C} \cong \mathbb{C}S_n$ as a $\mathbb{C}S_n$-module, this is isomorphic simply to tensor space $\widehat{V}^{\otimes n}$, but it carries an interesting grading induced from the natural grading on C. Composing the direct sum over all ν of the isomorphisms from Lemma 3.2 with the isomorphism $\kappa_{\widetilde{C}}$ from Lemma 3.1, we obtain an isomorphism

$$\varphi : \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C} \xrightarrow{\sim} \textstyle\bigoplus_\nu C_\nu. \tag{3.8}$$

Using this, we transport the natural $\hat{\mathfrak{g}}$-action on $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}$ to the vector space $\bigoplus_\nu C_\nu$, to make the latter into a $\hat{\mathfrak{g}}$-module whose ν-weight space is equal to the partial coinvariant algebra C_ν. Because this definition involves push-forward, the action of $\hat{\mathfrak{g}}$ does not preserve the grading on $\bigoplus_\nu C_\nu$ in the usual sense. Instead, $\hat{\mathfrak{g}}$ leaves the subspaces $\bigoplus_\nu C_\nu(d_\nu - 2r)$ invariant for each $r \ge 0$.

Consider the following *key situation*. Fix $i \in \mathbb{Z}$, let ν be a composition of n with $\nu_i \neq 0$, and define ν' to be the composition of n obtained from ν

by replacing ν_i by $\nu_i - 1$ and ν_{i+1} by $\nu_{i+1}+1$. The Chevalley generators E_i and F_i of $\hat{\mathfrak{g}}$ define linear maps

$$C_\nu \underset{E_i}{\overset{F_i}{\rightleftarrows}} C_{\nu'}. \tag{3.9}$$

We are going to calculate these maps explicitly. It will be convenient to let $a := \nu'_i$, $b := \nu_{i+1}$ and $k := \sum_{j\leq i} \nu_j$, so that

$$x_{k-a} \overset{\nu}{\sim} \cdots \overset{\nu}{\sim} x_{k-1} \overset{\nu}{\sim} x_k, \quad x_{k+1} \overset{\nu}{\sim} \cdots \overset{\nu}{\sim} x_{k+b}, \tag{3.10}$$

$$x_{k-a} \overset{\nu'}{\sim} \cdots \overset{\nu'}{\sim} x_{k-1}, \quad x_k \overset{\nu'}{\sim} x_{k+1} \overset{\nu'}{\sim} \cdots \overset{\nu'}{\sim} x_{k+b}. \tag{3.11}$$

The notation $\overset{\nu}{\sim}$ and $\overset{\nu'}{\sim}$ being used here was introduced after (3.7).

Let

$$F_{\nu,\nu'} := \left\{((V_j)_{j\in\mathbb{Z}}, (V'_j)_{j\in\mathbb{Z}}) \in F_\nu \times F_{\nu'} \,\middle|\, V_j = V'_j \text{ for } j \neq i,\, V'_i \subset V_i\right\}. \tag{3.12}$$

This is obviously isomorphic to the partial flag manifold whose cohomology algebra has been identified with $C_{\nu,\nu'} := C^{S_\nu \cap S_{\nu'}}$. Both $C_\nu = C^{S_\nu}$ and $C_{\nu'} = C^{S_{\nu'}}$ are subalgebras of $C_{\nu,\nu'}$. By the sentence after (2.6), $C_{\nu,\nu'}$ is a free C_ν-module with basis $1, x_k, \dots, x_k^a$ and a free $C_{\nu'}$-module with basis $1, x_k, \dots, x_k^b$. Moreover, by (2.8), we have that for $r \geq 1$ that

$$x_k^{a+r} = \sum_{s=0}^{a}\sum_{t=0}^{s} (-1)^{s-t} e_{s-t}(\nu;i) h_{r+t}(\nu;i) x_k^{a-s}, \tag{3.13}$$

$$x_k^{b+r} = \sum_{s=0}^{b}\sum_{t=0}^{s} (-1)^{s-t} e_{s-t}(\nu';i+1) h_{r+t}(\nu';i+1) x_k^{b-s}. \tag{3.14}$$

Let $p : F_{\nu,\nu'} \to F_\nu$ and $p' : F_{\nu,\nu'} \to F_{\nu'}$ be the first and second projections. The pull-backs $p^* : C_\nu \to C_{\nu,\nu'}$ and $(p')^* : C_{\nu'} \to C_{\nu,\nu'}$ are simply the natural inclusions. To describe the push-forwards (again regarded as maps in cohomology via Poincaré duality), we define

$$\varepsilon_{\nu,\nu'} := \frac{1}{|S_\nu \cap S_{\nu'}|} \prod_{\substack{1\leq i<j\leq n \\ i \overset{\nu}{\sim} j \overset{\nu'}{\sim} i}} (x_i - x_j) \tag{3.15}$$

like in (3.7), so the fundamental class of a point of $F_{\nu,\nu'}$ is the canonical image of $\varepsilon/\varepsilon_{\nu,\nu'}$ in $C_{\nu,\nu'}$. Both ε_ν and $\varepsilon_{\nu'}$ are divisible by $\varepsilon_{\nu,\nu'}$, and we have that

$$\varepsilon_\nu/\varepsilon_{\nu,\nu'} = \frac{1}{a+1}\prod_{j=1}^{a}(x_{k-j} - x_k) = \frac{(-1)^a}{a+1}\sum_{s=0}^{a}(-1)^s e_s(\nu';i) x_k^{a-s}, \tag{3.16}$$

$$\varepsilon_{\nu'}/\varepsilon_{\nu,\nu'} = \frac{1}{b+1}\prod_{j=1}^{b}(x_k - x_{k+j}) = \frac{1}{b+1}\sum_{s=0}^{b}(-1)^s e_s(\nu;i+1) x_k^{b-s}. \tag{3.17}$$

Lemma 3.3. *Let notation be as above, so in particular $a = \nu_i - 1 = \nu'_i$ and $b = \nu_{i+1} = \nu'_{i+1} - 1$.*

(i) *The push-forward $p_* : C_{\nu,\nu'} \to C_\nu$ is the unique homogeneous C_ν-module homomorphism of degree $-a$ that maps $\varepsilon_\nu/\varepsilon_{\nu,\nu'}$ to 1. Equivalently, it is the unique C_ν-module homomorphism with $p_*(x_k^r) = (-1)^a h_{r-a}(\nu;i)$ for each $r \geq 0$.*

(ii) *The push-forward $p'_* : C_{\nu,\nu'} \to C_{\nu'}$ is the unique homogeneous $C_{\nu'}$-module homomorphism of degree $-b$ that maps $\varepsilon_{\nu'}/\varepsilon_{\nu,\nu'}$ to 1. Equivalently, it is the unique $C_{\nu'}$-module homomorphism with $p'_*(x_k^r) = h_{r-b}(\nu';i+1)$ for each $r \geq 0$.*

Proof. By the projection formula, p_* is a homogeneous C_ν-module homomorphism of degree $-a$, so it must map $\varepsilon_\nu/\varepsilon_{\nu,\nu'}$ to $c \cdot 1$ for some scalar c. Since it maps the fundamental class of a point to the fundamental class of a point, we know that $p_*(\varepsilon/\varepsilon_{\nu,\nu'}) = \varepsilon/\varepsilon_\nu$, from which we get that $c = 1$ as in the proof of Lemma 3.2. Now suppose that $f : C_{\nu,\nu'} \to C_\nu$ is any homogeneous C_ν-module homomorphism of degree $-a$ mapping $\varepsilon_\nu/\varepsilon_{\nu,\nu'}$ to 1. By (3.16) and (2.5),

$$\varepsilon_\nu/\varepsilon_{\nu,\nu'} = \frac{(-1)^a}{a+1}\sum_{s=0}^{a}\sum_{t=0}^{s}(-1)^{s+t}e_{s-t}(\nu;i)x_k^{a-s+t}.$$

Applying f to this equation and observing that $f(1) = \cdots = f(x_k^{a-1}) = 0$ by degree considerations, we deduce that $f(x_k^a) = (-1)^a$. Finally using (3.13), we get that $f(x_k^r) = (-1)^a h_{r-a}(\nu;i)$ for any $r \geq 0$ (recalling by convention that $h_{r-a}(\nu;i) = 0$ if $r < a$). Since the elements x_k^r generate $C_{\nu,\nu'}$ as a C_ν-module, there is clearly a unique such map. This proves (i), and (ii) is similar. □

Theorem 3.4. *Let notation be as above, so in particular $a = \nu_i - 1 = \nu'_i$ and $b = \nu_{i+1} = \nu'_{i+1} - 1$.*

(i) *The map $F_i : C_\nu \to C_{\nu'}$ is equal to the composite*

$$C_\nu \xrightarrow{p^*} C_{\nu,\nu'} \xrightarrow{m} C_{\nu,\nu'} \xrightarrow{p'_*} C_{\nu'}$$

where m is the map defined by multiplication by $\prod_{j=1}^{a}(x_{k-j} - x_k)$. Equivalently, it is the restriction to C_ν of the unique $C_{\nu'}$-module homomorphism $C_{\nu,\nu'} \to C_{\nu'}$ sending

$$x_k^r \mapsto (-1)^a \sum_{s=0}^{a}(-1)^s e_s(\nu';i)h_{r-s+a-b}(\nu';i+1)$$

for each $r \geq 0$.

(ii) *The map $E_i : C_{\nu'} \to C_\nu$ is equal to the composite*

$$C_{\nu'} \xrightarrow{(p')^*} C_{\nu,\nu'} \xrightarrow{m'} C_{\nu,\nu'} \xrightarrow{p_*} C_\nu$$

where m' is the map defined by multiplication by $\prod_{j=1}^{b}(x_k - x_{k+j})$. Equivalently, it is the restriction to $C_{\nu'}$ of the unique C_ν-module *homomorphism* $C_{\nu,\nu'} \to C_\nu$ *sending*

$$x_k^r \mapsto (-1)^a \sum_{s=0}^{b} (-1)^s e_s(\nu; i+1) h_{r-s+b-a}(\nu; i)$$

for each $r \geq 0$.

Proof. We just prove (i), the proof of (ii) being similar. Observe to start with that the action of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu C_\nu$ lifts to an action on $\bigoplus_\nu P_\nu$. To see this, let $\widetilde{P}$ denote the $\mathbb{C}S_n$-module obtained from P by twisting the natural permutation action by sign. Let $\widetilde{P}_\nu$ denote the S_ν-invariants in $\widetilde{P}$, or equivalently, the S_ν-anti-invariants in P. By another fundamental theorem of Chevalley, every element of $\widetilde{P}_\nu$ is divisible by ε_ν. Division by ε_ν defines a P_ν-module isomorphism $\hat{\pi}_* : \widetilde{P}_\nu \xrightarrow{\sim} P_\nu$ lifting the isomorphism $\pi_* : \widetilde{C}_\nu \xrightarrow{\sim} C_\nu$ from Lemma 3.2. Composing the direct sum of these isomorphisms over all ν with the isomorphism $\kappa_{\widetilde{P}}$ from Lemma 3.1, we obtain an isomorphism $\hat{\varphi}$ making the following diagram commute:

$$\begin{array}{ccc} \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{P} & \xrightarrow{\hat{\varphi}} & \bigoplus_\nu P_\nu \\ \downarrow & & \downarrow \\ \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C} & \xrightarrow[\varphi]{} & \bigoplus_\nu C_\nu \end{array}$$

where the vertical maps here (and later on) are the canonical quotient maps. The natural action of $\hat{\mathfrak{g}}$ on $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{P}$ clearly lifts the action of $\hat{\mathfrak{g}}$ on $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}$. Transporting this action through the isomorphism $\hat{\varphi}$ gives the desired action of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu P_\nu$.

Now we compute the effect of F_i on a polynomial $f \in P_\nu$. The inverse image of f under $\hat{\varphi}$ is $(\cdots \otimes v_i^{\otimes(a+1)} \otimes v_{i+1}^{\otimes b} \otimes \cdots) \otimes \varepsilon_\nu f$. Applying the Lie algebra element F_i (which maps one v_i to v_{i+1} in all possible ways), we get

$$(\cdots \otimes v_i^{\otimes a} \otimes v_{i+1}^{\otimes(b+1)} \otimes \cdots)(1 + \sum_{j=1}^{a} (k{-}j\ \ k)) \otimes \varepsilon_\nu f.$$

Since $-(k{-}j\ k)\varepsilon_\nu f = \varepsilon_\nu f$, this equals $(\cdots \otimes v_i^{\otimes a} \otimes v_{i+1}^{\otimes(b+1)} \otimes \cdots) \otimes (a+1)\varepsilon_\nu f$, which is the same as $(\cdots \otimes v_i^{\otimes a} \otimes v_{i+1}^{\otimes(b+1)} \otimes \cdots) \otimes \varepsilon_{\nu,\nu'} \prod_{j=1}^a (x_{k-j} - x_k) f$ by (3.16). Applying $\hat{\varphi}$, we get

$$\frac{1}{\varepsilon_{\nu'}} \frac{1}{|S_{\nu'}|} \sum_{w \in S_{\nu'}} \mathrm{sgn}(w) w \left(\varepsilon_{\nu,\nu'} \textstyle\prod_{j=1}^a (x_{k-j} - x_k) f \right).$$

Setting $P_{\nu,\nu'} := P^{S_\nu \cap S_{\nu'}}$, we have proved that the map $F_i : P_\nu \to P_{\nu'}$ is equal to the composite of the inclusion $\hat{p}^* : P_\nu \hookrightarrow P_{\nu,\nu'}$, then the map $\hat{m} : P_{\nu,\nu'} \to P_{\nu,\nu'}$ defined by multiplication by $\prod_{j=1}^a (x_{k-j} - x_k)$, and finally the map $\hat{p}'_* : P_{\nu,\nu'} \to P_{\nu'}$ defined by

$$f \mapsto \frac{1}{\varepsilon_{\nu'}} \frac{1}{|S_{\nu'}|} \sum_{w \in S_{\nu'}} \mathrm{sgn}(w) w \left(\varepsilon_{\nu,\nu'} f \right).$$

To complete the proof of the first statement of (i), it just remains to descend back down to partial coinvariant algebras, which amounts to checking that all three squares in the following diagram commute:

$$\begin{array}{ccccccc} P_\nu & \xrightarrow{\hat{p}^*} & P_{\nu,\nu'} & \xrightarrow{\hat{m}} & P_{\nu,\nu'} & \xrightarrow{\hat{p}'_*} & P_{\nu'} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ C_\nu & \xrightarrow[p^*]{} & C_{\nu,\nu'} & \xrightarrow[m]{} & C_{\nu,\nu'} & \xrightarrow[p'_*]{} & C_{\nu'}. \end{array}$$

The commutativity of the left hand two squares is obvious. For the right hand square, note from its definition that $\hat{p}'_*$ is a homogeneous $P_{\nu'}$-module homomorphism of degree $-b$ mapping $\varepsilon_{\nu'}/\varepsilon_{\nu,\nu'}$ to 1. Moreover, for any $f \in P_{\nu,\nu'}$ and any homogeneous symmetric function g of positive degree, we have that $\hat{p}'_*(fg) = \hat{p}'_*(f)g$, so $\hat{p}'_*$ factors through the quotients to induce a homogeneous $C_{\nu'}$-module homomorphism $C_{\nu,\nu'} \to C_{\nu'}$ of degree $-b$ mapping $\varepsilon_{\nu'}/\varepsilon_{\nu,\nu'}$ to 1. This coincides with the map p'_* by Lemma 3.3(ii).

Finally, to deduce the second description of F_i, take an element $z \in C_\nu$ and write $z = \sum_{r=0}^{b} z_r x_k^r$ for (unique) elements $z_r \in C_{\nu'}$. Using (3.16) and the second description of p'_* from Lemma 3.3(ii), $p'_* \circ m \circ p^*$ maps z to

$$\sum_{r=0}^{b} z_r \cdot (-1)^a \sum_{s=0}^{a} (-1)^s e_s(\nu'; i) h_{r-s+a-b}(\nu'; i+1).$$

This is also the image of z under the given $C_{\nu'}$-module homomorphism $C_{\nu,\nu'} \to C_{\nu'}$. □

4. The algebras C_ν^μ

Throughout the section, we fix a composition μ of n with transpose partition λ. We will now regard I_ν^μ as an ideal of C_ν rather than of P_ν, generated by the canonical images of the elements (2.16). So, recalling (2.17), we are now viewing the algebra C_ν^μ as the quotient C_ν/I_ν^μ, and $\bigoplus_\nu C_\nu^\mu$ is the quotient of $\bigoplus_\nu C_\nu$ by the subspace $\bigoplus_\nu I_\nu^\mu$. The following lemma implies that the action of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu C_\nu$ factors through this quotient to induce a well-defined action of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu C_\nu^\mu$.

Lemma 4.1. *For each $i \in \mathbb{Z}$, the Chevalley generators E_i and F_i of $\hat{\mathfrak{g}}$ leave the subspace $\bigoplus_\nu I_\nu^\mu$ of $\bigoplus_\nu C_\nu$ invariant.*

Proof. Fix $i \in \mathbb{Z}$ and let ν, ν' be compositions as in the key situation (3.9). We also let $a := \nu'_i, b := \nu_{i+1}$ and $k := \sum_{j \leq i} \nu_j$. It suffices to show that $F_i(I_\nu^\mu) \subseteq I_{\nu'}^\mu$ and $E_i(I_{\nu'}^\mu) \subseteq I_\nu^\mu$. We just verify the first containment, the second being entirely similar. Since $C_\nu \subseteq C_{\nu,\nu'}$ and $C_{\nu,\nu'}$ is generated as a $C_{\nu'}$-module by the elements x_k^p for $p \geq 0$, every element of I_ν^μ is a $C_{\nu'}$-linear combination of terms of the form $x_k^p h_r(\nu; i_1, \dots, i_m)$ for $p \geq 0$, $m \geq 1$, distinct integers $i_1, \dots, i_m$ and $r > \lambda_1 + \cdots + \lambda_m - \nu_{i_1} - \cdots - \nu_{i_m}$. Recalling the second description of F_i from Theorem 3.4(i), it therefore suffices to show that F_i maps all these $x_k^p h_r(\nu; i_1, \dots, i_m)$'s into $I_{\nu'}^\mu$.

If neither i nor $(i+1)$ belongs to the set $\{i_1, \dots, i_m\}$, then $h_r(\nu; i_1, \dots, i_m)$ equals $h_r(\nu'; i_1, \dots, i_m) \in C_{\nu'}$. So applying Theorem 3.4(i), we see that F_i maps $x_k^p h_r(\nu; i_1, \dots, i_m)$ to

$$(-1)^a \sum_{s=0}^{a} (-1)^s e_s(\nu'; i) h_{p-s+a-b}(\nu'; i+1) h_r(\nu'; i_1, \dots, i_m),$$

which belongs to $I_{\nu'}^{\mu}$ because $h_r(\nu'; i_1, \dots, i_m)$ does already according to the definition (2.16). The same argument applies if both i and $(i+1)$ belong to the set $\{i_1, \dots, i_m\}$. So it remains to consider the situations when exactly one of i and $(i+1)$ belongs to $\{i_1, \dots, i_m\}$. We may as well assume either that $i_m = i$ or that $i_m = i+1$.

Suppose first that $i_m = i$ and $i_1, \dots, i_{m-1} \neq i+1$. By (2.2), we have that $x_k^p h_r(\nu; i_1, \dots, i_m) = \sum_{t=0}^{r} h_t(\nu'; i_1, \dots, i_m) x_k^{p+r-t}$. Applying F_i using Theorem 3.4(i), we get the element

$$(-1)^a \sum_{s=0}^{a} \sum_{t=0}^{r} (-1)^s e_s(\nu'; i) h_{p+r-t-s+a-b}(\nu'; i+1) h_t(\nu'; i_1, \dots, i_m).$$

Since $r > \lambda_1 + \dots + \lambda_m - \nu_{i_1} - \dots - \nu_{i_m} = \lambda_1 + \dots + \lambda_m - \nu'_{i_1} - \dots - \nu'_{i_m} - 1$, we have that $h_t(\nu'; i_1, \dots, i_m) \in I_{\nu'}^{\mu}$ if $t > r$ by (2.16), so another application of (2.2) gives that

$$\sum_{t=0}^{r} h_{p+r-t-s+a-b}(\nu'; i+1) h_t(\nu'; i_1, \dots, i_m) \equiv h_{p+r-s+a-b}(\nu; i_1, \dots, i_m, i+1) \pmod{I_{\nu'}^{\mu}}.$$

It remains to show that $\sum_{s=0}^{a} (-1)^s e_s(\nu'; i) h_{p+r-s+a-b}(\nu'; i_1, \dots, i_m, i+1)$ belongs to $I_{\nu'}^{\mu}$. By (2.4), it equals $h_{p+r+a-b}(\nu'; i_1, \dots, i_{m-1}, i+1)$ which does indeed lie in $I_{\nu'}^{\mu}$ since $p + r + a - b = p + r + \nu_i - \nu'_{i+1} > \lambda_1 + \dots + \lambda_m - \nu'_{i_1} - \dots - \nu'_{i_{m-1}} - \nu'_{i+1}$.

Finally suppose that $i_m = i+1$ and $i_1, \dots, i_{m-1} \neq i$. By (2.4), we can rewrite $x_k^p h_r(\nu; i_1, \dots, i_m)$ as $x_k^p h_r(\nu'; i_1, \dots, i_m) - x_k^{p+1} h_{r-1}(\nu'; i_1, \dots, i_m)$. The image of this under F_i is

$$(-1)^a \sum_{s=0}^{a} (-1)^s e_s(\nu'; i) \big(h_{p-s+a-b}(\nu'; i+1) h_r(\nu'; i_1, \dots, i_m) - h_{p+1-s+a-b}(\nu'; i+1) h_{r-1}(\nu'; i_1, \dots, i_m) \big).$$

Since $r > \lambda_1 + \dots + \lambda_m - \nu'_{i_1} - \dots - \nu'_{i_m} + 1$ both of the terms $h_r(\nu'; i_1, \dots, i_m)$ and $h_{r-1}(\nu'; i_1, \dots, i_m)$ belong to $I_{\nu'}^{\mu}$. □

Recall C^μ denotes the algebra C_ν^μ when ν is regular. Let $\widetilde{C}^\mu$ denote the $\mathbb{C}S_n$-module obtained from C^μ by twisting the natural action by sign. Let $\widetilde{C}_\nu^\mu$ denote the subspace of all S_ν-invariants in $\widetilde{C}^\mu$, i.e. the space of all S_ν-anti-invariants in C^μ. The restriction of the canonical quotient map $C \twoheadrightarrow C^\mu$ defines a surjective linear map $\widetilde{C}_\nu \twoheadrightarrow \widetilde{C}_\nu^\mu$.

Lemma 4.2. *The isomorphism $\pi_* : \widetilde{C}_\nu \xrightarrow{\sim} C_\nu$ from Lemma 3.2 factors through the quotients to induce an isomorphism $\bar{\pi}_*$ making the following diagram commute:*

$$\begin{array}{ccc} \widetilde{C}_\nu & \xrightarrow{\pi_*} & C_\nu \\ \downarrow & & \downarrow \\ \widetilde{C}^\mu_\nu & \xrightarrow[\bar{\pi}_*]{} & C^\mu_\nu \end{array}$$

(Here, the vertical maps are the canonical quotient maps.) Composing the direct sum of these isomorphisms over all ν with the isomorphism $\kappa_{\widetilde{C}^\mu}$ from Lemma 3.1, we obtain a $\hat{\mathfrak{g}}$-module isomorphism $\bar{\varphi}$ such that the following diagram commutes:

$$\begin{array}{ccc} \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C} & \xrightarrow{\varphi} & \bigoplus_\nu C_\nu \\ \downarrow & & \downarrow \\ \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu & \xrightarrow[\bar{\varphi}]{} & \bigoplus_\nu C^\mu_\nu . \end{array}$$

(Here, φ is as in (3.8) and the vertical maps are the canonical quotients.)

Proof. Writing α, β and γ for the maps induced by the canonical quotient homomorphisms, the left hand square of the following diagram commutes:

$$\begin{array}{ccccc} \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C} & \xrightarrow{\kappa_{\widetilde{C}}} & \bigoplus_\nu \widetilde{C}_\nu & \xrightarrow{\pi_*} & \bigoplus_\nu C_\nu \\ \alpha\downarrow & & \beta\downarrow & & \downarrow\gamma \\ \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu & \xrightarrow[\kappa_{\widetilde{C}^\mu}]{} & \bigoplus_\nu \widetilde{C}^\mu_\nu & \xrightarrow[\bar{\pi}_*]{} & \bigoplus_\nu C^\mu_\nu . \end{array}$$

We have not yet defined the map $\bar{\pi}_*$ appearing in the right hand square.

Let us first show that the maps α and $\gamma \circ \pi_* \circ \kappa_{\widetilde{C}}$ have the same kernels. Both of these maps are $\hat{\mathfrak{g}}$-module homomorphisms. Moreover it is quite obvious that the restrictions of these maps to a regular weight space have the same kernels. Since any polynomial $\hat{\mathfrak{g}}$-module of degree n is generated by any one of its regular weight spaces, this implies that the kernels are equal everywhere. Using the commutativity of the left hand square, we deduce that the maps $\gamma \circ \pi_*$ and β also have the same kernels. Since they are both surjective, this means that π_* factors through the quotients to induce an isomorphism $\bar{\pi}_* : \bigoplus_\nu \widetilde{C}^\mu_\nu \to \bigoplus_\nu C^\mu_\nu$ making the right hand square commute. The rest of the lemma follows since $\varphi = \pi_* \circ \kappa_{\widetilde{C}}$ and $\bar{\varphi} = \bar{\pi}_* \circ \kappa_{\widetilde{C}^\mu}$. □

Now we can identify the $\hat{\mathfrak{g}}$-module $\bigoplus_\nu C^\mu_\nu$ explicitly. This result is a natural extension of Theorem 2.4, replacing S_n with $\hat{\mathfrak{g}}$.

Theorem 4.3. *Let μ be a composition of n with transpose partition λ.*

(i) *As a $\hat{\mathfrak{g}}$-module, $\bigoplus_\nu C^\mu_\nu$ is isomorphic to $\bigwedge^\mu(\widehat{V})$.*

(ii) *Assuming $C^\mu_\nu \neq 0$, i.e. $\lambda \geq \nu^+$, the top graded component of C^μ_ν is in degree*

$$d^\mu_\nu := \sum_{i \geq 1} \lambda_i(\lambda_i - 1) - \sum_{i \in \mathbb{Z}} \nu_i(\nu_i - 1) \tag{4.1}$$

(which is twice the dimension of the Spaltenstein variety F^μ_ν).

(iii) *As a $\hat{\mathfrak{g}}$-module, the direct sum $\bigoplus_\nu C^\mu_\nu(d^\mu_\nu)$ of the top graded components of all C^μ_ν is isomorphic to $P^\lambda(\widehat{V})$.*

(iv) *Given a non-zero vector $x \in C^\mu_\nu$, a regular composition ω and a composition γ with $\gamma^+ = \lambda$, there exist operators $u, v \in U(\hat{\mathfrak{g}})$ and $y \in C^\mu_\omega$ such that $vx \in C^\mu_\omega$, $y(vx) \in C^\mu_\omega(d^\mu_\omega)$ and $u(y(vx))$ is the identity element of the one-dimensional algebra C^μ_γ.*

Proof. Consider the $\hat{\mathfrak{g}}$-module isomorphism $\bar{\varphi} : \widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu \to \bigoplus_\nu C^\mu_\nu$ from Lemma 4.2. Combined with Theorem 2.4(i) and (3.3), it implies that $\bigoplus_\nu C^\mu_\nu$ is isomorphic to $\bigwedge^\mu(\widehat{V})$ as a $\hat{\mathfrak{g}}$-module, giving (i). Declaring that $\widehat{V}^{\otimes n}$ is concentrated in degree 0, the natural grading on $\widetilde{C}^\mu$ extends to a grading on $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu$, and the action of $\hat{\mathfrak{g}}$ preserves this grading. By Theorem 2.4(ii) the top graded component of $\widetilde{C}^\mu$ is in degree $\sum_{i\geq 1} \lambda_i(\lambda_i - 1)$, and by Theorem 2.4(iii) it is isomorphic to S^λ as a $\mathbb{C}S_n$-module. Hence the top graded component of $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu$ is also in degree $\sum_{i\geq 1} \lambda_i(\lambda_i - 1)$ and by (3.4) it is isomorphic to $P^\lambda(\widehat{V})$ as a $\hat{\mathfrak{g}}$-module. If $C^\mu_\nu \neq 0$, which means $\lambda \geq \nu^+$ by Lemma 2.3, the ν-weight space of $P^\lambda(\widehat{V})$ is non-zero, so this is also the degree of the top graded component of the ν-weight space of $\widehat{V}^{\otimes n} \otimes_{\mathbb{C}S_n} \widetilde{C}^\mu$. Since the restriction of the isomorphism $\bar{\varphi}$ to ν-weight spaces is a homogeneous map of degree $-\sum_{i \in \mathbb{Z}} \nu_i(\nu_i - 1)$, this proves (ii) and (iii).

Finally, we prove (iv). The ω-weight space of every irreducible polynomial representation of degree n is non-zero. So any vector in any polynomial representation can be mapped to a non-zero vector of weight ω by applying some element of $U(\hat{\mathfrak{g}})$. So given a non-zero vector $x \in C^\mu_\nu$, there exists $v \in U(\hat{\mathfrak{g}})$ such that vx is a non-zero element of C^μ_ω. Then by Theorem 2.4(iv) there is an element $y \in C^\mu_\omega$ so that $y(vx)$ is a non-zero element of the top graded component of C^μ_ω. Finally by the irreducibility of $\bigoplus_\nu C^\mu_\nu(d^\mu_\nu)$, we can find $u \in U(\hat{\mathfrak{g}})$ such that $u(y(vx))$ is a non-zero vector of C^μ_γ. The latter algebra is one dimensional as the γ-weight space of $\bigwedge^\mu(\widehat{V})$ is one dimensional, so we can even ensure that $u(y(vx)) = 1$. □

Corollary 4.4. *The dimension of C^μ_ν is equal to the number of column strict λ-tableaux of type ν.*

Proof. The usual monomial basis of the ν-weight space of $\bigwedge^\mu(\widehat{V})$ is parametrized in an obvious way by column strict λ-tableaux of type ν. □

Remark 4.5. We have shown that $\bigoplus_\nu C^\mu_\nu = \bigoplus_{r \geq 0} \bigoplus_\nu C^\mu_\nu(d^\mu_\nu - 2r)$, with each $\bigoplus_\nu C^\mu_\nu(d^\mu_\nu - 2r)$ being a $\hat{\mathfrak{g}}$-submodule. Letting κ be a partition of n with transpose τ, the above arguments combined with Remark 2.5 show moreover that

$$\sum_{r\geq 0}\left[\bigoplus_\nu C^\mu_\nu(d^\mu_\nu - 2r) : P^\kappa(\widehat{V})\right] t^r = K_{\tau,\mu}(t), \tag{4.2}$$

the Kostka-Foulkes polynomial. Hence, the Hilbert polynomial of the graded algebra C^μ_ν is given by the formula

$$\sum_{r\geq 0} \dim C^\mu_\nu(r)\, t^r = t^{d^\mu_\nu} \sum_{(\kappa,\tau)} K_{\kappa,\nu} K_{\tau,\mu}(t^{-2}), \tag{4.3}$$

where the sum is over all pairs (κ, τ) of mutually transpose partitions of n.

Remark 4.6. Let $i : F^\mu \hookrightarrow F$ and $j : F^\mu_\nu \hookrightarrow F_\nu$ be the natural inclusions. Recall we have identified $H^*(F, \mathbb{C})$ with the coinvariant algebra C. In turn, by [T], we can identify $H^*(F^\mu, \mathbb{C})$ with C^μ so that the pull-back homomorphism $i^* : H^*(F, \mathbb{C}) \to H^*(F^\mu, \mathbb{C})$ coincides with the natural quotient map $C \twoheadrightarrow C^\mu$. Thus, the graded vector spaces $\widetilde{C}_\nu$ and $\widetilde{C}^\mu_\nu$ are identified with the S_ν-anti-invariants in $H^*(F, \mathbb{C})$ and in $H^*(F^\mu, \mathbb{C})$, respectively. In [BM, Corollary 3.4(b)], Borho and MacPherson proved that there exists a graded vector space isomorphism $\widetilde{C}^\mu_\nu \xrightarrow{\sim} H^*(F^\mu_\nu, \mathbb{C})$ that is homogeneous of degree $-\sum_{i\in\mathbb{Z}} \nu_i(\nu_i - 1)$. In fact, there is a unique such isomorphism $\bar{\pi}_*$ making the following diagram commute:

$$\begin{array}{ccc} \widetilde{C}_\nu & \xrightarrow{\pi_*} & H^*(F_\nu, \mathbb{C}) \\ {\scriptstyle i^*}\downarrow & & \downarrow{\scriptstyle j^*} \\ \widetilde{C}^\mu_\nu & \xrightarrow[\bar{\pi}_*]{} & H^*(F^\mu_\nu, \mathbb{C}). \end{array} \tag{4.4}$$

This last statement, which does not seem to follow directly from [BM], will be proved in [BO]. Comparing with the first statement of Lemma 4.2, it implies that the algebra C^μ_ν is canonically isomorphic to the cohomology algebra of the Spaltenstein variety F^μ_ν.

Example 4.7. Let us give one small example where everything can be worked out by hand. Let $\mu = \nu = (\dots, 0, 1, 2, 1, 0, \dots)$. So the nilpotent matrix x_μ is simply equal to the matrix unit $e_{2,3}$, and the Spaltenstein variety F^μ_ν is the space of all partial flags $(L \subset H)$ consisting of an $e_{2,3}$-invariant line L and an $e_{2,3}$-invariant hyperplane H in $\mathbb{C}^4$ such that $e_{2,3}H \subseteq L$. As a variety, F^μ_ν is isomorphic to two copies of $\mathbb{P}^2$ glued at a point, and its cohomology algebra is isomorphic to the algebra

$$\mathbb{C}[x, y]/\langle x^3, y^3, xy\rangle$$

with basis $\{1, x, x^2, y, y^2\}$; both these statements make good exercises. On the other hand, using Lemma 2.2, our algebra C^μ_ν is the quotient of P_ν by the ideal generated by all the elementary symmetric functions in x_1, x_2, x_3, x_4 of positive degree together with the elements $\{x_1x_4, x_1x_2x_3, x_2x_3x_4\}$. It is straightforward now by explicitly checking relations to see that there is an isomorphism $H^*(F^\mu_\nu, \mathbb{C}) \to C^\mu_\nu$ sending $x \mapsto x_1, y \mapsto x_4$. In particular, $\{1, x_1, x_1^2, x_4, x_4^2\}$ is a basis[4] for C^μ_ν.

5. Trace maps

In this section, we give a quite different algebraic interpretation of the actions of the Chevalley generators E_i and F_i of $\hat{\mathfrak{g}}$ on $\bigoplus_\nu C_\nu$. Fix $i \in \mathbb{Z}$ and compositions ν, ν' as in the key situation (3.9), and set $a := \nu'_i, b := \nu_{i+1}$ and $k := \sum_{j \leq i} \nu_i$. Recall that the algebra $C^{S_\nu \cap S_{\nu'}}$ is free both as a C_ν-module with basis $1, x_k, \dots, x_k^a$ and as a $C_{\nu'}$-module with basis $1, x_k, \dots, x_k^b$. From now on, we reserve the notation $C_{\nu,\nu'}$ for this algebra when viewed as a $(C_\nu, C_{\nu'})$-bimodule, and we introduce the new notation $C_{\nu',\nu}$ for the same algebra when viewed as a $(C_{\nu'}, C_\nu)$-bimodule. Of course since all our algebras are commutative, this distinction is quite artificial, but it helps to keep track of the actions (left or right) later on. Tensoring with these bimodules defines exact functors

$$C_{\nu,\nu'} \otimes_{C_{\nu'}} ? : C_{\nu'}\text{-mod} \to C_\nu\text{-mod}, \tag{5.1}$$

$$C_{\nu',\nu} \otimes_{C_\nu} ? : C_\nu\text{-mod} \to C_{\nu'}\text{-mod}, \tag{5.2}$$

where A-mod denotes the category of finite dimensional left A-modules. The immediate goal is to prove that these functors are both left and right adjoint to one another in canonical ways; see also [FKS, Proposition 3.5].

Lemma 5.1. *There is a unique $(C_{\nu'}, C_\nu)$-bimodule isomorphism*

$$\delta : C_{\nu',\nu} \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$$

such that $(\delta(x_k^r))(x_k^s) = (-1)^a h_{r+s-a}(\nu; i)$ for $r, s \geq 0$. The inverse isomorphism sends $f \mapsto (-1)^a \sum_{r=0}^a (-1)^r e_r(\nu'; i) f(x_k^{a-r})$.

Proof. For $0 \leq r \leq a$, let $\delta_r : C_{\nu,\nu'} \to C_\nu$ be the unique C_ν-module homomorphism such that

$$\delta_r(x_k^{a-s}) = \begin{cases} 0 & \text{if } s < a, \\ (-1)^a & \text{if } s = a. \end{cases}$$

The maps $\delta_0, \delta_1, \dots, \delta_a$ form a basis for $\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$ as a free C_ν-module. Moreover, by (3.13), we have that $\delta_0(x_k^r) = (-1)^a h_{r-a}(\nu; i)$ for any $r \geq 0$. Hence, writing $x_k^r \delta_0$ for the map $x \mapsto \delta_0(x x_k^r)$, we get that

$$x_k^r \delta_0 = \sum_{s=0}^a \delta_s h_{r-s}(\nu; i),$$

since both sides map x_k^{a-s} to $h_{r-s}(\nu; i)$. Inverting using (2.1) we get that

$$\delta_r = \sum_{s=0}^r (-1)^{r-s} (x_k^s \delta_0) e_{r-s}(\nu; i)$$

for $0 \leq r \leq a$. These equations imply that the maps $\delta_0, x_k \delta_0, \dots, x_k^a \delta_0$ also form a basis for $\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$ as a free C_ν-module. Moreover, since

[4]A similar algebraic basis for the algebras C^μ_ν in general will be constructed in [BO].

$\sum_{s=0}^{r}(-1)^{r-s}x_k^s e_{r-s}(\nu;i) = (-1)^r e_r(\nu';i)$ by a special case of (2.5), we have shown that

$$\delta_r = (-1)^r e_r(\nu';i)\delta_0$$

for $0 \leq r \leq a$.

Now we know enough to prove the lemma. Recalling that $C^{S_\nu \cap S_{\nu'}} = C_{\nu,\nu'} = C_{\nu',\nu}$, there is a well-defined $C^{S_\nu \cap S_{\nu'}}$-module homomorphism

$$\delta : C_{\nu',\nu} \to \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu), \qquad 1 \mapsto \delta_0.$$

This is automatically a $(C_{\nu'}, C_\nu)$-bimodule homomorphism. The elements $1, x_k, \dots, x_k^a$ form a basis for $C_{\nu',\nu}$ as a free right C_ν-module, and δ maps them to to the functions $\delta_0, x_k\delta_0, \dots, x_k^a\delta_0$ which we showed in the previous paragraph give a basis for $\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$ as a free right C_ν-module. Hence, δ is an isomorphism. Moreover,

$$(\delta(x_k^r))(x_k^s) = (x_k^r\delta_0)(x_k^s) = \delta_0(x_k^{r+s}) = (-1)^a h_{r+s-a}(\nu;i),$$

so the isomorphism δ just constructed is precisely the map in the statement of the lemma. Finally, take any $f \in \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$. We have that

$$f = (-1)^a \sum_{r=0}^{a} \delta_r f(x_k^{a-r}) = (-1)^a \sum_{r=0}^{a} (-1)^r e_r(\nu';i)\delta_0 f(x_k^{a-r}).$$

Hence, $\delta^{-1}(f) = (-1)^a \sum_{r=0}^{a}(-1)^r e_r(\nu';i) f(x_k^{a-r})$ as claimed. □

Corollary 5.2. *For any C_ν-module M, there is a natural $C_{\nu'}$-module isomorphism*

$$C_{\nu',\nu} \otimes_{C_\nu} M \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, M)$$

such that $x_k^r \otimes m$ for any $r \geq 0$ and $m \in M$ maps to the unique C_ν-module homomorphism sending x_k^s to $(-1)^a h_{r+s-a}(\nu;i)m$ for each $s \geq 0$. The inverse isomorphism sends $f \mapsto (-1)^a \sum_{r=0}^{a}(-1)^r e_r(\nu';i) \otimes f(x_k^{a-r})$.

Proof. Let δ be the isomorphism from Lemma 5.1. We obtain the desired natural isomorphism from the composite

$$C_{\nu',\nu} \otimes_{C_\nu} M \xrightarrow{\delta \otimes \mathrm{id}_M} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu) \otimes_{C_\nu} M \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, M),$$

where the second map is the obvious natural isomorphism. Now compute this composite map and its inverse using the explicit descriptions of δ and δ^{-1} from Lemma 5.1. □

Corollary 5.2 combined with adjointness of tensor and hom implies that $(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$ is an adjoint pair of functors. Let us write down the unit and counit of the canonical adjunction explicitly; see also the discussion following (6.7) below. The unit is the natural transformation

$$\iota' : \mathrm{Id}_{C_{\nu'}\text{-mod}} \to C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'} \otimes_{C_{\nu'}} ? \tag{5.3}$$

defined by $m \mapsto (-1)^a \sum_{r=0}^{a}(-1)^r e_r(\nu';i) \otimes x_k^{a-r} \otimes m$ for all $C_{\nu'}$-modules M and $m \in M$. (The prime in the notation ι' is intended to help remember that it defines $C_{\nu'}$-module homomorphisms.) The counit is the natural transformation

$$\varepsilon : C_{\nu,\nu'} \otimes_{C_{\nu'}} C_{\nu',\nu} \otimes_{C_\nu} ? \to \mathrm{Id}_{C_\nu\text{-mod}} \tag{5.4}$$

defined by $x_k^r \otimes x_k^s \otimes m \mapsto (-1)^a h_{r+s-a}(\nu; i)m$ for $r, s \geq 0$, $m \in M$ and any C_ν-module M.

Repeating the proof of Lemma 5.1 with the roles of C_ν and $C_{\nu'}$ reversed, one shows instead that there exists a unique $(C_\nu, C_{\nu'})$-bimodule isomorphism

$$\delta' : C_{\nu,\nu'} \to \mathrm{Hom}_{C_{\nu'}}(C_{\nu',\nu}, C_{\nu'}) \tag{5.5}$$

such that $(\delta'(x_k^r))(x_k^s) = h_{r+s-b}(\nu'; i+1)$ for each $r, s \geq 0$. The inverse map sends $f \mapsto \sum_{r=0}^b (-1)^r e_r(\nu; i+1) f(x_k^{b-r})$. Hence, just like in Corollary 5.2, the functors $C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ and $\mathrm{Hom}_{C_{\nu'}}(C_{\nu',\nu}, ?)$ are isomorphic. This means that $(C_{\nu',\nu} \otimes_{C_\nu} ?, C_{\nu,\nu'} \otimes_{C_\nu} ?)$ is an adjoint pair too. This way round, the unit of the canonical adjunction is the natural transformation

$$\iota : \mathrm{Id}_{C_\nu\text{-mod}} \to C_{\nu,\nu'} \otimes_{C_{\nu'}} C_{\nu',\nu} \otimes_{C_\nu} ? \tag{5.6}$$

defined by $m \mapsto \sum_{r=0}^b (-1)^r e_r(\nu; i+1) \otimes x_k^{b-r} \otimes m$ for all C_ν-modules M and $m \in M$. The counit is the natural transformation

$$\varepsilon' : C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'} \otimes_{C_{\nu'}} ? \to \mathrm{Id}_{C_{\nu'}\text{-mod}} \tag{5.7}$$

defined by $x_k^r \otimes x_k^s \otimes m \mapsto h_{r+s-b}(\nu'; i+1)m$ for $r, s \geq 0$, $m \in M$ and any $C_{\nu'}$-module M.

Using the adjoint pairs just constructed and the general construction of [Be, §3], we can define some natural *trace maps*

$$Z(C_\nu\text{-mod}) \underset{E_i}{\overset{F_i}{\rightleftarrows}} Z(C_{\nu'}\text{-mod}), \tag{5.8}$$

between the centers of the module categories. To define these maps explicitly, take $z \in Z(C_\nu\text{-mod})$ and $z' \in Z(C_{\nu'}\text{-mod})$. Then $E_i(z')$ and $F_i(z)$ are the unique elements of $Z(C_\nu\text{-mod})$ and $Z(C_{\nu'}\text{-mod})$, respectively, defined by the equations

$$E_i(z') := \varepsilon \circ (\mathbf{1}_{C_{\nu,\nu'} \otimes_{C_{\nu'}} ?}) z' (\mathbf{1}_{C_{\nu',\nu} \otimes_{C_\nu} ?}) \circ \iota, \tag{5.9}$$

$$F_i(z) := \varepsilon' \circ (\mathbf{1}_{C_{\nu',\nu} \otimes_{C_\nu} ?}) z (\mathbf{1}_{C_{\nu,\nu'} \otimes_{C_{\nu'}} ?}) \circ \iota'. \tag{5.10}$$

Since C_ν is a commutative algebras, the center $Z(C_\nu\text{-mod})$ is canonically isomorphic to the algebra C_ν itself, via the map $C_\nu \xrightarrow{\sim} Z(C_\nu\text{-mod})$ arising from multiplication. Similarly, $Z(C_{\nu'}\text{-mod})$ is canonically isomorphic to $C_{\nu'}$. Making these identifications, the maps E_i and F_i just defined become linear maps between C_ν and $C_{\nu'}$.

Theorem 5.3. *The linear maps E_i and F_i just defined are the same as the maps arising from the Chevalley generators E_i and F_i of $\hat{\mathfrak{g}}$ that were computed in Theorem 3.4.*

Proof. We just check this in the case of F_i, the argument for E_i being similar. We need the maps

$$\iota' : C_{\nu'} \to C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'}, \qquad \varepsilon' : C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'} \to C_{\nu'}$$

arising from (5.3) and (5.7) applied to the module $M = C_{\nu'}$. The former maps $1 \mapsto (-1)^a \sum_{r=0}^a (-1)^r e_r(\nu'; i) \otimes x_k^{a-r}$. The latter is the $C_{\nu'}$-module homomorphism mapping $x_k^r \otimes x_k^s \mapsto h_{r+s-b}(\nu'; i+1)$ for all $r, s \geq 0$. Now take any $z \in C_\nu$ and write it as $z = \sum_{r=0}^{b} z_r x_k^r$ for (unique) elements $z_r \in C_{\nu'}$. By the definition (5.10), $F_i(z)$ is the image of $1 \in C_{\nu'}$ under the composite first of $\iota' : C_{\nu'} \to C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'}$, then the endomorphism of $C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'}$ defined by left multiplication by $1 \otimes z$, and finally the map $\varepsilon' : C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'} \to C_{\nu'}$. As z belongs to C_ν, left multiplication by $1 \otimes z$ is the same thing as left multiplication by $z \otimes 1$. Using these facts, an elementary calculation now gives that

$$F_i(z) = \sum_{r=0}^{b} z_r \cdot (-1)^a \sum_{s=0}^{a} (-1)^s e_s(\nu'; i) h_{r-s+a-b}(\nu'; i+1).$$

This agrees with the second description of the action of the Chevalley generator F_i on z from Theorem 3.4(i). □

6. Category $\mathcal{O}$

At last it is time to bring category $\mathcal{O}$ into the picture. The basic notation concerning $\mathcal{O}$ was introduced already in the introduction. We continue to write $\bigoplus_\nu$ for the direct sum over all compositions ν of n, so $\bigoplus_\nu \mathcal{O}_\nu$ is the sum of all the integral blocks of $\mathcal{O}$. We also fix throughout the section a composition μ of n and let λ be the transpose partition. We will soon need the following result, which was proved in [B, Theorem 2]. For the final part of (ii), we refer the reader to [BK2, §4] where an explicit parametrization of the irreducible modules in parabolic category $\mathcal{O}$ for $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ is explained in terms of column strict tableaux.

Lemma 6.1. *Let ν be a composition of n.*

(i) *The natural multiplication map $m_\nu^\mu : Z(\mathfrak{g}) \to Z(\mathcal{O}_\nu^\mu)$ is surjective.*

(ii) *The dimension of $Z(\mathcal{O}_\nu^\mu)$ is equal to the number of isomorphism classes of irreducible modules in $\mathcal{O}_\nu^\mu$, which is the same as the number of column strict λ-tableaux of type ν.*

Let us recall some of Soergel's results from [S]. Fix a composition ν of n. Let Q_ν denote the antidominant projective indecomposable module in $\mathcal{O}_\nu$, that is, the projective cover of the irreducible module $L(\alpha)$ where $\alpha = \sum_{i=1}^{n} a_i \varepsilon_i$ is the unique weight such that $a_1 \leq \cdots \leq a_n$ and exactly ν_i of the integers $a_1, \dots, a_n$ are equal to i for each $i \in \mathbb{Z}$. Let $p_\nu : Z(\mathfrak{g}) \to \mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}}$ be the homomorphism induced by multiplication; the op here indicates that we are for once viewing Q_ν as a *right* module over $\mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}}$. Also let $q_\nu : Z(\mathfrak{g}) \to C_\nu$ be the homomorphism sending the generator $z_r \in Z(\mathfrak{g})$ to $e_r(x_1 + a_1, \dots, x_n + a_n) \in C_\nu$ for each $r = 1, \dots, n$, as in the introduction. With this notation, we can now formulate Soergel's fundamental theorem [S, Endomorphismensatz 7] for the Lie algebra $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ as follows: *The maps p_ν and q_ν are surjective and have the same kernels.*

Hence, there is a unique isomorphism c_ν making the following diagram commute:

$$\begin{array}{ccc} & Z(\mathfrak{g}) & \\ {}^{p_\nu}\swarrow & & \searrow^{q_\nu} \\ \mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}} & \xrightarrow[c_\nu]{\sim} & C_\nu. \end{array} \tag{6.1}$$

We also need the following known lemma; see for example [MS, Theorem 5.2(2)] where a more general result than this is proved (for regular blocks). For completeness, we include a self-contained proof based on Lemma 6.1.

Lemma 6.2. *The natural multiplication map $f_\nu : Z(\mathcal{O}_\nu) \to \mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}}$ is an isomorphism.*

Proof. As well as being the projective cover of $L(\alpha)$, the self-dual module Q_ν is its injective hull. Every Verma module in $\mathcal{O}_\nu$ has irreducible socle isomorphic to $L(\alpha)$. Every projective module in $\mathcal{O}_\nu$ has a Verma flag, so embeds into a direct sum of copies of Q_ν. Hence every module in $\mathcal{O}_\nu$ is a quotient of a submodule of a direct sum of copies of Q_ν. This means that if z belongs to $\ker f_\nu$, i.e. it acts as zero on Q_ν, it also acts as zero on every module in $\mathcal{O}_\nu$, so in fact $z = 0$. Hence f_ν is injective. It is surjective because the natural multiplication map $m_\nu : Z(\mathfrak{g}) \to Z(\mathcal{O}_\nu)$ is surjective by Lemma 6.1(i), and the surjection p_ν from (6.1) factors as $p_\nu = f_\nu \circ m_\nu$. □

Using the isomorphism c_ν from (6.1) we will from now on identify the algebra $\mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}}$ with C_ν, making Q_ν into a right C_ν-module. Using the isomorphism f_ν from Lemma 6.2 we will also identify $Z(\mathcal{O}_\nu)$ with $\mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}}$. So it makes sense to write simply $Z(\mathcal{O}_\nu) = \mathrm{End}_{\mathcal{O}_\nu}(Q_\nu)^{\mathrm{op}} = C_\nu$, and the maps p_ν and c_ν in (6.1) have been identified with a surjection $p_\nu : Z(\mathfrak{g}) \to Z(\mathcal{O}_\nu)$ and an isomorphism $c_\nu : Z(\mathcal{O}_\nu) \to C_\nu$, as we wrote in the introduction.

We introduce Soergel's combinatorial functor

$$\mathbb{V}_\nu : \mathcal{O}_\nu \to C_\nu\text{-mod}, \qquad \mathbb{V}_\nu := \mathrm{Hom}_{\mathcal{O}_\nu}(Q_\nu, ?). \tag{6.2}$$

By [S, Struktursatz 9], given any projective module P in $\mathcal{O}_\nu$ and any other module M, the functor $\mathbb{V}_\nu$ defines an isomorphism

$$\mathbb{V}_\nu : \mathrm{Hom}_{\mathcal{O}_\nu}(M, P) \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(\mathbb{V}_\nu M, \mathbb{V}_\nu P). \tag{6.3}$$

So if we let P_ν denote a minimal projective generator for $\mathcal{O}_\nu$ and set

$$A_\nu := \mathrm{End}_{\mathcal{O}_\nu}(P_\nu)^{\mathrm{op}}, \tag{6.4}$$

then the functor $\mathbb{V}_\nu$ defines an algebra isomorphism $A_\nu \xrightarrow{\sim} \mathrm{End}_{C_\nu}(\mathbb{V}_\nu P_\nu)^{\mathrm{op}}$. It is often convenient to identify A_ν with $\mathrm{End}_{C_\nu}(\mathbb{V}_\nu P_\nu)^{\mathrm{op}}$ in this way. We can also identify Q_ν with a unique indecomposable summand of P_ν, so there exists an idempotent $e_\nu \in A_\nu$ such that $Q_\nu = P_\nu e_\nu$. It is then the case that $C_\nu = e_\nu A_\nu e_\nu$.

Suppose now that we are given compositions ν, ν' of n and exact functors $\mathbb{E} : \mathcal{O}_{\nu'} \to \mathcal{O}_\nu$ and $\mathbb{F} : \mathcal{O}_\nu \to \mathcal{O}_{\nu'}$ that commute with direct sums. Assume in addition that we are given a $(C_{\nu'}, C_\nu)$-bimodule $C_{\nu',\nu}$, a $(C_\nu, C_{\nu'})$-bimodule $C_{\nu,\nu'}$, and a pair of isomorphisms of functors

$$\tau : \mathbb{V}_\nu \circ \mathbb{E} \xrightarrow{\sim} C_{\nu,\nu'} \otimes_{C_{\nu'}} ? \circ \mathbb{V}_{\nu'}, \tag{6.5}$$

$$\tau' : \mathbb{V}_{\nu'} \circ \mathbb{F} \xrightarrow{\sim} C_{\nu',\nu} \otimes_{C_\nu} ? \circ \mathbb{V}_\nu. \tag{6.6}$$

Let $\mathrm{Adj}(\mathbb{E}, \mathbb{F})$ and $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$ denote the sets of all (not necessarily graded) adjunctions making $(\mathbb{E}, \mathbb{F})$ and $(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$, respectively, into adjoint pairs of functors. Given an adjunction belonging to $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$, there is an induced $(C_{\nu'}, C_\nu)$-bimodule isomorphism

$$\delta : C_{\nu',\nu} = \mathrm{Hom}_{C_{\nu'}}(C_{\nu'}, C_{\nu',\nu} \otimes_{C_\nu} C_\nu) \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'} \otimes_{C_{\nu'}} C_{\nu'}, C_\nu) = \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu), \tag{6.7}$$

where the middle isomorphism is defined by the adjunction. Conversely, any such $(C_{\nu'}, C_\nu)$-bimodule isomorphism δ defines an isomorphism between the functors $C_{\nu',\nu} \otimes_{C_\nu} ?$ and $\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, ?)$ as in the proof of Corollary 5.2. Hence by adjointness of tensor and hom, δ induces an adjunction between $C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ and $C_{\nu',\nu} \otimes_{C_\nu} ?$, i.e. an element of $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$. The unit ι and counit ε of this induced adjunction are characterized as follows:

- ι is the natural transformation $\mathrm{Id}_{C_{\nu'}\text{-mod}} \to C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ defining the map $1 \mapsto \sum_{i=1}^k a_i \otimes b_i$ on the regular module $C_{\nu'}$, where $\sum_{i=1}^k a_i \otimes b_i$ is the unique element of $C_{\nu',\nu} \otimes_{C_\nu} C_{\nu,\nu'}$ such that $\sum_{i=1}^k (\delta(a_i))(x) b_i = x$ for all $x \in C_{\nu,\nu'}$;
- ε is the natural transformation $C_{\nu,\nu'} \otimes_{C_{\nu'}} C_{\nu',\nu} \otimes ? \to \mathrm{Id}_{C_\nu\text{-mod}}$ defining the map $C_{\nu,\nu'} \otimes_{C_{\nu'}} C_{\nu',\nu} \to C_\nu, a \otimes b \mapsto (\delta(b))(a)$ on the regular module C_ν.

The two constructions just described give mutually inverse bijections between $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$ and the set of all $(C_{\nu'}, C_\nu)$-bimodule isomorphisms from $C_{\nu',\nu}$ to $\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu)$. In similar fashion, an element of $\mathrm{Adj}(\mathbb{E}, \mathbb{F})$ defines an $(A_{\nu'}, A_\nu)$-bimodule isomorphism

$$\hat{\delta} : \mathrm{Hom}_{\mathcal{O}_{\nu'}}(P_{\nu'}, \mathbb{F} P_\nu) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{O}_\nu}(\mathbb{E} P_{\nu'}, P_\nu). \tag{6.8}$$

This gives a bijection between $\mathrm{Adj}(\mathbb{E}, \mathbb{F})$ and the set of all such $(A_{\nu'}, A_\nu)$-bimodule isomorphisms. The following lemma explaining how adjunctions between $\mathbb{E}$ and $\mathbb{F}$ induce adjunctions between $C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ and $C_{\nu',\nu} \otimes_{C_\nu} ?$ is a general result about quotient functors in the sense of [G, §III.1]. We include a sketch of the proof since we will exploit the explicit description of the induced unit and counit later on.

Lemma 6.3. *There is a well-defined map*

$$T : \mathrm{Adj}(\mathbb{E}, \mathbb{F}) \to \mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$$

sending the adjunction in $\mathrm{Adj}(\mathbb{E}, \mathbb{F})$ *with unit* $\hat{\iota}$ *and counit* $\hat{\varepsilon}$ *to the adjunction in* $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$ *with unit* ι *and counit* ε *determined by the* *property that the following diagrams commute:*

$$\begin{array}{ccc}
\mathbb{V}_{\nu'} & \xrightarrow{\mathbf{1}_{\mathbb{V}_{\nu'}}\hat{\iota}} & \mathbb{V}_{\nu'}\mathbb{F}\mathbb{E} \\
{\scriptstyle \iota\mathbf{1}_{\mathbb{V}_{\nu'}}}\big\downarrow & & \big\downarrow{\scriptstyle \tau'\mathbf{1}_{\mathbb{E}}} \\
C_{\nu',\nu}\otimes_{C_\nu} C_{\nu,\nu'}\otimes_{C_{\nu'}}\mathbb{V}_{\nu'} & \xleftarrow[(\mathbf{1}_{C_{\nu',\nu}\otimes ?})\tau]{} & C_{\nu',\nu}\otimes_{C_\nu}\mathbb{V}_\nu\mathbb{E},
\end{array}$$

$$\begin{array}{ccc}
\mathbb{V}_{\nu} & \xleftarrow{\mathbf{1}_{\mathbb{V}_{\nu}}\hat{\varepsilon}} & \mathbb{V}_{\nu}\mathbb{E}\mathbb{F} \\
{\scriptstyle \varepsilon\mathbf{1}_{\mathbb{V}_{\nu}}}\big\uparrow & & \big\downarrow{\scriptstyle \tau\mathbf{1}_{\mathbb{F}}} \\
C_{\nu,\nu'}\otimes_{C_{\nu'}} C_{\nu',\nu}\otimes_{C_{\nu}}\mathbb{V}_{\nu} & \xleftarrow[(\mathbf{1}_{C_{\nu,\nu'}\otimes ?})\tau']{} & C_{\nu,\nu'}\otimes_{C_{\nu'}}\mathbb{V}_{\nu'}\mathbb{F}.
\end{array}$$

Proof. We first construct the map T. Take an adjunction between $\mathbb{E}$ and $\mathbb{F}$ with unit $\hat{\iota}$ and counit $\hat{\varepsilon}$, i.e. an element of $\mathrm{Adj}(\mathbb{E},\mathbb{F})$. It defines an isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'},\mathbb{F}Q_\nu)\xrightarrow{\sim}\mathrm{Hom}_{\mathcal{O}_\nu}(\mathbb{E}Q_{\nu'},Q_\nu),\qquad g\mapsto\hat{\varepsilon}_{Q_\nu}\circ\mathbb{E}g.$$

Composing on the right with the inverse of the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'},\mathbb{F}Q_\nu)\xrightarrow{\mathbb{V}_{\nu'}}\mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'}Q_{\nu'},\mathbb{V}_{\nu'}(\mathbb{F}Q_\nu))\xrightarrow{\tau'}$$
$$\mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'}Q_{\nu'},C_{\nu',\nu}\otimes_{C_\nu}\mathbb{V}_\nu Q_\nu)=\mathrm{Hom}_{C_{\nu'}}(C_{\nu'},C_{\nu',\nu}\otimes_{C_\nu}C_\nu)=C_{\nu',\nu}$$

mapping $g\mapsto\tau'_{Q_\nu}\circ\mathbb{V}_{\nu'}g$ and on the left with the isomorphism

$$\mathrm{Hom}_{\mathcal{O}_\nu}(\mathbb{E}Q_{\nu'},Q_\nu)\xrightarrow{\mathbb{V}_\nu}\mathrm{Hom}_{C_\nu}(\mathbb{V}_\nu(\mathbb{E}Q_{\nu'}),\mathbb{V}_\nu Q_\nu)\xrightarrow{\tau}$$
$$\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}\otimes_{C_{\nu'}}\mathbb{V}_{\nu'}Q_{\nu'},\mathbb{V}_\nu Q_\nu)=\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'},C_\nu)$$

mapping $h\mapsto\mathbb{V}_\nu h\circ\tau^{-1}_{Q_{\nu'}}$, we get a $(C_{\nu'},C_\nu)$-bimodule isomorphism $\delta:C_{\nu',\nu}\to\mathrm{Hom}_{C_\nu}(C_{\nu,\nu'},C_\nu)$. As explained just after (6.7), this defines an element of $\mathrm{Adj}(C_{\nu,\nu'}\otimes_{C_{\nu'}}?,C_{\nu',\nu}\otimes_{C_\nu}?)$.

Now we need to verify that the two diagrams in the statement of the lemma commute; we just sketch the argument for the second one. We claim for $g\in\mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'},\mathbb{F}Q_\nu)$ and ε defined via the second diagram that

$$\mathbb{V}_\nu(\hat{\varepsilon}_{Q_\nu}\circ\mathbb{E}g)=\varepsilon_{\mathbb{V}_\nu Q_\nu}\circ\Big(\mathrm{id}_{C_{\nu,\nu'}}\otimes(\tau'_{Q_\nu}\circ\mathbb{V}_{\nu'}g)\Big)\circ\tau_{Q_{\nu'}},$$

equality in $\mathrm{Hom}_{C_\nu}(\mathbb{V}_\nu\mathbb{E}Q_{\nu'},\mathbb{V}_\nu Q_\nu)$. Well, by the naturality of τ, we have that $\tau_{\mathbb{F}Q_\nu}\circ\mathbb{V}_\nu\mathbb{E}g=(\mathrm{id}_{C_{\nu,\nu'}}\otimes\mathbb{V}_{\nu'}g)\circ\tau_{Q_{\nu'}}$. Hence, using the commuting diagram defining ε applied to the module Q_ν, we get that

$$\begin{aligned}\mathbb{V}_\nu(\hat{\varepsilon}_{Q_\nu}\circ\mathbb{E}g)&=\mathbb{V}_\nu\hat{\varepsilon}_{Q_\nu}\circ\mathbb{V}_\nu\mathbb{E}g=\varepsilon_{\mathbb{V}_\nu Q_\nu}\circ\mathrm{id}_{C_{\nu,\nu'}}\otimes\tau'_{Q_\nu}\circ\tau_{\mathbb{F}Q_\nu}\circ\mathbb{V}_\nu\mathbb{E}g\\&=\varepsilon_{\mathbb{V}_\nu Q_\nu}\circ\mathrm{id}_{C_{\nu,\nu'}}\otimes\tau'_{Q_\nu}\circ\mathrm{id}_{C_{\nu,\nu'}}\otimes\mathbb{V}_{\nu'}g\circ\tau_{Q_{\nu'}}\\&=\varepsilon_{\mathbb{V}_\nu Q_\nu}\circ\Big(\mathrm{id}_{C_{\nu,\nu'}}\otimes(\tau'_{Q_\nu}\circ\mathbb{V}_{\nu'}g)\Big)\circ\tau_{Q_{\nu'}}.\end{aligned}$$

This proves the claim. Now take any $f\in\mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'}Q_{\nu'},C_{\nu',\nu}\otimes_{C_\nu}\mathbb{V}_\nu Q_\nu)=C_{\nu',\nu}$. We can write $f=\tau'_{Q_\nu}\circ\mathbb{V}_{\nu'}g$ for a unique $g\in\mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'},\mathbb{F}Q_\nu)$. The map δ defined in the previous paragraph maps f to $\mathbb{V}_\nu(\hat{\varepsilon}_{Q_\nu} \circ \mathbb{E}g) \circ \tau^{-1}_{Q_{\nu'}}$. By the claim, this is the same as $\varepsilon_{\mathbb{V}_\nu Q_\nu} \circ \Big(\mathrm{id}_{C_{\nu,\nu'}} \otimes (\tau'_{Q_\nu} \circ \mathbb{V}_{\nu'} g)\Big)$. This is the image of the adjunction with counit ε under the map δ from (6.7), which is what we were trying to check. $\square$

Conversely, every adjunction between $C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ and $C_{\nu',\nu} \otimes_{C_\nu} ?$ lifts in a canonical way to an adjunction between $\mathbb{E}$ and $\mathbb{F}$, thanks to the next lemma.

Lemma 6.4. *There exists a map*

$$R : \mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?) \to \mathrm{Adj}(\mathbb{E}, \mathbb{F})$$

such that $T \circ R = \mathrm{id}$*, where* T *is the map from the preceeding lemma.*

Proof. Recall that there are idempotents $e_\nu \in A_\nu$ and $e_{\nu'} \in A_{\nu'}$ such that $Q_\nu = P_\nu e_\nu, Q_{\nu'} = P_{\nu'} e_{\nu'}$ and $C_\nu = e_\nu A_\nu e_\nu, C_{\nu'} = e_{\nu'} A_{\nu'} e_{\nu'}$. We have that

$$e_{\nu'} \mathrm{Hom}_{\mathcal{O}_{\nu'}}(P_{\nu'}, \mathbb{F}P_\nu) e_\nu = \mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'}, \mathbb{F}Q_\nu),$$
$$e_{\nu'} \mathrm{Hom}_{\mathcal{O}_{\nu'}}(\mathbb{E}P_{\nu'}, P_\nu) e_\nu = \mathrm{Hom}_{\mathcal{O}_{\nu'}}(\mathbb{E}Q_{\nu'}, Q_\nu),$$
$$e_{\nu'} \mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'} P_{\nu'}, C_{\nu',\nu} \otimes_{C_\nu} \mathbb{V}_\nu P_\nu) e_\nu = \mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'} Q_{\nu'}, C_{\nu',\nu} \otimes_{C_\nu} \mathbb{V}_\nu Q_\nu),$$
$$e_{\nu'} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'} \otimes_{C_{\nu'}} \mathbb{V}_{\nu'} P_{\nu'}, \mathbb{V}_\nu P_\nu) e_\nu = \mathrm{Hom}_{C_{\nu'}}(C_{\nu,\nu'} \otimes_{C_{\nu'}} \mathbb{V}_{\nu'} Q_{\nu'}, \mathbb{V}_\nu Q_\nu).$$

Consider the following diagram:

$$\begin{array}{ccc}
\left\{\begin{array}{l} (A_{\nu'}, A_\nu)\text{-bimodule isomorphisms} \\ \mathrm{Hom}_{\mathcal{O}_{\nu'}}(P_{\nu'}, \mathbb{F}P_\nu) \to \mathrm{Hom}_{\mathcal{O}_\nu}(\mathbb{E}P_{\nu'}, P_\nu) \end{array}\right\} & \searrow & \\
\downarrow & & \left\{\begin{array}{l} (A_{\nu'}, A_\nu)\text{-bimodule isomorphisms} \\ \mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'} P_{\nu'}, C_{\nu',\nu} \otimes_{C_\nu} \mathbb{V}_\nu P_\nu) \\ \quad \to \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'} \otimes_{C_{\nu'}} \mathbb{V}_{\nu'} P_{\nu'}, \mathbb{V}_\nu P_\nu) \end{array}\right\} \\
\left\{\begin{array}{l} (C_{\nu'}, C_\nu)\text{-bimodule isomorphisms} \\ \mathrm{Hom}_{\mathcal{O}_{\nu'}}(Q_{\nu'}, \mathbb{F}Q_\nu) \to \mathrm{Hom}_{\mathcal{O}_\nu}(\mathbb{E}Q_{\nu'}, Q_\nu) \end{array}\right\} & & \Big\downarrow T' \\
& \searrow & \left\{\begin{array}{l} (C_{\nu'}, C_\nu)\text{-bimodule isomorphisms} \\ \mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'} Q_{\nu'}, C_{\nu',\nu} \otimes_{C_\nu} \mathbb{V}_\nu Q_\nu) \\ \quad \to \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'} \otimes_{C_{\nu'}} \mathbb{V}_{\nu'} Q_{\nu'}, \mathbb{V}_\nu Q_\nu) \end{array}\right\}
\end{array}$$

where the vertical maps arise by multiplying on the left by $e_{\nu'}$ and on the right by e_ν, and the diagonal maps are the isomorphisms defined by composing on the left and right by isomorphisms arising from (6.3) and (6.5)–(6.6) exactly like in the proof of Lemma 6.3. Using the identifications of the spaces of adjunctions with spaces of bimodule isomorphisms, T is by definition the composite of the two left hand maps. The diagram commutes, so T is also the composite of the two right hand maps. So we just need to observe that the map T' possesses a right inverse R'. To see that, take a bimodule isomorphism δ in the codomain of T', that is, a bimodule isomorphism

$$\delta : C_{\nu',\nu} \xrightarrow{\sim} \mathrm{Hom}_{C_\nu}(C_{\nu,\nu'}, C_\nu).$$

It defines an adjunction between the functors $(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$, from which we get a bimodule isomorphism

$$\hat{\delta} : \mathrm{Hom}_{C_{\nu'}}(\mathbb{V}_{\nu'} P_{\nu'}, C_{\nu,\nu'} \otimes_{C_{\nu'}} \mathbb{V}_\nu P_\nu) \stackrel{\sim}{\longrightarrow} \mathrm{Hom}_{C_\nu}(C_{\nu',\nu} \otimes_{C_\nu} \mathbb{V}_{\nu'} P_{\nu'}, \mathbb{V}_\nu P_\nu).$$

Moreover, $T'(\hat{\delta}) = \delta$, so we get the desired map R' by setting $R'(\delta) := \hat{\delta}$. □

There is just one more essential ingredient: the special translation functors $\mathbb{E}_i, \mathbb{F}_i : \mathcal{O} \to \mathcal{O}$ which we define here following the approach of [CR, §7.4]; see also [BFK, §3.1] for a special case and [BK1, §4.4] for a detailed discussion of the combinatorics of these functors in general. Let V and V^* denote the natural $\mathfrak{g}$-module and its dual, respectively. For $\mathfrak{g}$-modules M and N, multiplication by $\Omega := \sum_{i,j=1}^n e_{i,j} \otimes e_{j,i} \in \mathfrak{g} \otimes \mathfrak{g}$ defines a $\mathfrak{g}$-module endomorphism of $M \otimes N$. For $i \in \mathbb{Z}$, let $\mathbb{F}_i$ be the functor defined on a module M first by tensoring with the natural module V, then taking the generalized i-eigenspace of the endomorphism Ω. Similarly, let $\mathbb{E}_i$ be the functor defined first by tensoring with V^*, then taking the generalized $-(n+i)$-eigenspace of the endomorphism Ω. This defines functors

$$\mathbb{E}_i, \mathbb{F}_i : \bigoplus_\nu \mathcal{O}_\nu \to \bigoplus_\nu \mathcal{O}_\nu \tag{6.9}$$

for each $i \in \mathbb{Z}$. It is well known that these functors are both left and right adjoint to each other, for example one gets adjunctions induced by the canonical adjunctions between the functors of tensoring with V and with V^*. In particular, both of the functors are exact and commute with direct sums. It is also known that $\mathbb{E}_i$ is zero on modules belonging to $\mathcal{O}_{\nu'}$ if $\nu'_{i+1} = 0$, and $\mathbb{F}_i$ is zero on modules belonging to $\mathcal{O}_\nu$ if $\nu_i = 0$. Moreover, given compositions ν, ν' related to each other as in the key situation (3.9), the functor $\mathbb{E}_i$ maps modules belonging to $\mathcal{O}_{\nu'}$ into $\mathcal{O}_\nu$ and $\mathbb{F}_i$ maps modules belonging to $\mathcal{O}_\nu$ into $\mathcal{O}_{\nu'}$. Hence, $\mathbb{E}_i$ and $\mathbb{F}_i$ restrict to well-defined functors

$$\mathcal{O}_\nu \underset{\mathbb{E}_i}{\overset{\mathbb{F}_i}{\rightleftarrows}} \mathcal{O}_{\nu'}. \tag{6.10}$$

From now on, $C_{\nu,\nu'}$ denotes the algebra $C^{S_\nu \cap S_{\nu'}}$ viewed as a $(C_\nu, C_{\nu'})$-bimodule and $C_{\nu',\nu}$ denotes $C^{S_\nu \cap S_{\nu'}}$ viewed as a $(C_{\nu'}, C_\nu)$-bimodule, like in the previous section. The following important lemma is proved in [FKS, Proposition 3.3].

Lemma 6.5. *There are isomorphisms of functors*

$$\tau : \mathbb{V}_\nu \circ \mathbb{E}_i \stackrel{\sim}{\longrightarrow} C_{\nu,\nu'} \otimes_{C_{\nu'}} ? \circ \mathbb{V}_{\nu'},$$
$$\tau' : \mathbb{V}_{\nu'} \circ \mathbb{F}_i \stackrel{\sim}{\longrightarrow} C_{\nu',\nu} \otimes_{C_\nu} ? \circ \mathbb{V}_\nu.$$

In the previous section, we constructed an explicit adjunction between $C_{\nu,\nu'} \otimes_{C_{\nu'}} ?$ and $C_{\nu',\nu} \otimes_{C_\nu} ?$, i.e. an element of $\mathrm{Adj}(C_{\nu,\nu'} \otimes_{C_{\nu'}} ?, C_{\nu',\nu} \otimes_{C_\nu} ?)$, with unit ι' and counit ε defined by (5.3)–(5.4). Applying Lemmas 6.3–6.4 with $\mathbb{E} = \mathbb{E}_i$ and $\mathbb{F} = \mathbb{F}_i$ we lift this adjunction to an element of $\mathrm{Adj}(\mathbb{E}_i, \mathbb{F}_i)$. Denoting the unit and counit of this lift by $\hat{\iota}'$ and $\hat{\varepsilon}$, respectively, the appropriate analogues of the diagrams in the statement of Lemma 6.3 commute.

Similarly, this time taking $\mathbb{E} = \mathbb{F}_i$ and $\mathbb{F} = \mathbb{E}_i$, we lift the adjunction in $\mathrm{Adj}(C_{\nu',\nu} \otimes_{C_\nu} ?, C_{\nu,\nu'} \otimes_{C_{\nu'}} ?)$ with unit ι and counit ε' defined by (5.6)–(5.7) to an adjunction in $\mathrm{Adj}(\mathbb{F}_i, \mathbb{E}_i)$, whose unit $\hat{\iota}$ and counit $\hat{\varepsilon}'$ are again defined by the appropriate analogues of the diagrams from Lemma 6.3. We remark that the adjunctions making $(\mathbb{E}_i, \mathbb{F}_i)$ and $(\mathbb{F}_i, \mathbb{E}_i)$ into adjoint pairs that we have just defined are definitely *not* in general the same as the adjunctions induced by the canonical adjunctions between tensoring with V and V^* mentioned earlier.

Now we can repeat the definitions (5.9)–(5.10) in the present setting to get induced trace maps

$$Z(\mathcal{O}_\nu) \underset{E_i}{\overset{F_i}{\rightleftarrows}} Z(\mathcal{O}_{\nu'}). \tag{6.11}$$

Thus, E_i and F_i are the maps defined on $z' \in Z(\mathcal{O}_{\nu'})$ and $z \in Z(\mathcal{O}_\nu)$ by

$$E_i(z') := \hat{\varepsilon} \circ \mathbf{1}_{\mathbb{E}_i}\, z'\, \mathbf{1}_{\mathbb{F}_i} \circ \hat{\iota} \in Z(\mathcal{O}_\nu), \tag{6.12}$$

$$F_i(z) := \hat{\varepsilon}' \circ \mathbf{1}_{\mathbb{F}_i}\, z\, \mathbf{1}_{\mathbb{E}_i} \circ \hat{\iota}' \in Z(\mathcal{O}_{\nu'}), \tag{6.13}$$

respectively. Also define $D_i : Z(\mathcal{O}_\nu) \to Z(\mathcal{O}_\nu)$ to be multiplication by the scalar ν_i. Taking the direct sum of these linear maps over all compositions of n, interpreting E_i as the zero map on $Z(\mathcal{O}_{\nu'})$ if $\nu'_{i+1} = 0$ and F_i as the zero map on $Z(\mathcal{O}_\nu)$ if $\nu_i = 0$, we obtain linear maps

$$D_i, E_i, F_i : \textstyle\bigoplus_\nu Z(\mathcal{O}_\nu) \to \bigoplus_\nu Z(\mathcal{O}_\nu) \tag{6.14}$$

for each $i \in \mathbb{Z}$. The following theorem shows that these maps define actions of the generators D_i, E_i and F_i of $\hat{\mathfrak{g}}$ making $\bigoplus_\nu Z(\mathcal{O}_\nu)$ into a well-defined $\hat{\mathfrak{g}}$-module.

Theorem 6.6. *Under the identification of $\bigoplus_\nu Z(\mathcal{O}_\nu)$ with $\bigoplus_\nu C_\nu$, the endomorphisms D_i, E_i and F_i just defined coincide with the maps arising from the actions of the generators D_i, E_i and F_i of $\hat{\mathfrak{g}}$ defined just after (3.8).*

Proof. It is obvious that the D_i's are equal. So in view of Theorem 5.3 we just need to check for fixed $i \in \mathbb{Z}$ and ν, ν' as above that the maps E_i and F_i from (5.8) coincide with the maps E_i and F_i from (6.11). We explain the argument just for E_i, since the other case is similar. As C_ν is commutative, we can identify C_ν with $Z(C_\nu\text{-mod})$ as before. Consider the following commutative diagram:

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{q_\nu} & Z(C_\nu\text{-mod}) \\ {\scriptstyle p_\nu}\big\downarrow & & \big\downarrow{\scriptstyle y_\nu} \\ Z(\mathcal{O}_\nu) & \xrightarrow[x_\nu]{} & \mathrm{End}(\mathbb{V}_\nu), \end{array}$$

where x_ν is the map sending a natural transformation $z \in \mathrm{End}(\mathrm{Id}_{\mathcal{O}_\nu})$ to the natural transformation $\mathbf{1}_{\mathbb{V}_\nu} z \in \mathrm{End}(\mathbb{V}_\nu)$ and y_ν is the map sending a natural transformation $z \in \mathrm{End}(\mathrm{Id}_{C_\nu\text{-mod}})$ to the natural transformation $z\mathbf{1}_{\mathbb{V}_\nu} \in \mathrm{End}(\mathbb{V}_\nu)$. We note that y_ν is injective. Indeed, if $z \in \ker y_\nu$, then in particular z acts as zero on $\mathrm{End}_{C_\nu}(\mathbb{V}_\nu Q_\nu) = \mathrm{End}_{C_\nu}(C_\nu)$, hence $z = 0$. Moreover, the diagram commutes. To see this, take $z \in Z(\mathfrak{g})$ and any $M \in \mathcal{O}_\nu$. We need to show $x_\nu(p_\nu(z))$ and $y_\nu(q_\nu(z))$ both define the same endomorphism of $\mathbb{V}_\nu M = \mathrm{Hom}_{\mathcal{O}_\nu}(Q_\nu, M)$. Well, $x_\nu(p_\nu(z))$ defines the endomomorphism $f \mapsto \hat f$ where $\hat f(q) = zf(q)$ and $y_\nu(q_\nu(z))$ defines the endomorphism $f \mapsto \tilde f$ where $\tilde f(q) = f(zq)$. Since f is a $\mathfrak{g}$-module homomorphism we do indeed have that $\hat f = \tilde f$.

The facts established in the previous paragraph imply that $x \in Z(\mathcal{O}_\nu)$ is equal to $y \in Z(C_\nu\text{-mod})$ under all our identifications if and only if $\mathbf{1}_{\mathbb{V}_\nu} x = y\mathbf{1}_{\mathbb{V}_\nu}$ in $\mathrm{End}(\mathbb{V}_\nu)$. Similarly, $x \in Z(\mathcal{O}_{\nu'})$ is equal to $y \in Z(C_{\nu'}\text{-mod})$ if and only if $\mathbf{1}_{\mathbb{V}_{\nu'}} x = y\mathbf{1}_{\mathbb{V}_{\nu'}}$ in $\mathrm{End}(\mathbb{V}_{\nu'})$. So take $x \in Z(\mathcal{O}_{\nu'})$ and $y \in Z(C_{\nu'}\text{-mod})$ such that $\mathbf{1}_{\mathbb{V}_{\nu'}} x = y\mathbf{1}_{\mathbb{V}_{\nu'}}$. To complete the proof of the theorem, we need to show that

$$\mathbf{1}_{\mathbb{V}_\nu} E_i(x) = E_i(y)\mathbf{1}_{\mathbb{V}_\nu}.$$

Recalling (5.9) and (6.12), this is the statement that

$$\mathbf{1}_{\mathbb{V}_\nu}\hat\varepsilon \circ \mathbf{1}_{\mathbb{V}_\nu}\mathbf{1}_{\mathbb{E}_i} x \mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota = \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?}) y (\mathbf{1}_{C_{\nu',\nu}\otimes_{C_\nu}?})\mathbf{1}_{\mathbb{V}_\nu} \circ \iota\mathbf{1}_{\mathbb{V}_\nu}.$$

Recalling Lemma 6.5, naturality implies that

$$\tau \circ \mathbf{1}_{\mathbb{V}_\nu}\mathbf{1}_{\mathbb{E}_i} x = (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\mathbf{1}_{\mathbb{V}_{\nu'}} x \circ \tau,$$
$$\tau' \circ y\mathbf{1}_{\mathbb{V}_{\nu'}}\mathbf{1}_{\mathbb{F}_i} = y(\mathbf{1}_{C_{\nu',\nu}\otimes_{C_\nu}?})\mathbf{1}_{\mathbb{V}_\nu} \circ \tau'.$$

Also by the commuting squares from Lemma 6.3 that are satisfied by the special adjunctions fixed above, we have that

$$\mathbf{1}_{\mathbb{V}_\nu}\hat\varepsilon = \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ \tau\mathbf{1}_{\mathbb{F}_i},$$
$$\iota\mathbf{1}_{\mathbb{V}_\nu} = (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ \tau\mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota.$$

Now we calculate:

$$\begin{aligned}
&\mathbf{1}_{\mathbb{V}_\nu}\hat\varepsilon \circ \mathbf{1}_{\mathbb{V}_\nu}\mathbf{1}_{\mathbb{E}_i} x \mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota\\
&\quad= \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ \tau\mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\mathbf{1}_{\mathbb{E}_i} x \mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota\\
&\quad= \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\mathbf{1}_{\mathbb{V}_{\nu'}} x \mathbf{1}_{\mathbb{F}_i} \circ \tau\mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota\\
&\quad= \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?}) y\mathbf{1}_{\mathbb{V}_{\nu'}}\mathbf{1}_{\mathbb{F}_i} \circ \tau\mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota\\
&\quad= \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?}) y (\mathbf{1}_{C_{\nu',\nu}\otimes_{C_\nu}?})\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?})\tau' \circ \tau\mathbf{1}_{\mathbb{F}_i} \circ \mathbf{1}_{\mathbb{V}_\nu}\hat\iota\\
&\quad= \varepsilon\mathbf{1}_{\mathbb{V}_\nu} \circ (\mathbf{1}_{C_{\nu,\nu'}\otimes_{C_{\nu'}}?}) y (\mathbf{1}_{C_{\nu',\nu}\otimes_{C_\nu}?})\mathbf{1}_{\mathbb{V}_\nu} \circ \iota\mathbf{1}_{\mathbb{V}_\nu}.
\end{aligned}$$

This is what we wanted. □

Now we have all the necessary machinery set up, we can quite quickly prove the Main Theorem. Let $\mathcal{O}^\mu_\nu$ be the integral block of parabolic category $\mathcal{O}$ parametrized by the fixed compositions μ and ν, as in the introduction[5]. Because $\mathcal{O}^\mu_\nu$ is a full subcategory of $\mathcal{O}_\nu$, restriction defines an algebra

[5]Although not needed here, we remark that Soergel's combinatorial functor $\mathbb{V}$ has been considered recently in the parabolic setting in [S1, §10] and (from a quite different point of view) in [BK2, §5].

homomorphism

$$r_\nu^\mu : Z(\mathcal{O}_\nu) \to Z(\mathcal{O}_\nu^\mu). \tag{6.15}$$

Since the surjection m_ν^μ from Lemma 6.1(i) factors as $m_\nu^\mu = r_\nu^\mu \circ p_\nu$, this map r_ν^μ is surjective, i.e. $Z(\mathcal{O}_\nu^\mu)$ is a quotient of $Z(\mathcal{O}_\nu)$. The functors $\mathbb{E}_i$ and $\mathbb{F}_i$ from (6.10) restrict to well-defined functors

$$\mathcal{O}_\nu^\mu \underset{\mathbb{E}_i}{\overset{\mathbb{F}_i}{\rightleftarrows}} \mathcal{O}_{\nu'}^\mu. \tag{6.16}$$

Working with the same adjunctions as before (viewed now as adjunctions between the restricted functors), we define endomorphisms

$$D_i, E_i, F_i : \bigoplus_\nu Z(\mathcal{O}_\nu^\mu) \to \bigoplus_\nu Z(\mathcal{O}_\nu^\mu) \tag{6.17}$$

for each $i \in \mathbb{Z}$ in exactly the same way as (6.14). It is then immediate that the map $\oplus_\nu r_\nu^\mu : \bigoplus_\nu Z(\mathcal{O}_\nu) \twoheadrightarrow \bigoplus_\nu Z(\mathcal{O}_\nu^\mu)$ intertwines the endomorphisms from (6.14) and (6.17). So the latter maps define actions of the generators D_i, E_i and F_i of $\hat{\mathfrak{g}}$ making $\bigoplus_\nu Z(\mathcal{O}_\nu^\mu)$ into a $\hat{\mathfrak{g}}$-module. Let $s_\nu^\mu : C_\nu \twoheadrightarrow C_\nu^\mu$ denote the canonical quotient map for each ν and recall that $\bigoplus_\nu C_\nu^\mu$ is a $\hat{\mathfrak{g}}$-module described by Theorem 4.3.

Theorem 6.7. *For each composition ν of n, there exists a unique algebra isomorphism c_ν^μ making the following diagram commute:*

$$\begin{array}{ccc} & Z(\mathcal{O}_\nu) = C_\nu & \\ {\scriptstyle r_\nu^\mu}\swarrow & & \searrow{\scriptstyle s_\nu^\mu} \\ Z(\mathcal{O}_\nu^\mu) & \xrightarrow[c_\nu^\mu]{} & C_\nu^\mu. \end{array} \tag{6.18}$$

Moreover, the map $\oplus_\nu c_\nu^\mu : \bigoplus_\nu Z(\mathcal{O}_\nu^\mu) \xrightarrow{\sim} \bigoplus_\nu C_\nu^\mu$ is a $\hat{\mathfrak{g}}$-module isomorphism.

Proof. Both of the maps $\oplus_\nu r_\nu^\mu$ and $\oplus_\nu s_\nu^\mu$ are surjective $\hat{\mathfrak{g}}$-module homomorphisms. Moreover, we know that $\dim Z(\mathcal{O}_\nu^\mu) = \dim C_\nu^\mu$ by Corollary 4.4 and Lemma 6.1(ii). So it suffices to check that $\ker r_\nu^\mu \subseteq \ker s_\nu^\mu$ for each ν.

We first treat the special case that $\nu^+ = \lambda$, when there is just one column strict λ-tableau of type ν. Let $a_1 \leq \cdots \leq a_n$ be the integers such that ν_i of them are equal to i for each $i \in \mathbb{Z}$. Since there is just one isomorphism class of simple modules in the highest weight category $\mathcal{O}_\nu^\mu$, it is a semisimple category. So $z_r \in Z(\mathfrak{g})$ acts by the scalar $e_r(a_1, \dots, a_n)$ on every module in $\mathcal{O}_\nu^\mu$. It follows easily that $\ker r_\nu^\mu$ is generated by the elements $p_\nu(z_r) - e_r(a_1, \dots, a_n)$ for all $r \geq 1$. We therefore need to show that $q_\nu(z_r) - e_r(a_1, \dots, a_n) = e_r(x_1 + a_1, \dots, x_n + a_n) - e_r(a_1, \dots, a_n)$ belongs to $\ker s_\nu^\mu$ for each $r \geq 1$. This is clear since C_ν^μ is a one dimensional graded algebra and each of these elements involves only strictly positive degree terms.

Now take an arbitrary ν. Also let γ be any composition with $\gamma^+ = \lambda$ and let ω be any regular composition. Let $x \in C_\nu$ be an element that is not contained in the kernel of s_ν^μ. We need to show that $r_\nu^\mu(x) \neq 0$. By Theorem 4.3(iv), we can find $y \in C_\omega$ and $u, v \in U(\hat{\mathfrak{g}})$ such that $vx \in C_\omega$,

$u(y(vx)) \in C_\gamma$ and $s_\gamma^\mu(u(y(vx))) \neq 0$. Since $\gamma^+ = \lambda$, the previous paragraph implies that $r_\gamma^\mu(u(y(vx)) \neq 0$. But

$$r_\gamma^\mu(u(y(vx)) = ur_\omega^\mu(y(vx)) = u(r_\omega^\mu(y)r_\omega^\mu(vx)) = u(r_\omega^\mu(y)(vr_\nu^\mu(x))),$$

so we deduce that $r_\nu^\mu(x) \neq 0$ too. □

This completes the proof of the Main Theorem from the introduction.

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