Modular branching rules and the Mullineux map for Hecke algebras of type A

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1 Introduction

In a series of papers [21, 22, 23], Kleshchev has recently proved a branching rule for modular representations of the symmetric group $\mathfrak{S}(r)$ on r symbols. Fix a field $\mathbb{F}$ of arbitrary characteristic p. Then, Kleshchev's branching rule gives a precise description of the socle of the restriction of an irreducible $\mathbb{F}\mathfrak{S}(r)$-module to $\mathbb{F}\mathfrak{S}(r-1)$.

In [24], Kleshchev applies this branching rule to give a purely combinatorial description of the ***Mullineux map***, which describes the irreducible $\mathbb{F}\mathfrak{S}(r)$-module obtained by tensoring an arbitrary irreducible $\mathbb{F}\mathfrak{S}(r)$-module with the 1-dimensional sign representation. In [13] (see also [3]), Ford and Kleshchev show, with some technical combinatorics, that Kleshchev's description of the Mullineux map is equivalent to a quite different algorithm conjectured by Mullineux [27] in 1979, thus proving the so-called Mullineux conjecture.

The main purpose of this paper is to prove the quantum analogues of these results, replacing the group algebra $\mathbb{F}\mathfrak{S}(r)$ with the corresponding Hecke algebra $\mathcal{H}(r)$ over $\mathbb{F}$ at an arbitrary root of unity $\bar{q} \in \mathbb{F}$. Our main result is the Hecke algebra analogue of Kleshchev's modular branching rule. It turns out that the branching rule for $\mathcal{H}(r)$ only depends on the integer ℓ, where ℓ is the smallest natural number such that $1 + \bar{q} + \cdots + \bar{q}^{\ell-1} = 0$ in $\mathbb{F}$. If $\bar{q} = 1$, ℓ is precisely the characteristic p of $\mathbb{F}$, and our branching rule specializes to the classical case considered by Kleshchev.

As a consequence of this branching rule, combining Kleshchev's argument in [24] with the known block structure of $\mathcal{H}(r)$ from [9], we are able to deduce the quantum analogue of Kleshchev's description of the corresponding Mullineux map for $\mathcal{H}(r)$. The proof by Ford and Kleshchev in [13] that this is equivalent to Mullineux's original algorithm does not depend on p being prime, so our result is equivalent by [13] to a quantum version of the Mullineux conjecture, replacing p by ℓ.

We remark that there is now a quite different proof that Kleshchev's algorithm gives the Mullineux map using crystal bases. This was first observed for fields of characteristic 0 in [25], modulo a conjecture (which has now been proved [17, 2]) relating crystal bases for a certain affine quantum group to decomposition matrices of $\mathcal{H}(r)$ over fields of characteristic 0. The result in arbitrary characteristic can be deduced from the characteristic 0 case by an argument due to Richards [28, 2.13]. However, this alternative approach to the Mullineux map does not give a proof of the branching rule that is the main subject of this paper.

In Kleshchev's original proof of the modular branching rule, he first proves a branching rule for representations of the hyperalgebra of the algebraic group $SL(n)$ over $\mathbb{F}$, then applies a Schur functor to deduce the result for symmetric groups. The strategy here is very similar: we shall work mainly with the "quantum hyperalgebra" $U(n)$ corresponding to $GL(n)$, then apply a Schur functor to deduce results about $\mathcal{H}(r)$.

The main difficulty in generalizing Kleshchev's arguments to the quantum hyperalgebra is to find an appropriate analogue of the lowering operators $T_{r,s}(M)$ introduced by Kleshchev in [23]. We show how to do this, defining operators $S_{i,j}(A)$ in $U(n)$ for all $1 \leq i \leq j \leq n$ and all subsets A of the interval $\{i+1, \ldots, j-1\}$. The lowering operators introduced here have other applications to the representation theory of quantum $GL(n)$. In the special case $A = \{i+1, \ldots, j-1\}$, the operator $S_{i,j}(A)$ defined in this paper is the quantum analogue of the operator $S_{i,j}$ defined by Carter in [6]. Further properties of these operators can be found in [4, Chapters 7-8] and [5].

We now describe the layout of the paper. In section 2, we state our main results, Theorem 2.5 and Theorem 2.6, for the Hecke algebra $\mathcal{H}(r)$. We show how to apply these results to construct the Mullineux map, following the original argument of Kleshchev in [24]. In section 3, we define the quantum hyperalgebra $U(n)$ corresponding to $GL(n)$, over $\mathbb{F}$ and at an arbitrary root unity, by base change starting from an integral form for the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$. We use R. Green's quantum analogue from [16] of the Carter-Lusztig semistandard basis theorem for standard (Weyl) modules to prove a quantum analogue of the classical branching rule. In section 4, we define the lowering operators $S_{i,j}(A)$ and prove some basic properties. These are the required quantum analogues of Kleshchev's operators $T_{r,s}(M)$. In section 5, we use these operators to prove modular branching rules for $U(n)$, in Theorem 5.3 and Theorem 5.4. Finally, we show how to deduce Theorem 2.5 and Theorem 2.6 from these two results by an identical Schur functor argument to the original classical case in [22].

2 The Main Results for the Hecke algebra

2.1 Let r be a natural number. Throughout the paper, we will be working with a fixed partition $\lambda \vdash r$. In this section, n denotes a fixed integer such that $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n = 0$ and $\lambda_1 + \cdots + \lambda_n = r$. We denote the ***transpose*** of the partition λ by $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$, where $m = \lambda_1$; by definition, λ'_i is equal to the number of λ_j $(1 \leq j < n)$ with $\lambda_j \geq i$.

Let $\bar{\mathcal{A}}$ be the ring of Laurent polynomials $\mathbb{Z}[q, q^{-1}]$ in an indeterminate q. We write $\mathcal{H}(r)_{\bar{\mathcal{A}}}$ for the generic Hecke algebra corresponding to the symmetric group $\mathfrak{S}(r)$. By definition, $\mathcal{H}(r)_{\bar{\mathcal{A}}}$ is the free $\bar{\mathcal{A}}$-algebra with basis $\{T_w \mid w \in \mathfrak{S}(r)\}$ and multiplication defined by

$$T_w T_s := \begin{cases} T_{ws} & \text{if } l(w) = l(ws) + 1 \\ qT_{ws} + (q-1)T_w & \text{otherwise,} \end{cases}$$

for all $w, s \in \mathfrak{S}(r)$ with $l(s) = 1$ (here, $l(u)$ denotes the usual ***length*** of $u \in \mathfrak{S}(r)$). The algebra $\mathcal{H}(r)_{\bar{\mathcal{A}}}$ is generated by the elements T_s for all basic transpositions $s \in \mathfrak{S}(r)$.

Let $\mathbb{F}$ be an arbitrary field and fix $\bar{q} \in \mathbb{F}^{\times}$. Define $\mathcal{H}(r)$ to be the Hecke algebra over $\mathbb{F}$ corresponding to $\mathcal{H}(r)_{\bar{\mathcal{A}}}$; regarding $\mathbb{F}$ as an $\bar{\mathcal{A}}$-module by letting $q \in \bar{\mathcal{A}}$ act on $\mathbb{F}$ by multiplication by $\bar{q} \in \mathbb{F}$, $\mathcal{H}(r)$ is the $\mathbb{F}$-algebra $\mathcal{H}(r)_{\bar{\mathcal{A}}} \otimes_{\bar{\mathcal{A}}} \mathbb{F}$. Let $\mathcal{H}(r-1)$ be the naturally embedded subalgebra of $\mathcal{H}(r)$ corresponding to $\mathfrak{S}(r-1)$.

Given an arbitrary $\mu \vdash r$, let S^{μ} be the (right) q-Specht module for $\mathcal{H}(r)$ corresponding to μ, as defined in [8, Section 4]. It will be more convenient for us to parametrize Specht modules instead with the transpose partition μ', so we define S_{μ} to be $S^{\mu'}$. Define the integer ℓ as follows:

(i) if $\bar{q} = 1$, define ℓ to be the characteristic of the field $\mathbb{F}$;

(ii) if $\bar{q} \neq 1$ is a root of unity in $\mathbb{F}$, let ℓ be the smallest positive integer such that $\bar{q}^{\ell} = 1$;

(iii) otherwise, let $\ell = 0$.

If $\ell = 0$, it is known that the algebra $\mathcal{H}(r)$ is semisimple. By definition, λ is ***ℓ-restricted*** if $\ell = 0$ or $\lambda_i - \lambda_{i+1} < \ell$ for all $1 \leq i < n$. If λ is ℓ-restricted, then S_{λ} has simple head which we denote by D_{λ}. Note D_{λ} is the module $D^{\lambda'}$ of [8]. By [8, 6.3, 6.8], the set of all D_{λ} for all ℓ-restricted partitions $\lambda \vdash r$ is a complete set of non-isomorphic irreducible $\mathcal{H}(r)$-modules, for arbitrary ℓ.

2.2 We introduce some non-standard notation that we shall use repeatedly. For integers $1 \leq h \leq k \leq n$, we shall write $(h..k), [h..k), (h..k]$ and $[h..k]$ for the corresponding open/closed intervals of $\mathbb{N}$, so that $(h..k)$ is the open interval $\{l \in \mathbb{N} \mid h < l < k\}$, $(h..k]$ is the interval $\{l \in \mathbb{N} \mid h < l \leq k\}$ and so on. If $A \subseteq [1..n]$, $A_{h..k}$ denotes the intersection of A with the open interval $(h..k)$, so $A_{h..k} := A \cap (h..k)$.

We define two partial orders on subsets of $[1..n]$, which we call the ***lattice orders***, denoted by $\downarrow$ and $\uparrow$ respectively. Let $A, B \subseteq [1..n]$. Then, $A \downarrow B$ if there exists an injection $\theta : A \hookrightarrow B$ such that $\theta(a) \leq a$ for all $a \in A$. Similarly, $A \uparrow B$ if there exists an injection $\theta : A \hookrightarrow B$ such that $\theta(a) \geq a$ for all $a \in A$.

There are two equivalent ways of stating these definitions. First, $A \downarrow B$ if and only if $|A \cap [1..k]| \leq |B \cap [1..k]|$ for all $k \in [1..n]$, and $A \uparrow B$ if and only if $|A \cap [k..n]| \leq |B \cap [k..n]|$ for all $k \in [1..n]$. Second, let $(s_1, \ldots, s_n)$ be the sequence where

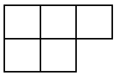
$$s_h = \begin{cases} 1 & \text{if } h \in B \setminus A \\ -1 & \text{if } h \in A \setminus B \\ 0 & \text{otherwise} \end{cases}$$

for all $h \in [1..n]$. Then, $A \downarrow B$ is equivalent to $\sum_{h=1}^{k} s_h \geq 0$ for all $k \in [1..n]$, and $A \uparrow B$ is equivalent to $\sum_{h=k}^{n} s_h \geq 0$ for all $k \in [1..n]$.

2.3 We write $[\lambda]$ for the Young diagram of λ; by definition,

$$[\lambda] := \{(i, j) \in \mathbb{N} \times \mathbb{N} \mid \lambda_i > 0, 1 \leq j \leq \lambda_i\}.$$

We represent this set of coordinates by an array of boxes, with $(i,j) \in [\lambda]$ corresponding to the box in the ith row and jth column. For instance if $\lambda = (3,2)$, the corresponding diagram is:



Given $(i,j) \in [\lambda]$, define the corresponding ***ℓ-residue*** $\mathrm{res}_\ell(i,j)$ to be $(i-j)$ regarded as an element of the ring $\mathbb{Z}/\ell\mathbb{Z}$. In the above example, the 3-residues are:

0	2	1
1	0	

Say a node $(i,j) \in [\lambda]$ is a ***removable node*** if $[\lambda] \setminus \{(i,j)\}$ is the diagram of a partition. Say i is a ***removable row*** if the node (i, λ_i) is a removable node. Let $R(\lambda) := \{1 \leq i < n \mid \lambda_i \neq \lambda_{i+1}\}$ denote the set of all removable rows. If $i \in R(\lambda)$, let $\lambda(i) \vdash (r-1)$ be the partition with Young diagram obtained from the diagram of λ by removing the node (i, λ_i).

For $1 \leq i \leq j \leq n$, let

$$\begin{aligned} B_{i,j}(\lambda) &:= \{k \in [i..j) \mid \mathrm{res}_\ell(i,\lambda_i) = \mathrm{res}_\ell(k+1, \lambda_{k+1}+1)\}, \\ C_{i,j}(\lambda) &:= \{k \in (i..j) \mid \mathrm{res}_\ell(i,\lambda_i) = \mathrm{res}_\ell(k, \lambda_k)\}. \end{aligned}$$

Let $R_{\mathrm{normal}}(\lambda)$ denote the set of all $i \in R(\lambda)$ such that $B_{i,n}(\lambda) \downarrow C_{i,n}(\lambda)$. Let $R_{\mathrm{good}}(\lambda)$ denote the set of all $i \in R_{\mathrm{normal}}(\lambda)$ such that there is no $j \in R_{\mathrm{normal}}(\lambda)$ with $j < i$ and $\mathrm{res}_\ell(i,\lambda_i) = \mathrm{res}_\ell(j,\lambda_j)$.

2.4 Remarks (I) Note that if $j \in (i..n)$ is not a removable row then $j \in B_{i,n}(\lambda)$ if and only if $j \in C_{i,n}(\lambda)$ (because $\lambda_j = \lambda_{j+1}$). By the definition of $\downarrow$, for any $B, C \subseteq [1..n]$, $B \downarrow C$ if and only if $B \setminus C \downarrow C \setminus B$. Hence, $B_{i,n}(\lambda) \downarrow C_{i,n}(\lambda)$ if and only if $B_{i,n}(\lambda) \cap R(\lambda) \downarrow C_{i,n}(\lambda) \cap R(\lambda)$. This observation is useful when computing $R_{\mathrm{normal}}(\lambda)$ and $R_{\mathrm{good}}(\lambda)$ in practise.

(II) In the introduction of [23], Kleshchev defines normal and good in a slightly different way to here. Using the observation in (I), it is not hard to show in the case that $\ell = p$ is prime, $i \in R_{\mathrm{normal}}(\lambda)$ in our notation if and only if (λ_i, i) is a normal node for the transpose partition λ' in Kleshchev's notation, and similarly for good nodes.

We give an example illustrating these definitions in Example 2.8. We can now state the main results of the paper, proved in section 5. The first result is a generalization of [23, Theorem 0.4] to arbitrary $\bar{q}$.

2.5 Theorem *Let $\lambda \vdash r$, $\mu \vdash (r-1)$ be ℓ-restricted partitions. Then,*

$$\mathrm{Hom}_{\mathcal{H}(r-1)}(S_\mu, D_\lambda \downarrow_{\mathcal{H}(r-1)}) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda(i) \text{ for some } i \in R_{\mathrm{normal}}(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

The second main result is a generalization of [23, Theorem 0.5]. Recall that the socle of an $\mathcal{H}(r)$-module is the largest semisimple submodule.

2.6 Theorem *Let $\lambda \vdash r$, $\mu \vdash (r-1)$ be ℓ-restricted partitions. Then,*

$$\mathrm{Hom}_{\mathcal{H}(r-1)}(D_\mu, D_\lambda \downarrow_{\mathcal{H}(r-1)}) = \begin{cases} \mathbb{F} & \textit{if } \mu = \lambda(i) \textit{ for some } i \in R_{\mathrm{good}}(\lambda) \\ 0 & \textit{otherwise.} \end{cases}$$

Hence, the socle of the restriction of D_λ to $\mathcal{H}(r-1)$ is $\displaystyle\bigoplus_{i \in R_{\mathrm{good}}(\lambda)} D_{\lambda(i)}$.

2.7 Remark It is also true (but not proved here) that if $\lambda \vdash r$, $\mu \vdash (r-1)$ are ℓ-restricted partitions, then

$$\mathrm{Hom}_{\mathcal{H}(r-1)}(S_\mu, S^*_\lambda \downarrow_{\mathcal{H}(r-1)}) = \begin{cases} \mathbb{F} & \text{if } \mu = \lambda(i) \text{ for some } i \in R(\lambda) \\ 0 & \text{otherwise.} \end{cases}$$

Note that by the definition, there is at most one $i \in R_{\mathrm{good}}(\lambda)$ with $\mathrm{res}_\ell(i, \lambda_i) = \rho$ for each $\rho \in \mathbb{Z}/\ell\mathbb{Z}$. Hence, if $\ell \neq 0$, $|R_{\mathrm{good}}(\lambda)| \leq \ell$. So as an immediate and rather surprising consequence of Theorem 2.6, first observed by Kleshchev in [24] in the classical case, if λ is ℓ-restricted ($\ell \neq 0$) then the restriction of D_λ to $\mathcal{H}(r-1)$ splits as a direct sum of at most ℓ indecomposable summands. In fact, the restriction splits as a direct sum of precisely $|R_{\mathrm{good}}(\lambda)|$ indecomposable summands, since each $D_{\lambda(i)}$ for $i \in R_{\mathrm{good}}(\lambda)$ lies in a different block for $\mathcal{H}(r-1)$.

We now illustrate the definitions with an example.

2.8 Example Consider $\lambda = (6, 4, 4, 3, 2)$ and $\ell = 3$ or 4; λ is ℓ-restricted in either case. The ℓ-residues and the subsets $B_{i,n}(\lambda) \cap R(\lambda)$ and $C_{i,n}(\lambda) \cap R(\lambda)$ are listed in the table below, for each $i \in R(\lambda)$.

$\ell = 3$:

0	2	1	0	2	1
1	0	2	1		
2	1	0	2		
0	2	1			
1	0				

i	$\mathrm{res}_\ell(i, \lambda_i)$	$B_{i,n}(\lambda) \cap R(\lambda)$	$C_{i,n}(\lambda) \cap R(\lambda)$
1	1	$\varnothing$	$\{4\}$
3	2	$\{4, 5\}$	$\varnothing$
4	1	$\varnothing$	$\varnothing$
5	0	$\varnothing$	$\varnothing$

Here, using the observation in Remark 2.4, $R_{\mathrm{normal}}(\lambda) = \{1, 4, 5\}$ and $R_{\mathrm{good}}(\lambda) = \{1, 5\}$.

$\ell = 4$:

0	3	2	1	0	3
1	0	3	2		
2	1	0	3		
3	2	1			
0	3				

i	$\mathrm{res}_\ell(i, \lambda_i)$	$B_{i,n}(\lambda) \cap R(\lambda)$	$C_{i,n}(\lambda) \cap R(\lambda)$
1	3	$\varnothing$	$\{3, 5\}$
3	3	$\varnothing$	$\{5\}$
4	1	$\{5\}$	$\varnothing$
5	3	$\varnothing$	$\varnothing$

Here, $R_{\mathrm{normal}}(\lambda) = \{1, 3, 5\}$ and $R_{\mathrm{good}}(\lambda) = \{1\}$.

Consequently, by Theorem 2.6, the restriction of D_λ to $\mathcal{H}(r-1)$ is decomposable if $\ell = 3$ but indecomposable if $\ell = 4$.

2.9 We now assume that we have proved Theorem 2.6, and show how to deduce Kleshchev's algorithm for computing the Mullineux map from this branching rule. Recall from [10] that there is an involution $\# : \mathcal{H}(r) \to \mathcal{H}(r)$ defined on generators by $T_s \mapsto -T_s + \bar{q} - 1$ for all basic transpositions $s \in \mathfrak{S}(r)$. Given any $\mathcal{H}(r)$-module V, define the module $V^{\#}$ to be V as a vector space, with action $v.h = vh^{\#}$ for all $v \in V, h \in \mathcal{H}(r)$. In the case $\bar{q} = 1$, $V^{\#}$ is precisely the module $V \otimes \mathrm{sgn}$, where sgn is the 1-dimensional sign representation of $\mathfrak{S}(r)$.

Let $\lambda \vdash r$ be ℓ-restricted, so that D_λ is a well-defined irreducible. Then, $(D_\lambda)^{\#}$ is also an irreducible $\mathcal{H}(r)$-module, so $(D_\lambda)^{\#} = D_{m(\lambda)}$ for some ℓ-restricted partition $m(\lambda)$. The map $\lambda \mapsto m(\lambda)$ is an involution (possibly the identity) on ℓ-restricted partitions of r. We refer to this involution as the ***Mullineux map***.

There are two algorithms for computing the Mullineux map. The first was conjectured by Mullineux (in the case ℓ prime) in [27]. The second algorithm is due (in the case ℓ prime) to Kleshchev [24, Algorithm 4.8]. In both cases the algorithm is purely combinatorial and does not depend on the primality of ℓ. We show here that Kleshchev's algorithm is correct for arbitrary ℓ. It is known by [13] that this algorithm is equivalent to Mullineux's original algorithm for arbitrary ℓ.

The argument is identical to Kleshchev's original argument in [23] in the case $\bar{q} = 1$. We need two lemmas. The first uses the parametrization of the blocks of $\mathcal{H}(r)$ from [9]. We define the ***residue content*** of a partition λ to be the set of ℓ-residues in the diagram $[\lambda]$, counted with multiplicities.

2.10 Lemma *For λ ℓ-restricted, the residue contents of $m(\lambda)$ and of λ' are equal.*

Proof. Recall that the dual V^* of a right $\mathcal{H}(r)$-module V is naturally a right $\mathcal{H}(r)$-module as in [8, p. 35]. By [10, p. 25] and [8, Lemma 4.7] respectively,

$$S_\lambda^* \cong (S_{\lambda'})^{\#},$$
$$D_\lambda^* \cong D_\lambda.$$

By definition, D_λ is the head of S_λ. Dualizing and applying the previous remarks, it follows that D_λ is the socle of $(S_{\lambda'})^{\#}$. So, $(D_\lambda)^{\#}$ lies in the socle of $(S_{\lambda'})^{\#\#}$, or, equivalently, $D_{m(\lambda)}$ lies in the socle of of $S_{\lambda'}$. Hence, λ and $m(\lambda)'$ lie in the same block. Now the result follows by [9], which shows that λ and $m(\lambda)'$ are in the same block if and only if they have the same residue content.

The second lemma is purely combinatorial, and is proved by Kleshchev in [24].

2.11 Lemma ([24, Lemma 1.4]) *Suppose that $\mu \vdash (r-1)$ is ℓ-restricted. Let $\rho \in \mathbb{Z}/\ell\mathbb{Z}$. Then, there is at most one ℓ-restricted partition $\lambda \vdash r$ such that $\mu = \lambda(i)$ for some $i \in R_{\mathrm{good}}(\lambda)$ with $\mathrm{res}_\ell(i, \lambda_i) = \rho$.*

Now we can give Kleshchev's combinatorial description of the Mullineux map. Fix an ℓ-restricted partition $\lambda \vdash r$. It is clear that if $r = 1$, then $m(\lambda) = \lambda$. So suppose that $r > 1$

and that the involution m has been constructed inductively for all smaller r. To define $m(\lambda)$, choose an arbitrary $i \in R_{\text{good}}(\lambda)$, and let $\rho = \text{res}_\ell(i, \lambda_i)$. By Theorem 2.6, $D_{\lambda(i)}$ is in the socle of $D_\lambda \downarrow_{\mathcal{H}(r-1)}$. Hence, $D_{m(\lambda(i))}$ is in the socle of $D_{m(\lambda)} \downarrow_{\mathcal{H}(r-1)}$. We know $m(\lambda(i))$ by induction. By Lemma 2.10, $m(\lambda)$ is some partition of r such that $m(\lambda(i))$ is obtained by removing a node of ℓ-residue $-\rho$ from the jth row of $m(\lambda)$, for some $j \in R(m(\lambda))$. Moreover by Theorem 2.6, $j \in R_{\text{good}}(m(\lambda))$. By Lemma 2.11, $m(\lambda)$ is uniquely determined by this property.

This proves the required algorithm for computing the Mullineux map. We refer the reader to [25] for a reinterpretation of this algorithm in terms of the "ℓ-good lattice".

The remainder of the paper is taken up with the proof of Theorem 2.5 and Theorem 2.6. We will always assume for the rest of the paper that $\bar{q} \neq 1$, since the case $\bar{q} = 1$ is precisely the classical case proved by Kleshchev in [23] (and can be deduced from our proof by a careful specialization argument). We will also assume that $\bar{q}$ has a square root $\bar{v}$ in $\mathbb{F}$, which we may do by adjoining a square root if necessary without loss of generality, since $\mathbb{F}$ is a splitting field for $\mathcal{H}(r)$.

3 The Quantum Hyperalgebra

3.1 In this section, we define the quantum hyperalgebra $U(n)$ corresponding to $GL(n)$, over $\mathbb{F}$ and at an arbitrary root of unity. The definition is by base change starting from the integral form constructed in [26, 12] for the quantized enveloping algebra $U_q(\mathfrak{gl}_n)$. The main result of the section is a short proof of (the quantum analogue of) the classical branching rule, showing that the restriction of any standard module for $U(n)$ to $U(n-1)$ has a filtration by $U(n-1)$-standard modules.

Let v be an indeterminate. Let $\mathcal{A}$ be the ring $\mathbb{Z}[v, v^{-1}]$ of Laurent polynomials in v. Let $\mathcal{F} := \mathbb{Q}(v)$ denote the field of fractions of $\mathcal{A}$. Define the ***quantized enveloping algebra*** $U(n)_{\mathcal{F}}$ to be the $\mathcal{F}$-algebra with generators

$$E_i, F_i, K_j, K_j^{-1} \qquad (1 \le i < n, 1 \le j \le n)$$

and relations

$$\begin{array}{ll} K_iK_j = K_jK_i, & K_iK_i^{-1} = K_i^{-1}K_i = 1, \\ K_iE_j = v^{\delta_{i,j}-\delta_{i,j+1}}E_jK_i, & K_iF_j = v^{\delta_{i,j+1}-\delta_{i,j}}F_jK_i, \end{array}$$

$$E_iF_j - F_jE_i = \delta_{i,j}\frac{K_{i,i+1} - K_{i,i+1}^{-1}}{v - v^{-1}},$$

$$E_iE_j = E_jE_i, \quad F_iF_j = F_jF_i \qquad \text{if } |i-j| > 1,$$

$$\begin{array}{ll} E_i^2E_j - (v+v^{-1})E_iE_jE_i + E_jE_i^2 = 0, & \\ F_i^2F_j - (v+v^{-1})F_iF_jF_i + F_jF_i^2 = 0 & \text{if } |i-j| = 1. \end{array}$$

Here, for any $1 \le i < j \le n$, $K_{i,j}$ denotes $K_iK_j^{-1}$. We regard $U(n)_{\mathcal{F}}$ as a Hopf algebra over $\mathcal{F}$ with comultiplication $\Delta : U(n)_{\mathcal{F}} \to U(n)_{\mathcal{F}} \otimes U(n)_{\mathcal{F}}$ defined on generators by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_{i,i+1}, \quad \Delta(F_i) = K_{i,i+1}^{-1} \otimes F_i + F_i \otimes 1, \quad \Delta(K_i) = K_i \otimes K_i.$$

The Hopf algebra structure on $U(n)_{\mathcal{F}}$ is not canonical. The choice Δ used here is as in [12, 16], but is different from the choice used in [26, 19]; in particular, this choice affects the construction of standard modules described below in (3.9).

For $t, u \in \mathbb{N}$, define the ***quantum factorial*** and the ***quantum binomial coefficient*** by

$$[t]! := \prod_{s=1}^{t} \frac{v^s - v^{-s}}{v - v^{-1}}, \qquad \left[\begin{array}{c} t \\ u \end{array} \right] := \prod_{s=1}^{u} \frac{v^{t-s+1} - v^{-t+s-1}}{v^s - v^{-s}}.$$

Let $U(n)_{\mathcal{A}}$ be the Hopf $\mathcal{A}$-subalgebra of $U(n)_{\mathcal{F}}$ generated by

$$E_i^{(s)}, F_i^{(s)}, K_j, K_j^{-1}, \left[\begin{array}{c} K_j \\ u \end{array} \right] \quad (s, u \in \mathbb{N}, c \in \mathbb{Z}, 1 \leq i < n, 1 \leq j \leq n),$$

where for $X \in U(n)_{\mathcal{F}}$, $X^{(s)}$ denotes the ***divided power*** $X^s/[s]!$ and

$$\left[\begin{array}{c} K_j; 0 \\ u \end{array} \right] = \left[\begin{array}{c} K_j \\ u \end{array} \right] := \prod_{s=1}^{u} \frac{K_j v^{-s+1} - K_j^{-1} v^{s-1}}{v^s - v^{-s}}.$$

The algebra $U(n)_{\mathcal{F}}$ admits an antiautomorphism τ defined on generators by

$$\tau(E_i) = F_i, \quad \tau(F_i) = E_i, \quad \tau(K_i) = K_i.$$

This antiautomorphism stabilizes $U(n)_{\mathcal{A}}$.

3.2 We now introduce elements $E_{i,j}, F_{i,j}$ for arbitrary $1 \leq i < j \leq n$ using the braid group action as in [26, 1.9]. There is no canonical way of doing this; the choice here – which is the same as [26, Example 4.4] – leads to quite pleasant notation later on, but other choices could also surely be used.

Let B_W be the braid group of type A_{n-1} on generators T_i, T_i^{-1} $(1 \leq i < n)$. Then, by [26], $U(n)_{\mathcal{F}}$ is a B_W-module with action defined by

$$T_i(E_j) = \left\{ \begin{array}{lll} -F_j K_{j,j+1} & \text{if } i = j & -K_{j,j+1}^{-1} F_j \\ E_j & \text{if } |i-j| > 1 & E_j \\ -E_i E_j + v^{-1} E_j E_i & \text{if } |i-j| = 1 & v^{-1} E_i E_j - E_j E_i \end{array} \right\} = T_i^{-1}(E_j),$$

$$T_i(F_j) = \left\{ \begin{array}{lll} -K_{j,j+1}^{-1} E_j & \text{if } i = j & -E_j K_{j,j+1} \\ F_j & \text{if } |i-j| > 1 & F_j \\ -F_j F_i + v F_i F_j & \text{if } |i-j| = 1 & v F_j F_i - F_i F_j \end{array} \right\} = T_i^{-1}(F_j),$$

$$T_i(K_j) = T_i^{-1}(K_j) = \left\{ \begin{array}{ll} K_{i+1} & \text{if } j = i \\ K_i & \text{if } j = i+1 \\ K_j & \text{otherwise.} \end{array} \right.$$

We remark that in Lusztig's later work, and in [19], a slightly different braid group action is used; see [19, 8.14 *Warning*] for an explanation of the relationship between the two.

Let $E_{i,i+1} := E_i, F_{i,i+1} := F_i$, and in general for $1 \leq i < j \leq n$ with $|i-j| > 1$, define

$$E_{i,j} := T_{j-1}(E_{i,j-1}) = T_{j-1}T_{j-2}\dots T_{i+1}(E_i),$$
$$F_{i,j} := T_{j-1}(F_{i,j-1}) = T_{j-1}T_{j-2}\dots T_{i+1}(F_i).$$

In fact, apart from in (3.4), we shall only ever use $F_{i,j}$. We record some simple properties of the braid group action.

3.3 Lemma (i) *If $|j-i| > 1$, $T_i(F_{i,j}) = F_{i+1,j}$.*

(ii) *If $i < l < j-1$, $l < i-1$ or $l > j$ then $T_l(F_{i,j}) = F_{i,j}$.*

Proof. (i) By definition, $T_i(F_{i,j}) = T_iT_{j-1}\dots T_{i+1}(F_i) = T_{j-1}\dots T_{i+2}T_iT_{i+1}(F_i)$. It therefore remains to show that $T_iT_{i+1}(F_i) = F_{i+1}$. This is immediate since by the definition of the braid group action, $T_{i+1}(F_i) = T_i^{-1}(F_{i+1})$.

(ii) The result is obvious unless $i < l < j-1$, when we need to apply the braid relations. We have that $T_l(F_{i,j}) = T_lT_{j-1}\dots T_l(F_{i,l}) = T_{j-1}\dots T_lT_{l+1}T_l(F_{i,l})$. Now apply the braid relation $T_lT_{l+1}T_l = T_{l+1}T_lT_{l+1}$. The conclusion follows easily.

3.4 Du [12, Section 2] and Lusztig [26, 4.5] have shown that $U(n)_{\mathcal{A}}$ is a free $\mathcal{A}$-subalgebra of $U(n)_{\mathcal{F}}$, and construct the following free $\mathcal{A}$-module basis for $U(n)_{\mathcal{A}}$:-

$$\underbrace{\prod_{1\leq i<j\leq n} F_{i,j}^{(N_{ij})}}_{U(n)^-_{\mathcal{A}}} \underbrace{\prod_{1\leq i\leq n}\left(K_i^{\delta_i}\begin{bmatrix} K_i \\ N_{ii}\end{bmatrix}\right)}_{U(n)^0_{\mathcal{A}}} \underbrace{\prod_{1\leq i<j\leq n} E_{i,j}^{(N_{ji})}}_{U(n)^+_{\mathcal{A}}}$$

as $N = (N_{ij})_{1\leq i,j\leq n}$ runs over all $n\times n$ matrices with entries in $\mathbb{Z}_{\geq 0}$ and $\delta = (\delta_i)_{1\leq i\leq n}$ runs over all vectors with entries in $\{0,1\}$. The order of multiplication in the first and last products in this expression needs to be fixed, but is otherwise arbitrary. We always choose the ordering for both products to be lexicographic (reading tuples from the right); that is, the order for the tuples (i,j) in the products is

$$(1,2);(1,3),(2,3);(1,4),\dots,(3,4);\dots;(1,n),\dots,(n-1,n).$$

Define $U(n)^-_{\mathcal{A}}, U(n)^0_{\mathcal{A}}, U(n)^+_{\mathcal{A}}$ to be the $\mathcal{A}$-subalgebras spanned by the subsets of this basis as indicated above, so that $U(n)_{\mathcal{A}} \cong U(n)^-_{\mathcal{A}} \otimes_{\mathcal{A}} U(n)^0_{\mathcal{A}} \otimes_{\mathcal{A}} U(n)^+_{\mathcal{A}}$. Finally, following the notation in [1], let $U(n)^{\flat}_{\mathcal{A}} := U(n)^-_{\mathcal{A}}U(n)^0_{\mathcal{A}}$ and $U(n)^{\sharp}_{\mathcal{A}} := U(n)^0_{\mathcal{A}}U(n)^+_{\mathcal{A}}$.

3.5 Lemma *The following relations hold in $U(n)_{\mathcal{F}}$.*

(i) *For $1 \leq i < j-1 < n$, $F_{i,j} = vF_{i+1,j}F_i - F_iF_{i+1,j}$.*

(ii) *For $1 \leq i \leq n, 1 \leq h < k \leq n$, $K_iF_{h,k} = v^{\delta_{i,k}-\delta_{i,h}}F_{h,k}K_i$.*

(iii) *For $1 \leq i < j < h < k \leq n$, $F_{i,j}$ and $F_{h,k}$ commute.*

(iv) *For* $1 \leq l < n, 1 \leq i < j \leq n$,

$$E_l F_{i,j} - F_{i,j} E_l = \begin{cases} \frac{K_{i,i+1} - K_{i,i+1}^{-1}}{v - v^{-1}} & \text{if } l = i, l+1 = j \\ -F_{i+1,j} K_{i,i+1}^{-1} & \text{if } l = i, l+1 \neq j \\ K_{j-1,j} F_{i,j-1} & \text{if } l \neq i, l+1 = j \\ 0 & \text{if } l \neq i, l+1 \neq j. \end{cases}$$

Proof. (i) By definition, $F_{i,i+2} = T_{i+1}(F_i) = vF_{i+1}F_i - F_iF_{i+1}$. The claim follows on applying $T_{j-1} \dots T_{i+2}$ to this equation.

(ii) This follows immediately from the defining relation $K_i F_h = v^{\delta_{i,h+1} - \delta_{i,h}} F_h K_i$ unless $h < i < k$. In that case, one first checks directly from the defining relations that the claim holds if $h + 1 = i = k - 1$, then applies the braid group action and Lemma 3.3 to obtain the general case.

(iii) follows immediately from the defining relations.

(iv) This is clear if $l < i, l \geq j$ or if $l = i, l + 1 = j$. To prove it in the remaining three cases $l = i, l + 1 < j; l > i, l + 1 = j; i < l < j - 1$ it is enough, by the braid group action and Lemma 3.3 to check the following three special cases.

(a) $E_i F_{i,i+2} - F_{i,i+2} E_i = -F_{i+1} K_{i,i+1}^{-1}$;

(b) $E_{i+1} F_{i,i+2} - F_{i,i+2} E_{i+1} = K_{i+1,i+2} F_i$;

(c) $E_{i+1} F_{i,i+3} = F_{i,i+3} E_{i+1}$.

The proofs of (a) and (b) are similar: in each case, expand $F_{i,i+2}$ as $-F_iF_{i+1} + vF_{i+1}F_i$ and compute the commutators one by one. For (c), compute:

$$\begin{aligned} E_{i+1} F_{i,i+3} &= E_{i+1} T_{i+2}(F_{i,i+2}) = T_{i+2}(T_{i+2}^{-1}(E_{i+1}) F_{i,i+2}) \\ &= T_{i+2}(-E_{i+1} E_{i+2} F_{i,i+2} + v^{-1} E_{i+2} E_{i+1} F_{i,i+2}) \\ &= T_{i+2}(F_{i,i+2} T_{i+2}^{-1}(E_{i+1}) - K_{i+1,i+2} F_i E_{i+2} + v^{-1} E_{i+2} K_{i+1,i+2} F_i) \\ &= T_{i+2}(F_{i,i+2} T_{i+2}^{-1}(E_{i+1})) = F_{i,i+3} E_{i+1}, \end{aligned}$$

as required.

3.6 In section 4, we will work with a renormalization of $F_{i,j}$. For $1 \leq i < j \leq n$, define

$$\hat{F}_{i,j} := v^{-j} K_j F_{i,j} K_i v^{-i}.$$

Also define $b_{i,j}$ and $c_{i,j} \in U(n)^0_{\mathcal{A}}$ by

$$\begin{aligned} b_{i,j} &:= \frac{v^{-2i-1} K_i^2 - v^{-2j-1} K_{j+1}^2}{v - v^{-1}} \\ c_{i,j} &:= \frac{v^{-2i-1} K_i^2 - v^{-2j-1} K_j^2}{v - v^{-1}} \end{aligned}$$

for $i \le j$. Note that for $i \le k \le j$, $c_{i,j} = c_{i,k} + c_{k,j}$ and $b_{i,j} = c_{i,k} + b_{k,j}$. Lemma 3.5(iv) translates into the following relations for the renormalized $\hat{F}_{i,j}$, for all $1 \le i < j \le n$ and all $1 \le l < n$:

$$
E_l \hat{F}_{i,j} = \begin{cases} \hat{F}_{i,j} E_l + v b_{i,i} & \text{if } l = i, l+1 = j \\ v^{-1} \hat{F}_{i,j} E_i - v \hat{F}_{i+1,j} & \text{if } l = i, l+1 \ne j \\ v \hat{F}_{i,j} E_i + \hat{F}_{i,j-1} & \text{if } l \ne i, l+1 = j \\ \hat{F}_{i,j} E_l & \text{if } l \notin \{i,j\}, l+1 \notin \{i,j\} \\ v \hat{F}_{i,j} E_l & \text{if } l+1 = i \\ v^{-1} \hat{F}_{i,j} E_l & \text{if } l = j. \end{cases} \tag{3.7}
$$

For $1 \le i < j \le n$ and $A \subseteq (i..j)$, define

$$
\begin{aligned}
F_{i,j}^A &:= F_{i,a_1} F_{a_1,a_2} \dots F_{a_r,j}, \\
\hat{F}_{i,j}^A &:= \hat{F}_{i,a_1} \hat{F}_{a_1,a_2} \dots \hat{F}_{a_r,j}.
\end{aligned}
$$

Notice that, in the notation of (2.2), if $t \in A$ then $\hat{F}_{i,j}^A = \hat{F}_{i,t}^{A_{i..t}} \hat{F}_{t,j}^{A_{t..j}}$ and similarly for $F_{i,j}^A$. We shall also need the following:

3.8 Lemma *Let $\mathfrak{n}^+$ denote the $\mathcal{A}$-submodule of $U(n)_\mathcal{A}$ generated by $\{E_{i,j} \mid 1 \le i < j \le n\}$. Then, for any $1 \le i < j \le n$ and any $A \subseteq (i..j)$,*

$$
E_i E_{i+1} \dots E_{j-1} . \hat{F}_{i,j}^A \equiv v \prod_{t \in \{i\} \cup A} b_{t,t} \qquad (\textit{modulo } U(n)_\mathcal{A} . \mathfrak{n}^+).
$$

Proof. If $j - i = 1$, the result follows by 3.7. So suppose that $j - i > 1$. Let $t = \max\{i\} \cup A$, so that $\hat{F}_{i,j}^A = \hat{F}_{i,t}^{A_{i..t}} \hat{F}_{t,j}$. By 3.7,

$$
E_i E_{i+1} \dots E_{j-1} . \hat{F}_{i,j}^A = v^{-1} E_i E_{i+1} \dots E_{t-1} \hat{F}_{i,t}^{A_{i..t}} E_t \dots E_{j-1} \hat{F}_{t,j}.
$$

The conclusion is immediate from this, using 3.7 and induction on $j - i$.

3.9 For $1 \le i \le n$, let $\varepsilon_i : U(n)_\mathcal{F}^0 \to \mathcal{F}$ denote the $\mathcal{F}$-algebra homomorphism defined by $K_j \mapsto v^{\delta_{ij}}$ for $1 \le j \le n$. Let $\mathcal{X}$ be the free abelian group generated by $\varepsilon_1, \dots, \varepsilon_n$. We shall call $\mathcal{X}$ the ***weight lattice***, and elements of $\mathcal{X}$ are ***weights***. We shall often restrict elements of $\mathcal{X}$ to $U(n)_\mathcal{A}^0$ to obtain $\mathcal{A}$-algebra maps $U(n)_\mathcal{A}^0 \to \mathcal{A}$.

Fix a dominant weight $\lambda = \sum_{i=1}^n \lambda_i \varepsilon_i \in \mathcal{X}$, so $\lambda_1 \ge \dots \ge \lambda_n$, and assume in addition that $\lambda_n \ge 0$. We identify λ with the partition $(\lambda_1, \dots, \lambda_n)$; as in (2.3), $[\lambda]$ denotes the diagram of the partition λ. Let $r = \lambda_1 + \dots + \lambda_n$.

We now define the (left) standard module $\Delta_n(\lambda)_\mathcal{A}$ over $\mathcal{A}$. Let $E(n)_\mathcal{F}$ be the natural n-dimensional $U(n)_\mathcal{F}$-module. This is the vector space over $\mathcal{F}$ with basis $e_1, \dots, e_n$ and $U(n)_\mathcal{F}$-action defined by

$$
\begin{aligned}
E_{i,i+1} . e_h &= \delta_{i+1,h} e_i, \\
F_{i,i+1} . e_h &= \delta_{i,h} e_{i+1}, \\
K . e_h &= \varepsilon_h(K) e_h
\end{aligned}
$$

for all $1 \leq i < n$ and all $K \in U(n)^0_{\mathcal{F}}$. Let $E(n)_{\mathcal{A}}$ be the $U(n)_{\mathcal{A}}$-submodule of $E(n)_{\mathcal{F}}$ generated (as an $\mathcal{A}$-module) by $e_1, \ldots, e_n$. Regard the tensor space $E(n)^{\otimes r}_{\mathcal{A}}$ as a $U(n)_{\mathcal{A}}$-module via the coassociative comultiplication Δ.

By a ***λ-tableau***, we mean a function $t : [\lambda] \to [1..n]$, which we usually regard just as the diagram $[\lambda]$ with the boxes filled with entries in $[1..n]$. Let $m = \lambda_1$. Let $i_1, \ldots, i_r$ be the sequence $1, 2, \ldots, \lambda'_1, 1, 2, \ldots, \lambda'_2, \ldots, 1, 2, \ldots, \lambda'_m$. Let $C(\lambda)$ denote the ***column stabilizer*** of the partition λ; by definition, $C(\lambda)$ is the intersection in $\mathfrak{S}(r)$ of the stabilizers of the sets $\{1, \ldots, \lambda'_1\}, \{\lambda'_1 + 1, \ldots, \lambda'_2\}, \ldots, \{\lambda'_1 + \cdots + \lambda'_m + 1, \ldots, r\}$. For example, if $\lambda = (3, 2)$, then $i_1, \ldots, i_5$ is the sequence $1, 2, 1, 2, 1$, obtained by reading down the columns of the tableau

1	1	1
2	2	

The column stabilizer is the stabilizer of the sets $\{1, 2\}, \{3, 4\}, \{5\}$, which are precisely the entries in the columns of the tableau

1	3	5
2	4	

Define $z_\lambda \in E(n)^{\otimes r}_{\mathcal{A}}$ as in [16, p. 80] to be

$$\sum_{w \in C(\lambda)} (-v)^{-l(w)} e_{i_{1.w}} \otimes \cdots \otimes e_{i_{r.w}}.$$

Define the ***standard module*** $\Delta_n(\lambda)_{\mathcal{A}}$ to be the left $U(n)_{\mathcal{A}}$-submodule of $E(n)^{\otimes r}_{\mathcal{A}}$ generated by z_λ. By [16, 5.1.1], the vector z_λ is annihilated by $U(n)^+_{\mathcal{A}}$, and for $K \in U(n)^0_{\mathcal{A}}$, $K.z_\lambda = \lambda(K).z_\lambda$. We say that $\Delta_n(\lambda)_{\mathcal{A}}$ is a ***high weight module*** of ***high weight*** λ, and call z_λ a ***high weight vector***.

3.10 There is an entirely different definition of $\Delta_n(\lambda)_{\mathcal{A}}$ obtained by lifting the Dipper-James q-Weyl module W^λ. We now explain this definition briefly. Let $\mathcal{H}(r)_{\mathcal{A}} := \mathcal{H}(r)_{\bar{\mathcal{A}}} \otimes_{\bar{\mathcal{A}}} \mathcal{A}$ be the generic Hecke algebra of (2.1) over $\mathcal{A}$, where $q = v^2$. As in [12, 1.2], we make $E(n)^{\otimes r}_{\mathcal{A}}$ into a $(U(n)_{\mathcal{A}}, \mathcal{H}(r)_{\mathcal{A}})$-bimodule by defining the right $\mathcal{H}(r)_{\mathcal{A}}$-action by

$$(e_{i_1} \otimes \cdots \otimes e_{i_r})T_{(j,j+1)} = \begin{cases} v e_{i_1} \otimes \cdots \otimes e_{i_{j+1}} \otimes e_{i_j} \otimes \cdots \otimes e_{i_r} & \text{if } i_j < i_{j+1} \\ v^2 e_{i_1} \otimes \cdots \otimes e_{i_r} & \text{if } i_j = i_{j+1} \\ (v^2 - 1) e_{i_1} \otimes \cdots \otimes e_{i_r} & \\ \qquad + v e_{i_1} \otimes \cdots \otimes e_{i_{j+1}} \otimes e_{i_j} \otimes \cdots \otimes e_{i_r} & \text{if } i_j > i_{j+1} \end{cases}$$

for all basic transpositions $(j, j+1)$ in $\mathfrak{S}(r)$. Define the ***q-Schur algebra***

$$S(n, r)_{\mathcal{A}} := \mathrm{End}_{\mathcal{H}(r)_{\mathcal{A}}}(E(n)^{\otimes r}_{\mathcal{A}}).$$

As remarked in [12, Remark 1.4], this is canonically isomorphic to the Dipper-James q-Schur algebra $S_q(n, r)$ (over $\mathcal{A}$) as defined in [10]. In [12], Du shows that the representation

$$\rho : U(n)_{\mathcal{A}} \to \mathrm{End}(E(n)^{\otimes r}_{\mathcal{A}})$$

has image $S(n,r)_{\mathcal{A}}$. This result is known as ***quantized Weyl reciprocity***. Now, Dipper-James define the q-Weyl module W^λ for $S(n,r)_{\mathcal{A}}$ to be a certain left ideal in $S(n,r)_{\mathcal{A}}$ generated by an element $z'_\lambda \in S(n,r)_{\mathcal{A}}$. In [16, 5.3.6], R. Green shows that W^λ (regarded as a $U(n)_{\mathcal{A}}$-module via the surjection ρ) is isomorphic, via the map $z_\lambda \mapsto z'_\lambda$, to our definition of $\Delta_n(\lambda)_{\mathcal{A}}$. We shall identify the modules $\Delta_n(\lambda)_{\mathcal{A}}$ and W^λ in this way.

3.11 We now describe the ***standard basis theorem*** for $\Delta_n(\lambda)_{\mathcal{A}}$. A λ-tableau is ***standard*** if the entries increase weakly along the rows from left to right and strictly down the columns. A λ-tableau is ***row standard*** if the entries increase weakly along the rows from left to right. Given an arbitrary row standard λ-tableau t such that all entries on the ith row are greater than or equal to i, let

$$F_t := \prod_{1\le i<j\le n} F_{i,j}^{(N_{i,j})}$$

where $N_{i,j}$ is equal to the number of entries equal to j on the ith row of t, and the order of terms in the product is as in (3.4).

3.12 Theorem (The standard basis theorem) *The module $\Delta_n(\lambda)_{\mathcal{A}}$ is a free $\mathcal{A}$-module with basis*

$$\{F_t.z_\lambda \mid \textit{for all standard } \lambda\textit{-tableaux } t : [\lambda] \to [1..n]\}\,.$$

In particular, $\Delta_n(\lambda)_{\mathcal{A}}$ has rank equal to the number of standard λ-tableaux with entries in $[1..n]$.

In the form stated here, this is proved in [16, 5.1.4]. We remark (as was pointed out to me by R. Green) that our definition of $F_{i,j}$ is not the same as in [16]; our $F_{i,j}$ differs from the $F_{i,j}$ defined in [16] by a multiple of some power of v. This does not affect the theorem.

It is also possible to deduce the standard basis theorem used here from the Dipper-James 'semistandard' basis theorem in [11]. In [16], Green shows that for a standard λ-tableau t, the image of $F_t.z_\lambda$ in the Dipper-James q-Weyl module W^λ is equal to the corresponding Dipper-James standard basis element of [11], multiplied by some power of v. Hence, the two versions of the standard basis theorem are in fact equivalent.

Later on, we shall appeal to the following technical lemma, which is a very special case of the straightening rule.

3.13 Lemma *Let $1 \le i < j \le n$, where $\lambda_{j-1} \ne 0, \lambda_j = 0$. For any $A \subseteq (i..j)$,*

$$F_{i,n}^A.z_\lambda = \pm F_{i,n}^B.z_\lambda$$

for some $A \subseteq B \subseteq (i..j)$ such that $F_{i,n}^B = F_t$ for some standard λ-tableau t.

Proof. Given $A \subseteq (i..j)$, there is a unique row standard λ-tableau $t(A)$, with all entries on row i of $t(A)$ greater than or equal to i, such that $F_{t(A)} = F_{i,n}^A$.

For the proof, take $A \subseteq (i..j)$. Let $w \in \mathfrak{S}([\lambda])$ be the permutation of the entries at the ends of the rows of $[\lambda]$ such that $t(A) \circ w$ is standard (recall that we regard λ-tableaux as functions $[\lambda] \to [1..n]$ so the composition of functions $t(A) \circ w$ is also a λ-tableau). We prove the result by induction on $l(w)$. If $l(w) = 0$, $t(A)$ is already standard, and the result is clear, taking $B = A$. Otherwise, write $w = sw_1$ where s is a basic transposition and $l(w_1) < l(w)$. Suppose that s corresponds to swapping the entries at the ends of the kth and $(k+1)$th rows of $t(A)$, so by construction, $k \in A, k+1 \notin A$ and $\lambda_k = \lambda_{k+1}$.

Now note that $t(A) \circ s = t(A \cup \{k+1\})$. So it suffices to show by induction that $F^A_{i,n}.z_\lambda = -F^B_{i,n}.z_\lambda$ where $B = A \cup \{k+1\}$. Let h be the smallest element of $A_{k..n} \cup \{n\}$, so that $h > k+1$. By Lemma 3.5(i),

$$F_{k,h} = vF_{k+1,h}F_k - F_kF_{k+1,h}.$$

Now, $F_{k,h}$ occurs as a term in the product $F^A_{i,n}$, and $F_kF_{k+1,h}$ occurs as a term in the product $F^B_{i,n}$. So the result follows from this identity providing $F_kF^{A_{h..n}}_{h,n}.z_\lambda = 0$. To see this, note that F_k commutes past $F_{h,t}$ for all $t > h > k+1$ by Lemma 3.5(iii), so it suffices to show that $F_k.z_\lambda = 0$. But since $\lambda_k = \lambda_{k+1}$, $\lambda - (\varepsilon_k - \varepsilon_{k+1})$ is not a weight of $\Delta_n(\lambda)_\mathcal{A}$, so $F_k.z_\lambda = 0$ as required.

3.14 Now we define the corresponding algebras to $U(n)_\mathcal{A}$ and $S(n,r)_\mathcal{A}$ over an arbitrary field (as we did in (2.1) to define $\mathcal{H}(r)_\mathcal{A}$ over $\mathbb{F}$). Fix an arbitrary field $\mathbb{F}$, and let $\bar{v} \in \mathbb{F}^\times$. Let $\bar{q} := \bar{v}^2$. Regard $\mathbb{F}$ as an $\mathcal{A}$-module, by letting $v \in \mathcal{A}$ act on $\mathbb{F}$ by multiplication by the fixed element $\bar{v} \in \mathbb{F}$. Then, the ***quantum hyperalgebra*** $U(n)$ over $\mathbb{F}$ is defined to be $U(n)_\mathcal{A} \otimes_\mathcal{A} \mathbb{F}$. Let $U(n)^-, U(n)^0, U(n)^+, U(n)^\flat, U(n)^\sharp$ be the images of $U(n)^-_\mathcal{A}, U(n)^0_\mathcal{A}, U(n)^+_\mathcal{A}, U(n)^\flat_\mathcal{A}, U(n)^\sharp_\mathcal{A}$ respectively in $U(n)$. For $X \in U(n)_\mathcal{A}$, we shall also write X for its image $X \otimes 1 \in U(n)$; it should always be clear whether we are working in $U(n)_\mathcal{A}$ or $U(n)$, so no confusion should arise.

Similarly, we define the q-Schur algebra $S(n,r)$ over $\mathbb{F}$ to be $S(n,r)_\mathcal{A} \otimes_\mathcal{A} \mathbb{F}$. The surjection $\rho : U(n)_\mathcal{A} \to S(n,r)_\mathcal{A}$ induces a surjection $\rho : U(n) \to S(n,r)$. The $(U(n)_\mathcal{A}, \mathcal{H}(r)_\mathcal{A})$-bimodule $E(n)^{\otimes r}_\mathcal{A}$ defines a $(U(n), \mathcal{H}(r))$-bimodule $E(n)^{\otimes r} := E(n)^{\otimes r}_\mathcal{A} \otimes_\mathcal{A} \mathbb{F}$. As before, $S(n,r) = \mathrm{End}_{\mathcal{H}(r)}(E(n)^{\otimes r})$. The standard module $\Delta_n(\lambda)$ is defined to be $\Delta_n(\lambda)_\mathcal{A} \otimes_\mathcal{A} \mathbb{F}$. The $U(n)$-module $\Delta_n(\lambda)$ has a unique maximal submodule $\mathrm{rad}\,\Delta_n(\lambda)$. Let $L_n(\lambda)$ denote the irreducible quotient $\Delta_n(\lambda)/\mathrm{rad}\,\Delta_n(\lambda)$. It is known [10] that

$$\{L_n(\lambda) \mid \text{for all } n\text{-part partitions } \lambda \vdash r\}$$

is a complete set of non-isomorphic irreducible $S(n,r)$-modules.

The antiautomorphism τ of $U(n)_\mathcal{A}$ from (3.1) induces an antiautomorphism of $U(n)$, which we again denote by τ. Given a finite dimensional $U(n)$-module W, we define its ***contravariant dual*** to be the dual space W^* with the natural right action of $U(n)$ made into a left action by composing with the antiautomorphism τ. (It is more usual to regard the dual space as a left module by composing with the antipode of the Hopf algebra $U(n)$.) We write $\nabla_n(\lambda)$ for the ***costandard*** module corresponding to λ. By definition, this is the contravariant dual of $\Delta_n(\lambda)$. The costandard module $\nabla_n(\lambda)$ has simple socle isomorphic to $L_n(\lambda)$.

Define ***polynomial representations of degree*** r for $U(n)$ over $\mathbb{F}$ to be the finite dimensional $U(n)$-modules that factor through the quotient $S(n,r)$; ***polynomial representations*** are just direct sums of polynomial representations of various different degrees. The modules $\Delta_n(\lambda), \nabla_n(\lambda)$ and $L_n(\lambda)$ are polynomial representations of degree r if $\lambda \vdash r$.

We now regard elements of $\mathcal{X}$ as $\mathbb{F}$-algebra maps $U(n)^0 \to \mathbb{F}$ in the obvious way. Given a $U(n)$-module W and a weight $\mu \in \mathcal{X}$, the corresponding ***μ-weight space*** is

$$W_\mu := \{w \in W \mid K.w = \mu(K)w \text{ for all } K \in U(n)^0\}.$$

We say that W splits as a direct sum of weight spaces if $W = \bigoplus_{\mu \in \mathcal{X}} W_\mu$. All polynomial $U(n)$-modules split as a direct sum of weight spaces.

We need one more well-known fact about standard modules.

3.15 Lemma (Universal property of standard modules) *Let W be any polynomial $U(n)$-module of degree r generated by a high weight vector of high weight λ, for some n-part partition $\lambda \vdash r$. Then, W is a quotient of the standard module $\Delta_n(\lambda)$.*

PROOF. Let $S(n,r)$ denote the q-Schur algebra over $\mathbb{F}$, so W is naturally an $S(n,r)$-module and $\Delta_n(\lambda) \cong W^\lambda$ is precisely the Dipper-James q-Weyl module for $S(n,r)$. Let $S(n,r)^+$ denote the image of $U(n)^\sharp$ in $S(n,r)$ under ρ; $S(n,r)^+$ is a ***Borel subalgebra*** of $S(n,r)$. Let $\mathbb{F}_\lambda$ denote the 1-dimensional $S(n,r)^+$-module corresponding to the weight λ. The argument of J. A. Green in [15, Theorem 8.1] generalizes to the q-Schur algebra using the quantized codeterminants of [16]. So,

$$\Delta_n(\lambda) \cong W^\lambda \cong S(n,r) \otimes_{S(n,r)^+} \mathbb{F}_\lambda. \tag{3.16}$$

Now the conclusion follows by a routine application of Frobenius reciprocity.

3.17 Let $U(n-1) < U(n)$ be the naturally embedded quantum hyperalgebra corresponding to $GL(n-1)$. So, $U(n-1)$ is defined by base change from the corresponding integral form for the $\mathcal{F}$-algebra generated by $\{E_i, F_i, K_j \mid 1 \le i < n-1, 1 \le j \le n-1\}$. Let $U(n-1)^-, U(n-1)^0, U(n-1)^+, U(n-1)^\flat, U(n-1)^\sharp$ be the corresponding subalgebras of $U(n-1)$. We will write $\Delta_{n-1}(\mu), \nabla_{n-1}(\mu)$ and $L_{n-1}(\mu)$ for the standard, costandard and irreducible modules for $U(n-1)$, for any partition μ with at most $(n-1)$ non-zero parts.

We can now prove the first important result. This is a quantum analogue of the classical branching rule, describing the restriction of $\Delta_n(\lambda)$ to $U(n-1)$. For the remainder of the section, we use the notation

$$\mu \longleftarrow \lambda$$

if $\mu = (\mu_1, \dots, \mu_{n-1})$ is an $(n-1)$-part partition such that $\lambda_{i+1} \le \mu_i \le \lambda_i$ for $i = 1, \dots, n-1$. So, if $\mu \longleftarrow \lambda$, then the diagram of μ is obtained by removing nodes from the bottom of columns of the diagram of λ. For example, if $\lambda = (3,2)$ and $n > 2$, then $\mu \longleftarrow \lambda$ if and only if μ equals $(3,2), (3,1), (3,0), (2,2), (2,1)$ or $(2,0)$. Given $\mu \longleftarrow \lambda$, define $t(\mu)$ to be the standard λ-tableau with hk-entry equal to h if $(h,k) \in [\mu]$, or n otherwise.

3.18 Lemma *The module $\Delta_n(\lambda)$ is generated as a $U(n-1)^-$-module by the vectors*

$$\{F_{t(\mu)}.z_\lambda \mid \mu \longleftarrow \lambda\}.$$

Proof. By the standard basis theorem, it suffices to show that given any standard λ-tableau t, there exists $\mu \longleftarrow \lambda$ and $X \in U(n-1)^-$ such that $F_t.z_\lambda = X.F_{t(\mu)}.z_\lambda$. Let s be the tableau obtained from t by deleting all entries equal to n, and let μ be the corresponding partition. Then, $F_t = F_s F_{t(\mu)}$. By definition of standard tableau, $\mu \longleftarrow \lambda$, and $F_s \in U(n-1)^-$, so the proof follows.

3.19 Theorem (The classical branching rule) *Let $\mu_1, \dots, \mu_N$ be all the partitions $\mu \longleftarrow \lambda$ ordered so that $\mu_i < \mu_j$ in the usual dominance order on $\mathcal{X}$ implies that $i > j$. Then*

(i) *$W = \Delta_n(\lambda)$ has a $U(n-1)$-stable filtration $0 = W_0 < W_1 < \cdots < W_N = W$ such that $W_i/W_{i-1} \cong \Delta_{n-1}(\mu_i)$ for all i.*

(ii) *The image of $F_{t(\mu_i)}.z_\lambda$ in W_i/W_{i-1} is a $U(n-1)$-high weight vector of weight μ_i.*

Proof. Define W_i to be the $U(n-1)^-$-module generated by $F_{t(\mu_i)}$ and W_{i-1}. This gives a filtration $0 = W_0 < W_1 < \cdots < W_N$ of $U(n-1)^-$-modules. By Lemma 3.18, $W_N = W$. We now prove by induction on i that W_i is $U(n-1)$-stable, and that the image of $F_{t(\mu_i)}$ is a $U(n-1)$-high weight vector of weight μ_i in W_i/W_{i-1}. The induction starts with $i = 0$. Suppose that $i > 0$ and that the result has been proved for all smaller i.

By the induction hypothesis and Lemma 3.18, the $U(n-1)$-module W/W_{i-1} is generated as a $U(n-1)^-$-module by the vectors

$$\{F_{t(\mu_j)}.z_\lambda \mid i \leq j \leq N\}.$$

By definition, μ_i is a maximal element of this set with respect to the dominance order. Hence, $F_{t(\mu_i)}$ is a $U(n-1)$-high weight vector. As $U(n-1) = U(n-1)^- U(n-1)^0 U(n-1)^+$, it follows that W_i is $U(n-1)$-stable. This completes the induction.

It remains to show that W_i/W_{i-1} is precisely the standard module $\Delta_{n-1}(\mu_i)$ for all i. Since W_i/W_{i-1} is a high weight module of high weight μ_i, Lemma 3.15 implies that W_i/W_{i-1} (which is a polynomial representation for $U(n-1)$) is certainly a homomorphic image of $\Delta_{n-1}(\mu_i)$. So suffices to show that W_i/W_{i-1} has the correct dimension d_i, the number of standard μ_i-tableaux with entries in $[1..n-1]$. But it is clear that the number of standard λ-tableaux with entries in $[1..n]$ is equal to the sum over all $\mu \longleftarrow \lambda$ of the number of standard μ-tableaux with entries in $[1..n-1]$; that is, $d = \dim W = \sum_{i=1}^N d_i$. Since $\dim W_i/W_{i-1}$ is at most d_i by Lemma 3.15, it follows that equality must hold for each i.

3.20 Corollary *Let μ be an $(n-1)$-part partition. Then,*

$$\mathrm{Hom}_{U(n-1)}(\Delta_{n-1}(\mu), \nabla_n(\lambda)\downarrow_{U(n-1)}) = \begin{cases} \mathbb{F} & \text{if } \mu \longleftarrow \lambda \\ 0 & \text{otherwise.} \end{cases}$$

Hence, each of the spaces

$$\begin{aligned}
&\mathrm{Hom}_{U(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)\downarrow_{U(n-1)}) \cong \mathrm{Hom}_{U(n-1)}(L_n(\lambda)\downarrow_{U(n-1)}, \nabla_{n-1}(\mu)),\\
&\mathrm{Hom}_{U(n-1)}(L_{n-1}(\mu), \nabla_n(\lambda)\downarrow_{U(n-1)}) \cong \mathrm{Hom}_{U(n-1)}(\Delta_n(\lambda)\downarrow_{U(n-1)}, L_{n-1}(\mu)),\\
&\mathrm{Hom}_{U(n-1)}(L_{n-1}(\mu), L_n(\lambda)\downarrow_{U(n-1)}) \cong \mathrm{Hom}_{U(n-1)}(L_n(\lambda)\downarrow_{U(n-1)}, L_{n-1}(\mu))
\end{aligned}$$

are at most 1*-dimensional, and they are non-zero only if* $\mu \longleftarrow \lambda$.

Proof. The second statement follows from the first applying the universal property of standard modules. In the case $\bar{q} = 1$, the first statement is immediate from Theorem 3.19 and standard properties of good filtrations [18, II.4.16]. This argument carries over to the quantum case, but there is also an elementary direct argument which easily generalizes to the quantum case, which we now sketch.

Applying Frobenius reciprocity (using 3.16) and taking contravariant duals,

$$\begin{aligned}
\mathrm{Hom}_{U(n-1)}(\Delta_{n-1}(\mu), \nabla_n(\lambda)\downarrow_{U(n-1)}) &\cong \mathrm{Hom}_{U(n-1)^\sharp}(\mathbb{F}_\mu, \nabla_n(\lambda)\downarrow_{U(n-1)})\\
&\cong \mathrm{Hom}_{U(n-1)^\flat}(\Delta_n(\lambda)\downarrow_{U(n-1)}, \mathbb{F}_\mu),
\end{aligned}$$

where $\mathbb{F}_\mu$ denotes the 1-dimensional $U(n-1)^\flat$-module. By Lemma 3.18, $\Delta_n(\lambda)$ is generated by the vectors $\{F_{t(\mu)}.z_\lambda \mid \mu \longleftarrow \lambda\}$ as a $U(n-1)^\flat$-module. Any $U(n-1)^\flat$-module homomorphism $\Delta_n(\lambda) \to \mathbb{F}_\mu$ is therefore determined (up to scalars) by the image each of these vectors in $\mathbb{F}_\mu$. In particular, this means that the Hom-space is at most 1-dimensional, and is non-zero only if $\mu \longleftarrow \lambda$. Finally, if $\mu \longleftarrow \lambda$, it is easy to check using Theorem 3.19 that the map defined on the generators by $F_{t(\mu)}.z_\lambda \mapsto 1$, $F_{t(\mu')}.z_\lambda \mapsto 0$ for $\mu' \longleftarrow \lambda$ with $\mu' \neq \mu$, is a well-defined homomorphism.

3.21 Definition Let $\mu \longleftarrow \lambda$.

(i) Say μ is ***normal*** if $\dim \mathrm{Hom}_{U(n-1)}(L_n(\lambda)\downarrow_{U(n-1)}, \nabla_{n-1}(\mu)) = 1$.

(ii) Say μ is ***conormal*** if $\dim \mathrm{Hom}_{U(n-1)}(\Delta_n(\lambda)\downarrow_{U(n-1)}, L_{n-1}(\mu)) = 1$.

(iii) Say μ is ***good*** if $\dim \mathrm{Hom}_{U(n-1)}(L_n(\lambda)\downarrow_{U(n-1)}, L_{n-1}(\mu)) = 1$, or if (equivalently) μ is both normal and conormal.

3.22 Note that if W is any polynomial $U(n)$-module, then W splits as a direct sum $W = \bigoplus_{z\geq 0} W^z$, where

$$W^z = \left\{ w \in W \mid K_n.w = v^z w, \begin{bmatrix} K_n \\ r \end{bmatrix}.w = \begin{bmatrix} z \\ r \end{bmatrix} w \text{ for all } r \in \mathbb{N} \right\}.$$

Since K_n and $\begin{bmatrix} K_n \\ r \end{bmatrix}$ centralize $U(n-1)$, this decomposition is $U(n-1)$-stable. Call W^z the zth level of W.

Assume now as in (2.1) that $\lambda_n = 0$ (the general case $\lambda_n \geq 0$ can be deduced easily from this). We specialize to the *first level* of $\Delta_n(\lambda)$ and $L_n(\lambda)$. So we only consider $\mu \longleftarrow \lambda$

obtained by removing a single node from the diagram of λ. We do this because the first level is all that is relevant to deducing results for $\mathcal{H}(r)$ by applying Schur functors.

In this case, μ is obtained from λ by removing precisely one node from the diagram of λ. Such nodes are precisely the removable nodes of (2.3), which we have parametrized by the set $R(\lambda)$. Recall that if $i \in R(\lambda)$, then $\lambda(i)$ denotes the partition obtained from the diagram of λ by removing (i, λ_i). Say i is normal (resp. conormal, resp. good) if $\lambda(i)$ is normal (resp. conormal, resp. good).

In Theorem 5.3, we will show that i is normal if and only if $i \in R_{\text{normal}}(\lambda)$. In Theorem 5.4, we will show that i is good if and only if $i \in R_{\text{good}}(\lambda)$. We will then show how to deduce Theorem 2.5 and Theorem 2.6 from these two results.

No criterion is known for an *arbitrary* μ corresponding to levels higher than the first level to be normal or good, nor is any criterion for conormal nodes known.

Before we can prove these statements, we need to introduce certain lowering operators $S_{i,j}(A)$ in $U(n)_{\mathcal{A}}$. These play precisely the same role as the operators $T_{r,s}(M)$ in Kleshchev's proof in [23] in the classical case. This is the subject of the next section.

4 Quantum Lowering Operators

4.1 In this section, we define the quantized lowering operators $S_{i,j}(A)$, for all $1 \leq i < j \leq n$ and all $A \subseteq (i..j)$. The quickest way to define these operators is to use the recurrence relation in Lemma 4.9, as described in Remark 4.10. We adopt a different approach which gives more information; in particular, our approach gives an explicit closed formula for the operator $S_{i,j}(A)$. As we have said before, the operator $S_{i,j}(A)$ is closely related (on specializing $v \mapsto 1$) to the operator $T_{r,s}(M)$ of [23]. The precise relationship between $S_{i,j}(A)$ and $T_{r,s}(M)$ is described in [5].

4.2 We begin the definition of $S_{i,j}(A)$ by defining elements $\mathcal{H}_{i,j}(A, B)$ of the free polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, for all $1 \leq i \leq j \leq n$ and all $A, B \subseteq (i..j)$. We will evaluate these polynomials at certain elements of $U(n)^0_{\mathcal{A}}$ when we come to define the lowering operator $S_{i,j}(A)$.

Recall from (2.2) that given any subset $A \subseteq (i..j)$, and $i \leq h < k \leq j$, we denote $A \cap (h..k)$ by $A_{h..k}$. For fixed i, j and any $A \subseteq (i..j)$, $\overline{A}$ will always denote its complement in $(i..j)$; that is, $\overline{A} := (i..j) \setminus A$. Given $i < k < j$ and any $D \subseteq (i..j)$, we use the notation $D_i(k)$ for the element immediately preceeding k in the set $D \cup \{i\}$; that is,

$$D_i(k) := \max\{t \in D \cup \{i\} \mid t < k\}.$$

We shall also use the following notation: given any property $\mathcal{P}$, define $\delta_{\mathcal{P}}$ to be 1 if $\mathcal{P}$ is true, or 0 if $\mathcal{P}$ is false.

4.3 Definition Let A, B be two arbitrary subsets of $(i..j)$. Let $x_1, \ldots, x_n$ be indeterminates. Define the rational function $\mathcal{H}_{i,j}(A, B) \in \mathbb{Q}(x_i, \ldots, x_{j-1})$ by

$$\mathcal{H}_{i,j}(A,B) := \sum_{D \subseteq B \setminus A} (-1)^{|D|} \frac{\prod_{t \in A} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}.$$

4.4 Remarks (I) There is no real need to include the subscript j in the notation $\mathcal{H}_{i,j}(A, B)$. This is done for uniformity with later notation.

(II) Note that if $i \leq k < j$ and A, B are subsets of $(k..j)$, then $\mathcal{H}_{k,j}(A, B)$ is precisely the function $\mathcal{H}_{i,j}(A, B)$ with x_k substituted for the indeterminate x_i.

(III) By cancelling terms in the product, $\mathcal{H}_{i,j}(A, B) = \mathcal{H}_{i,j}(A \setminus B, B \setminus A)$. We will often apply this fact without comment. In particular, replacing A by $A \setminus B$ and B by $B \setminus A$ in this way, we will often assume that A and B are disjoint. Also, since $A \setminus B = \overline{B} \setminus \overline{A}$, $\mathcal{H}_{i,j}(A, B) = \mathcal{H}_{i,j}(A \setminus B, B \setminus A) = \mathcal{H}_{i,j}(\overline{B} \setminus \overline{A}, \overline{A} \setminus \overline{B}) = \mathcal{H}_{i,j}(\overline{B}, \overline{A})$.

4.5 Example We list $\mathcal{H}_{i,i+3}(A, B)$ for all A, B. In the table, A indexes rows, B columns:

$\mathcal{H}_{i,i+3}(A,B)$	$\varnothing$	$\{i+1\}$	$\{i+2\}$	$\{i+1, i+2\}$
$\varnothing$	1	0	0	0
$\{i+1\}$	$x_{i+1} - x_i$	1	0	0
$\{i+2\}$	$x_{i+2} - x_i$	1	1	0
$\{i+1, i+2\}$	$(x_{i+1} - x_i)(x_{i+2} - x_i)$	$x_{i+2} - x_i$	$x_{i+1} - x_i$	1

The first lemma gives the basic properties of $\mathcal{H}_{i,j}(A, B)$.

4.6 Lemma *Let $A, B \subseteq (i..j)$.*

(i) $\mathcal{H}_{i,j}(A, B) \in \mathbb{Z}[x_i, \ldots, x_{j-1}]$.

(ii) $\mathcal{H}_{i,j}(A, B) \neq 0$ *if and only if $B \uparrow A$ in the lattice order.*

(iii) $\mathcal{H}_{i,j}(A, B)$ *satisfies the following recurrence relation. Firstly, if A is empty,*

$$\mathcal{H}_{i,j}(\varnothing, B) = \begin{cases} 1 & \text{if } B = \varnothing \\ 0 & \text{otherwise.} \end{cases}$$

Secondly, if A is not empty, let $k \in A$ and choose any $h \in [i..k) \setminus A$ such that $B_{h..k} \subseteq A_{h..k}$. Then

$$\mathcal{H}_{i,j}(A, B) = (x_k - x_h)\mathcal{H}_{i,j}(A \setminus \{k\}, B) + \delta_{h \neq i}\mathcal{H}_{i,j}(\{h\} \cup A \setminus \{k\}, B) \\ + \delta_{k \in B}\mathcal{H}_{i,k}(A_{i..k}, B_{i..k})\mathcal{H}_{k,j}(A_{k..j}, B_{k..j}).$$

PROOF. (i) We may assume that A, B are disjoint. Use induction on $|A|+|B|$. If $A \cup B = \varnothing$, $\mathcal{H}_{i,j}(A, B) = 1$ and the result is clear. Otherwise, let $h = \min(A \cup B)$. We consider two cases.

Case one: $h \in A$. By (4.2),

$$\mathcal{H}_{i,j}(A,B) = \sum_{D \subseteq B} (-1)^{|D|} \frac{(x_h - x_i) \prod_{t \in A \setminus \{h\}} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}$$
$$= (x_h - x_i)\mathcal{H}_{i,j}(A \setminus \{h\}, B).$$

This is in $\mathbb{Z}[x_i, \dots x_{j-1}]$ by induction.

Case two: $h \in B$. Splitting the summation in (4.2) into terms with $h \in D$ and $h \notin D$, one deduces that $\mathcal{H}_{i,j}(A,B)$ equals

$$\frac{1}{(x_h - x_i)} \sum_{\substack{D \subseteq B \setminus \{h\} \\ D' = D \cup \{h\}}} (-1)^{|D|} \left[\frac{\prod_{t \in A} (x_t - x_{D_i(t)})}{\prod_{t \in B \setminus \{h\}} (x_t - x_{D_i(t)})} - \frac{\prod_{t \in A} (x_t - x_{D'_i(t)})}{\prod_{t \in B \setminus \{h\}} (x_t - x_{D'_i(t)})} \right].$$

By definition, if $D' = D \cup \{h\}$ and $t > h$, then $D'_i(t) = D_h(t)$. So, by (4.2), this expression equals

$$\frac{1}{(x_h - x_i)} (\mathcal{H}_{i,j}(A, B \setminus \{h\}) - \mathcal{H}_{h,j}(A, B \setminus \{h\})).$$

By induction, $\mathcal{H}_{i,j}(A, B \setminus \{h\}) \in \mathbb{Z}[x_i, \dots, x_{j-1}]$. Note that from the definitions, $\mathcal{H}_{h,j}(A, B \setminus \{h\})$ is the polynomial $\mathcal{H}_{i,j}(A, B \setminus \{h\})$ with x_h substituted for x_i. Now, $x_h = (x_h - x_i) + x_i$. So, we may regard $\mathcal{H}_{h,j}(A, B \setminus \{h\})$ as a polynomial in $(x_h - x_i)$, with constant term precisely $\mathcal{H}_{i,j}(A, B \setminus \{h\})$. This constant term therefore cancels in the above expression for $\mathcal{H}_{i,j}(A,B)$, and so $\mathcal{H}_{i,j}(A,B) \in \mathbb{Z}[x_i, \dots, x_{j-1}]$ as required.

(ii) We compute the degree of $\mathcal{H}_{i,j}(A,B)$ as a polynomial in x_i. By the proof of (i), this degree satisfies the following recurrence relation:

(a) if $A \cup B = \varnothing$, $\mathcal{H}_{i,j}(A,B) = 1$ so $\deg_{x_i} \mathcal{H}_{i,j}(A,B) = 0$;

(b) if $A \cup B \neq \varnothing$ and $h = \min(A \cup B)$, then $\deg_{x_i} \mathcal{H}_{i,j}(A,B)$ equals

$$\begin{array}{ll}
\deg_{x_i} \mathcal{H}_{i,j}(A \setminus \{h\}, B \setminus \{h\}) & \text{if } h \in A \cap B; \\
\deg_{x_i} \mathcal{H}_{i,j}(A \setminus \{h\}, B) + 1 & \text{if } h \in A \setminus B; \\
\deg_{x_i} \mathcal{H}_{i,j}(A, B \setminus \{h\}) - 1 & \text{if } h \in B \setminus A \text{ and } \deg_{x_i} \mathcal{H}_{i,j}(A, B \setminus \{h\}) \neq 0; \\
-\infty & \text{if } h \in B \setminus A \text{ and } \deg_{x_i} \mathcal{H}_{i,j}(A, B \setminus \{h\}) = 0.
\end{array}$$

(We adopt the convention that the degree of 0 is $-\infty$).

Using this recurrence relation, a routine induction on $|A|+|B|$ shows that if $B \uparrow A$, then $\deg_{x_i} \mathcal{H}_{i,j}(A,B) = |A| - |B|$. In particular, if $B \uparrow A$, this degree is greater than or equal to 0 so $\mathcal{H}_{i,j}(A,B) \neq 0$.

Conversely, suppose $B \uparrow A$ is false. Then, by the first equivalent definition of $\uparrow$ in (2.2), we can choose $h < j$ maximal such that $|B_{h-1..j}| > |A_{h-1..j}|$. This implies that

$|B_{h..j}| = |A_{h..j}|$ and $B_{h..j} \uparrow A_{h..j}$, so $\deg_{x_i} \mathcal{H}_{i,j}(A_{h..j}, B_{h..j}) = 0$ by the previous paragraph. But now $h \in B, h \notin A$, so $\deg_{x_i} \mathcal{H}_{i,j}(A_{h-1..j}, B_{h-1..j}) = -\infty$. Now apply the recurrence relation again to deduce that $\deg_{x_i} \mathcal{H}_{i,j}(A, B) = -\infty$, so $\mathcal{H}_{i,j}(A, B) = 0$.

(iii) First, if $A = \varnothing$, it follows from (ii) that $\mathcal{H}_{i,j}(A, B) = 0$ unless $B \uparrow \varnothing$, so $B = \varnothing$. Also $\mathcal{H}_{i,j}(\varnothing, \varnothing) = 1$ by definition. Now suppose that $A \neq \varnothing$, and take h, k as in the statement. Replacing A by $A \setminus B_{h..k}$ and B by $B \setminus B_{h..k}$, we may assume that $B_{h..k} = \varnothing$. Note that

$$B \setminus (A \setminus \{k\}) = \begin{cases} B \setminus A & \text{if } k \notin B \\ (B \setminus A) \cup \{k\} & \text{if } k \in B. \end{cases}$$

Using this, we split the summation in (4.2) to show that

$$\mathcal{H}_{i,j}(A, B) = P_1 - \delta_{k \in B}\, P_2$$

where

$$P_1 = \sum_{D \subseteq B \setminus (A \setminus \{k\})} (-1)^{|D|}\, (x_k - x_{D_i(k)}) \frac{\prod_{t \in A \setminus \{k\}} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}$$

and

$$P_2 = \sum_{\substack{D \subseteq B \setminus (A \setminus \{k\}) \\ k \in D}} (-1)^{|D|}\, \frac{\prod_{t \in A \setminus \{k\}} (x_t - x_{D_i(t)})}{\prod_{t \in B \setminus \{k\}} (x_t - x_{D_i(t)})}.$$

Now we consider these two expressions separately. For P_1, write $(x_k - x_{D_i(k)})$ as $(x_k - x_h) + (x_h - x_{D_i(k)})$. Note $(x_h - x_{D_i(k)}) = 0$ if $h = i$, or if $h \neq i$ but $h \in D$. If $h \neq i$ and $h \notin D$, then $(x_h - x_{D_i(k)}) = (x_h - x_{D_i(h)})$. Using this, (i) equals

$$(x_k - x_h) \sum_{D \subseteq B \setminus (A \setminus \{k\})} (-1)^{|D|}\, \frac{\prod_{t \in A \setminus \{k\}} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}$$

$$+ \quad \delta_{h \neq i} \sum_{\substack{D \subseteq B \setminus (A \setminus \{k\}) \\ h \notin D}} (-1)^{|D|}\, (x_h - x_{D_i(h)}) \frac{\prod_{t \in A \setminus \{k\}} (x_t - x_{D_i(t)})}{\prod_{t \in B} (x_t - x_{D_i(t)})}.$$

This is precisely $(x_k - x_h)\, \mathcal{H}_{i,j}(A \setminus \{k\}, B) + \delta_{h \neq i}\, \mathcal{H}_{i,j}(\{h\} \cup A \setminus \{k\}, B)$.

Now consider P_2. Let $A' = A_{i..k}, A'' = A_{k..j}$ and define B', B'' similarly. We split the summation by replacing D with $D' \cup \{k\} \cup D''$ where $D' = D_{i..k}$ and $D'' = D_{k..j}$. Then,

$$
\begin{aligned}
P_2 &= \sum_{\substack{D' \subseteq B' \setminus A' \\ D'' \subseteq B'' \setminus A''}} (-1)^{|D'|+1+|D''|} \frac{\prod_{t \in A'} (x_t - x_{D'_i(t)})}{\prod_{t \in B'} (x_t - x_{D'_i(t)})} \frac{\prod_{t \in A''} (x_t - x_{D''_k(t)})}{\prod_{t \in B''} (x_t - x_{D''_k(t)})} \\
&= - \sum_{D' \subseteq B' \setminus A'} (-1)^{|D'|} \frac{\prod_{t \in A'} (x_t - x_{D'_i(t)})}{\prod_{t \in B'} (x_t - x_{D'_i(t)})} \sum_{D'' \subseteq B'' \setminus A''} (-1)^{|D''|} \frac{\prod_{t \in A''} (x_t - x_{D''_k(t)})}{\prod_{t \in B''} (x_t - x_{D''_k(t)})} \\
&= -\mathcal{H}_{i,k}(A', B')\mathcal{H}_{k,j}(A'', B'').
\end{aligned}
$$

This completes the proof.

4.7 Definition (i) For integers $1 \leq i < j \leq n$ and $A, B \subseteq (i..j)$, define $H_{i,j}(A, B) \in U(n)^0_{\mathcal{A}}$ by evaluating the polynomial $\mathcal{H}_{i,j}(A, B) \in \mathbb{Z}[x_i, \dots, x_{j-1}]$ from Definition 4.3 at

$$x_k := -\frac{v^{-2k-1} K_k^2}{v - v^{-1}}$$

for all $1 \leq k \leq n$. Note that if $i < j$ then $x_j - x_i$ evaluates to $c_{i,j}$ as defined in (3.6).

(ii) Given $A \subseteq (i..j)$, define $S_{i,j}(A) \in U(n)^\flat_{\mathcal{A}}$ by

$$S_{i,j}(A) := \sum_{B \subseteq (i..j)} \hat{F}^B_{i,j} H_{i,j}(A, B),$$

where $\hat{F}^B_{i,j}$ is as in (3.6). By convention, $S_{i,i}(\varnothing) = 1$.

4.8 Remark The appropriate definition for $S_{i,j}(A)$ in the specialization $v \mapsto 1$ is

$$S_{i,j}(A) := \sum_{B \subseteq (i..j)} F^B_{i,j} H_{i,j}(A, B)$$

where $H_{i,j}(A, B)$ is defined by evaluating the polynomial $\mathcal{H}_{i,j}(A, B)$ at $x_k := k - H_k$ for all k (and H_k denotes the diagonal matrix with a 1 in the kk-entry, zeros elsewhere in the Lie algebra $\mathfrak{gl}_n(\mathbb{Q})$.). See [4, Chapter 7].

The operators $S_{i,j}(A)$ are most easily computed using the following recurrence relation.

4.9 Lemma (Recurrence relation) *Let $A \subseteq (i..j)$. If $A = \varnothing$, then $S_{i,j}(A)$ equals $\hat{F}_{i,j}$. Otherwise, choose $k \in A$ and let $h = \max[i..k) \setminus A$. Then,*

$$S_{i,j}(A) = S_{i,j}(A \setminus \{k\})c_{h,k} + \delta_{h \neq i}\, S_{i,j}(\{h\} \cup A \setminus \{k\}) + S_{i,k}(A_{i..k})S_{k,j}(A_{k..j}).$$

Proof. First suppose that $A = \varnothing$. By Lemma 4.6(ii), $H_{i,j}(\varnothing, B)$ is zero unless $B = \varnothing$, when it is 1. Hence, $S_{i,j}(\varnothing) = \hat{F}_{i,j}$ as claimed. Now suppose that $A \neq \varnothing$, and choose h, k as in the statement. Apply Lemma 4.6(iii) to the definition of $S_{i,j}(A)$ to deduce that $S_{i,j}(A)$ equals

$$\sum_{B \subseteq (i..j)} \left(\begin{array}{r} \hat{F}^B_{i,j} H_{i,j}(A \setminus \{k\}, B) c_{h,k} + \hat{F}^B_{i,j} \delta_{h \neq i} H_{i,j}(\{h\} \cup A \setminus \{k\}, B) \\ + \delta_{k \in B} \hat{F}^B_{i,j} H_{i,k}(A_{i..k}, B_{i..k}) H_{k,j}(A_{k..j}, B_{k..j}) \end{array} \right).$$

The first two terms of this expression equal $S_{i,j}(A \setminus \{k\}) c_{h,k} + \delta_{h \neq i} S_{i,j}(\{h\} \cup A \setminus \{k\})$. For the third term split the summation to obtain

$$\sum_{\substack{B' \subseteq (i..k) \\ B'' \subseteq (k..j)}} \hat{F}^{B'}_{i,k} \hat{F}^{B''}_{k,j} H_{i,k}(A_{i..k}, B') H_{k,j}(A_{k..j}, B'').$$

Recall that $\mathcal{H}_{i,k}(A_{i..k}, B') \in \mathbb{Z}[x_i, \ldots, x_{k-1}]$. So, by Lemma 3.5(ii), $H_{i,k}(A_{i..k}, B')$ commutes with $\hat{F}^{B''}_{k,j}$. So this expression equals $S_{i,k}(A_{i..k}) S_{k,j}(A_{k..j})$ as required.

4.10 Remark This recurrence relation can in fact be used to define $S_{i,j}(A)$. The argument is as follows. Let $\mathrm{ht}(A) := \sum_{t \in A} t$. Define $S_{i,j}(A)$ by induction on height, by setting $S_{i,j}(\varnothing) = \hat{F}_{i,j}$ and, if $A \neq \varnothing$, letting $S_{i,j}(A)$ be the right hand side of the recurrence relation in Lemma 4.9 in the special case $k = \min A$. This gives a well-defined operator $S_{i,j}(A)$ for all $A \subseteq (i..j)$. One then needs to prove that the operator defined in this way satisfies the recurrence relation of Lemma 4.9 for *arbitrary* $k \in A$, which can be proved by induction on $\mathrm{ht}(A)$.

4.11 Lemma ('Commutators') *Let $A \subseteq (i..j)$ and $1 \leq l < n$. Define '$\equiv$' to be congruence modulo the left ideal $U(n)_{\mathcal{A}}.E_l$ of $U(n)_{\mathcal{A}}$.*

(i) *Suppose one of the following holds:*

(a) *$l + 1 \in A$;*

(b) *$l \notin \{i\} \cup A$ and $l + 1 \notin A \cup \{j\}$.*

Then, $E_l S_{i,j}(A) \equiv 0$.

(ii) *If $l \in \{i\} \cup A$ and $l + 1 \notin A \cup \{j\}$, then $E_l S_{i,j}(A) \equiv -v S_{i,l}(A_{i..l}) S_{l+1,j}(A_{l+1..j})$.*

(iii) *If $l = j - 1 \notin \{i\} \cup A$, then $E_{j-1} S_{i,j}(A) \equiv S_{i,j-1}(A)$.*

(iv) *If $l = j - 1 \in A$ and $k = \max[i..j-1) \setminus A$, then*

$$E_{j-1} S_{i,j}(A) \equiv S_{i,j-1}(A \setminus \{j-1\}) b_{k,j-1} + \delta_{k \neq i} S_{i,j-1}(\{k\} \cup A \setminus \{j-1\}).$$

Proof. (i) Use induction on height, where $\mathrm{ht}(A) = \sum_{a \in A} a$. If $\mathrm{ht}(A) = 0$, $S_{i,j}(A) = \hat{F}_{i,j}$ and the result is immediate from 3.7. If $\mathrm{ht}(A) = i + 1$, $A = \{i + 1\}$ and

$$S_{i,j}(A) = \hat{F}_{i,j} c_{i,i+1} + \hat{F}_{i,i+1} \hat{F}_{i+1,j}.$$

The conclusion is immediate from this and 3.7 if (b) holds. So, suppose (a) holds, so that $l = i$. Then, by 3.7

$$E_l S_{i,j}(A) \equiv -v\hat{F}_{i+1,j}c_{i,i+1} + vb_{i,i}\hat{F}_{i+1,j}.$$

Now one checks that $b_{i,i}\hat{F}_{i+1,j} = \hat{F}_{i+1,j}c_{i,i+1}$, so that $E_l S_{i,j}(A) \equiv 0$ as required. So now suppose that $\mathrm{ht}(A) > i+1$ and that the result has been proved for all A of smaller height.

Suppose first that $A = \{l+1\}$ where $i < l < j-1$. Applying Lemma 4.9 twice, $S_{i,j}(A)$ equals

$$\hat{F}_{i,l+1}\hat{F}_{l+1,j} + \hat{F}_{i,l}\hat{F}_{l,j} + \hat{F}_{i,j}c_{l-1,l+1} + \delta_{l-1,i}S_{i,j}(\{l-1\}).$$

By induction, $E_l S_{i,j}(\{l-1\}) \equiv 0$, and $E_l\hat{F}_{i,j} \equiv 0$. Also, by 3.7 again,

$$\begin{aligned} E_l\hat{F}_{i,l+1}\hat{F}_{l+1,j} + E_l\hat{F}_{i,l}\hat{F}_{l,j} &\equiv \hat{F}_{i,l}\hat{F}_{l+1,j} + v^{-1}\hat{F}_{i,l}E_l\hat{F}_{l,j} \\ &\equiv \hat{F}_{i,l}\hat{F}_{l+1,j} - v^{-1}\hat{F}_{i,l}v\hat{F}_{l+1,j} \equiv 0. \end{aligned}$$

So, we may assume that we can choose some $k \in A$ with $k \neq l+1$. Let $h = \max[i..k) \setminus A$ and apply Lemma 4.9. The conclusion follows in either case (a) or case (b) using the induction hypothesis.

(ii) It suffices to prove this in the case $l = i$; the general case follows easily from this by expanding $S_{i,j}(A)$ using Lemma 4.9 (with $k = l$). So, we prove that if $i+1 \notin A \cup \{j\}$ then

$$E_i S_{i,j}(A) \equiv -vS_{i+1,j}(A_{i+1..j}),$$

by induction on $\mathrm{ht}(A)$. If $\mathrm{ht}(A) = 0$, the result is just 3.7. Now suppose $\mathrm{ht}(A) > 0$ and that the result has been proved for all A of smaller height. We can find $k \in A$. Let $h = \max[i..k) \setminus A$, and notice that $h \neq i$ since $i+1 \notin A$. Apply Lemma 4.9:

$$S_{i,j}(A) = S_{i,j}(A \setminus \{k\})c_{h,k} + S_{i,j}(\{h\} \cup A \setminus \{k\}) + S_{i,k}(A_{i..k})S_{k,j}(A_{k..j}).$$

Applying the induction hypothesis, it follows that $E_i S_{i,j}(A)$ is congruent to

$$-S_{i+1,j}(A \setminus \{k\})c_{h,k} - S_{i+1,j}(\{h\} \cup A \setminus \{k\}) - S_{i+1,k}(A_{i..k})S_{k,j}(A_{k..j})$$

if $h \neq i+1$ or

$$-S_{i+1,j}(A \setminus \{k\})c_{h,k} - S_{i+1,k}(A_{i..k})S_{k,j}(A_{k..j})$$

if $h = i+1$, since in this case $E_i S_{i,j}(\{h\} \cup A \setminus \{k\}) \equiv 0$ by (i)(a). In either case, another application of Lemma 4.9 gives the conclusion.

(iii) The proof of this is almost identical to (ii) (in fact there is only one case to consider here, so it is slightly easier). We leave the details to the reader.

(iv) We deduce this from (iii). Let $k = \max[i..j-1) \setminus A$. By Lemma 4.9,

$$S_{i,j}(A) = S_{i,j}(A_{i..j-1})c_{k,j-1} + \delta_{k \neq i}S_{i,j}(\{k\} \cup A_{i..j-1}) + S_{i,j-1}(A_{i..j-1})\hat{F}_{j-1,j}.$$

Note that $E_{j-1}S_{i,j-1}(A_{i..j-1}) = v^{-1}S_{i,j-1}(A_{i..j-1})E_{j-1}$. So, applying (i),(iii) and 3.7, it follows that $E_{j-1}S_{i,j}(A)$ is congruent to

$$S_{i,j-1}(A_{i..j-1})c_{k,j-1} + \delta_{k\neq i}S_{i,j-1}(\{k\}\cup A_{i..j-1}) + v^{-1}S_{i,j-1}(A_{i..j-1})vb_{j-1,j-1}.$$

Since $b_{k,j-1} = c_{k,j-1} + b_{j-1,j-1}$, this gives the required expression.

4.12 We need one more property of $S_{i,j}(A)$ in order to prove the modular branching rules. As before, we begin by working with indeterminates then specialize to elements of $U(n)^0_{\mathcal{A}}$. For $A \subseteq (i..j)$, define $\mathcal{K}_{i,j}(A) \in \mathbb{Z}[x_i,\dots,x_{j-1};y_{i+1},\dots,y_j]$ by

$$\mathcal{K}_{i,j}(A) := \sum_{B\subseteq(i..j)}\left(\mathcal{H}_{i,j}(A,B)\prod_{t\in B\cup\{i\}}(y_{t+1}-x_t)\right).$$

The basic properties of $\mathcal{K}_{i,j}(A)$ that we shall need are given in the next lemma. In particular, we shall apply Lemma 4.13(ii) when we specialize $\mathcal{A}$ to an arbitrary field.

4.13 Lemma *Let $A \subseteq (i..j)$.*

(i) $\displaystyle \mathcal{K}_{i,j}(A) = \sum_{D\subseteq\overline{A}}(-1)^{|D|}\frac{\prod_{t\in(i..j]}(y_t - x_{D_i(t)})}{\prod_{t\in\overline{A}}(x_t - x_{D_i(t)})} = \sum_{D\subseteq\overline{A}}(-1)^{|D|}\frac{\prod_{t\in[i..j)}(y_{t+1} - x_{D_i(t+1)})}{\prod_{t\in\overline{A}}(x_t - x_{D_i(t)})}.$

(ii) *Let $B \subseteq (i..j)$ be any set such that $A \downarrow B$ and $|B| = |A|$. Then,*

$$\mathcal{K}_{i,j}(A) \equiv \prod_{t\in\{i\}\cup B}(y_{t+1}-x_i) \quad (\textit{modulo } J^{\theta}_{i,j}),$$

where $J^{\theta}_{i,j}$ is the ideal of $\mathbb{Z}[x_i,\dots,x_{j-1};y_{i+1},\dots,y_j]$ generated by $\{y_{\theta(t)+1} - x_t \mid t \in \overline{A}\}$, and θ is any bijection $\overline{A} \to \overline{B}$ such that $\theta(t) \geq t$ for all $t \in \overline{A}$.

Proof. (i) Expand the top product in the expression in (i) by writing $(y_{t+1} - x_{D_i(t+1)}) = (y_{t+1} - x_t) + (x_t - x_{D_i(t+1)})$, to obtain

$$\sum_{B\subseteq(i..j)}\prod_{t\in B\cup\{i\}}(y_{t+1}-x_t)\sum_{D\subseteq\overline{A}}(-1)^{|D|}\frac{\prod_{t\in\overline{B}}(x_t - x_{D_i(t+1)})}{\prod_{t\in\overline{A}}(x_t - x_{D_i(t)})}$$

Now observe that the top product vanishes unless $\overline{B}$ and D are disjoint, in which case $D_i(t+1) = D_i(t)$ for $t \in \overline{B}$. So, this equals

$$\sum_{B \subseteq (i..j)} \prod_{t \in B \cup \{i\}} (y_{t+1} - x_t) \sum_{D \subseteq \overline{A} \setminus \overline{B}} (-1)^{|D|} \frac{\prod_{t \in \overline{B}} (x_t - x_{D_i(t)})}{\prod_{t \in \overline{A}} (x_t - x_{D_i(t)})}$$

$$= \sum_{B \subseteq (i..j)} \prod_{t \in B \cup \{i\}} (y_{t+1} - x_t) \mathcal{H}_{i,j}(\overline{B}, \overline{A}).$$

Since $\mathcal{H}_{i,j}(\overline{B}, \overline{A}) = \mathcal{H}_{i,j}(A, B)$, this proves (i).

(ii) We use induction on $j - i$. If $A = (i..j)$, then by (i),

$$\mathcal{K}_{i,j}(A) = \prod_{t \in [i..j)} (y_{t+1} - x_i)$$

and the result holds. This includes the case $j - i = 1$, so now suppose that $j - i > 1$ and that the result has been proved for all smaller $j - i$. Let $A \subseteq (i..j)$. Let B, θ be as in the statement. The result has been proved if $A = (i..j)$. Otherwise, let $h = \min \overline{A}$. Let θ' be θ restricted to $(h..j)$, and B' equal $B_{h..j}$ if $h \notin B$ or $B_{h..j} \cup \{\theta(h)\}$ if $h \in B$. Note that $A_{h..j} \downarrow B'$ so by induction,

$$\mathcal{K}_{h,j}(A_{h..j}) = \prod_{t \in \{h\} \cup B'} (y_{t+1} - x_h) + \sum_{s \geq 0} a_s x_h^s$$

for some polynomials $a_s \in J_{h,j}^{\theta'}$ independent of x_h.

Now expand the expression in (i) by splitting the summation into terms with $h \in D$ and terms with $h \notin D$ (as we did in the proof of Lemma 4.6(i)):

$$\mathcal{K}_{i,j}(A) = \prod_{t \in [i..h)} (y_{t+1} - x_i) \frac{1}{x_h - x_i} \left[\mathcal{K}_{h,j}(A_{h..j})\big|_{x_h := x_i} - \mathcal{K}_{h,j}(A_{h..j}) \right].$$

Here, $\mathcal{K}_{h,j}(A_{h..j})\big|_{x_h := x_i}$ denotes the rational expression $\mathcal{K}_{h,j}(A_{h..j})$ with the indeterminate x_i substituted for x_h. Hence,

$$\mathcal{K}_{i,j}(A) = \prod_{t \in [i..h)} (y_{t+1} - x_i) \frac{1}{x_h - x_i} \Bigg[\prod_{t \in \{h\} \cup B'} (y_{t+1} - x_i) - \prod_{t \in \{h\} \cup B'} (y_{t+1} - x_h) + \sum_{s \geq 0} a_s (x_i^s - x_h^s) \Bigg].$$

Note $x_i^s - x_h^s$ is divisible by $x_h - x_i$ for all $s \geq 0$ and $a_s \in J_{i,j}^{\theta}$. So the third term in the bracket vanishes modulo $J_{i,j}^{\theta}$. The other two terms in the bracket may be written as

$$(y_{\theta(h)+1} - x_h) \left[\prod_{t \in B \cap [h..j)} (y_{t+1} - x_i) - \prod_{t \in B \cap [h..j)} (y_{t+1} - x_h) \right] + (x_h - x_i) \prod_{t \in B \cap [h..j)} (y_{t+1} - x_i).$$

Now $x_h - x_i$ divides the contents of the main bracket in this expression, by the same argument as Lemma 4.6(i). Hence, the first of these terms vanishes modulo $J^\theta_{i,j}$, as $y_{\theta(h)+1} - x_h \in J^\theta_{i,j}$. Hence,

$$\mathcal{K}_{i,j}(A) \equiv \prod_{t\in[i..h)} (y_{t+1} - x_i) \prod_{t\in B\cap[h..j)} (y_{t+1} - x_i) \quad (\text{modulo } J^\theta_{i,j}).$$

Finally note that $(i..h) \subseteq A$, so $(i..h) \subseteq B$. This completes the proof.

4.14 Now we evaluate $\mathcal{K}_{i,j}(A)$ to obtain an element of $U(n)^0_{\mathcal{A}}$. Let $K_{i,j}(A)$ be the element of $U(n)^0_{\mathcal{A}}$ defined by evaluating x_k and y_{k+1} at

$$x_k := -\frac{v^{-2k-1}K_k^2}{v - v^{-1}}, \qquad y_{k+1} := -\frac{v^{-2k-1}K_{k+1}^2}{v - v^{-1}}$$

for all $1 \le k \le n$. Note that $y_{j+1} - x_i$ evaluates to $b_{i,j}$ as defined in (3.6). The reason for introducing $\mathcal{K}_{i,j}(A)$ is explained by the next lemma.

4.15 Lemma *Let $\mathfrak{n}^+$ be the span of $\{E_{i,j} \mid 1 \le i < j \le n\}$. Then,*

$$E_iE_{i+1}\dots E_{j-1}.S_{i,j}(A) \equiv vK_{i,j}(A) \qquad (\textit{modulo } U(n)_{\mathcal{A}}.\mathfrak{n}^+).$$

Proof. This is immediate from the definition of $S_{i,j}(A)$, $K_{i,j}(A)$ and Lemma 3.8.

5 Proof of the Branching Rules

5.1 We now apply the lowering operators of section 4 to prove the main results of the paper. Throughout this section, we work over $\mathbb{F}$, and regard the operators $S_{i,j}(A)$ of section 4 as elements of the algebra $U(n)$ over $\mathbb{F}$ in the natural way. Fix $\lambda \vdash r$ with $\lambda_n = 0$ as in (2.1), and let $W = \Delta_n(\lambda)$ with maximal submodule $\operatorname{rad} W$. Let z_λ be a high weight vector in W. By the definition in (3.6),

$$\begin{aligned} b_{i,j}(\lambda) &= \frac{\bar v^{2(\lambda_i - i)} - \bar v^{2(\lambda_{j+1}-j)}}{\bar v^2 - 1} = \frac{\bar q^{(\lambda_i - i)} - \bar q^{(\lambda_{j+1}-j)}}{\bar q - 1}, \\ c_{i,j}(\lambda) &= \frac{\bar v^{2(\lambda_i - i)} - \bar v^{2(\lambda_j-j)}}{\bar v^2 - 1} = \frac{\bar q^{(\lambda_i - i)} - \bar q^{(\lambda_j-j)}}{\bar q - 1}. \end{aligned}$$

Notice in particular that $b_{i,j}(\lambda)$ vanishes (since we have assumed $\bar q \neq 1$) if and only if $\lambda_i - i \equiv \lambda_{j+1} - j \pmod{\ell}$. A similar argument applies for $c_{i,j}(\lambda)$. Now recall the definitions of $B_{i,j}(\lambda)$ and $C_{i,j}(\lambda)$ from (2.3). By the preceeding observations, these are equivalent to the following definitions:

$$\begin{aligned} B_{i,j}(\lambda) &:= \{k \in [i..j) \mid b_{i,k}(\lambda) = 0\}, \\ C_{i,j}(\lambda) &:= \{k \in (i..j) \mid c_{i,k}(\lambda) = 0\}. \end{aligned}$$

In this section, we shall always work with these definitions of $B_{i,j}(\lambda)$ and $C_{i,j}(\lambda)$. We will also often use the fact that if $c_{i,j+1}(\lambda)=0$ then $b_{i,j}(\lambda)\neq 0$. This follows since $c_{i,j+1}(\lambda)-b_{i,j}(\lambda)=\bar{q}^{(\lambda_{j+1}-j-1)}\neq 0$.

5.2 Lemma *Let $C\subseteq C_{i,j}(\lambda)$ and suppose that $S_{i,j}(\overline{C}).z_\lambda\notin \operatorname{rad} W$ (where $\overline{C}=(i..j)\setminus C$ as usual). Then, $B_{i,j}(\lambda)\downarrow C$ in the lattice order of (2.2).*

Proof. We prove this by induction on $j-i$. If $j-i=1$, then $S_{i,j}(\varnothing)=\hat{F}_{i,i+1}$ and in this case, $S_{i,j}(\varnothing).z_\lambda\notin \operatorname{rad} W$ if and only if $E_i\hat{F}_{i,i+1}.z_\lambda\notin \operatorname{rad} W$. Now, by 3.7, $E_i\hat{F}_{i,i+1}.z_\lambda=\bar{v}b_{i,i}(\lambda).z_\lambda$, so this is if and only if $b_{i,i}(\lambda)\neq 0$, or, equivalently, $B_{i,j}(\lambda)=\varnothing$. Thus the induction starts. Now suppose that $j-i>1$ and the result has been proved for all smaller $j-i$, and choose $C\subseteq C_{i,j}(\lambda)$ such that $S_{i,j}(A).z_\lambda\notin \operatorname{rad} W$, where $A=\overline{C}$.

We first claim that there exists some $1\leq l<n$ such that $E_lS_{i,j}(A).z_\lambda\notin \operatorname{rad} W$. For, suppose that the claim is false. Then by this assumption, together with an easy argument involving weights for $r>1$, $E_l^{(r)}S_{i,j}(A).z_\lambda=0$ for all $1\leq l<n$ and all $r\geq 1$. But this implies that $S_{i,j}(A).z_\lambda$ lies in a proper submodule of W, contrary to the assumption that $S_{i,j}(A).z_\lambda\notin \operatorname{rad} W$.

Hence, we can choose $1\leq l<n$ such that $E_lS_{i,j}(A).z_\lambda\notin \operatorname{rad} W$. We now consider the possibilities for l given by Lemma 4.11.

Case one. $l\in\{i\}\cup A, l+1\notin A\cup\{j\}$. By Lemma 4.11(ii),

$$S_{i,l}(A_{i..l})S_{l+1,j}(A_{l+1..j}).z_\lambda\notin \operatorname{rad} W.$$

Since $S_{i,l}(A_{i..l})$ and $S_{l+1,j}(A_{l+1..j})$ commute, this implies that both $S_{l+1,j}(A_{l+1..j}).z_\lambda$ and $S_{i,l}(A_{i..l}).z_\lambda$ are not elements of $\operatorname{rad} W$. So by induction, $B_{i,l}(\lambda)\downarrow C_{i..l}$ and $B_{l+1,j}(\lambda)\downarrow C_{l+1..j}$. Now notice that $l+1\in C$ so $c_{i,l+1}(\lambda)=0$. This implies that $B_{l+1,j}(\lambda)\subseteq B_{i,j}(\lambda)$ and that $b_{i,l}(\lambda)\neq 0$, so $l\notin B_{i,j}(\lambda)$. Hence $B_{i,j}(\lambda)=B_{i,l}(\lambda)\cup B_{l+1,j}(\lambda)$. Now it is immediate from the definition of $\downarrow$ that $B_{i,j}(\lambda)\downarrow C$.

Case two. $l=j-1, j-1\in C$. By Lemma 4.11(iii),

$$S_{i,j-1}(A_{i..j-1}).z_\lambda\notin \operatorname{rad} W.$$

Induction implies that $B_{i,j-1}(\lambda)\downarrow C_{i..j-1}$, so we can find a weakly decreasing injection $B_{i,j-1}(\lambda)\hookrightarrow C_{i..j-1}$. Since $j-1\in C$, this can always be extended to a weakly decreasing injection $B_{i,j}(\lambda)\hookrightarrow C$, so $B_{i,j}(\lambda)\downarrow C$ as required.

Case three. $l=j-1, j-1\notin C$. By Lemma 4.11(iv),

$$b_{k,j-1}(\lambda)S_{i,j-1}(A_{i..j-1}).z_\lambda+\delta_{k\neq i}S_{i,j-1}(\{k\}\cup A_{i..j-1}).z_\lambda\notin \operatorname{rad} W,$$

where $k=\max[i..j-1)\setminus A$, so $k\in C$. Note $b_{k,j-1}(\lambda)=b_{i,j-1}(\lambda)$. One of the terms on the right hand side of this expression must not be an element of $\operatorname{rad} W$. If the first term is not in $\operatorname{rad} W$, then $j-1\notin B_{i,j}(\lambda)$, and the result follows easily by induction as in case two. So suppose that $k\neq i$ and that by induction $B_{i,j-1}(\lambda)\downarrow C\setminus\{k\}$. Then it is easy to see by definition of $\downarrow$ that $B_{i,j}(\lambda)\downarrow C$ as required.

Now we prove the main results of this paper. The argument (iii)$\Rightarrow$(i) in the next theorem is due to Kleshchev in [23, Theorem 4.2]. The arguments for the other implications are new and rather simpler than in the original proof.

5.3 Theorem (Criterion for normal nodes) *Let $i \in R(\lambda)$. Then, the following are equivalent:*

(i) *i is normal;*

(ii) $S_{i,n}(A).z_\lambda \notin \operatorname{rad} W$*, where $A = (i..n) \setminus C_{i,n}(\lambda)$;*

(iii) $B_{i,n}(\lambda) \downarrow C_{i,n}(\lambda)$*;*

(iv) $i \in R_{\text{normal}}(\lambda)$*.*

Proof. (iii)$\Leftrightarrow$(iv) is just the definition of $R_{\text{normal}}(\lambda)$.

(i)$\Rightarrow$(ii). Let $f : W \to \nabla_{n-1}(\lambda(i))$ be a non-zero $U(n-1)$-homomorphism, as constructed in the proof of Corollary 3.20. By Corollary 3.20, f is unique up to a scalar. By definition, i is normal if and only if $\operatorname{rad} W \subseteq \ker f$. Suppose that (ii) is false. Let $w = S_{i,n}(A).z_\lambda$. By assumption, $w \in \operatorname{rad} W$, so it suffices to show that $w \notin \ker f$.

Suppose $A \subseteq (i..n)$ is such that $F^A_{i,n}.z_\lambda \neq 0$. If $A \neq \varnothing$, then by Lemma 3.13, $F^A_{i,n}.z_\lambda = \pm F^B_{i,n}.z_\lambda$ for some $A \subseteq B \subseteq (i..n)$ such that $F^B_{i,n}.z_\lambda$ is an element of the standard basis for W. But this lies in some factor strictly lower than the factor corresponding to $\lambda(i)$ in the filtration of Theorem 3.19(i). Hence, $f(F^A_{i,n}.z_\lambda) = 0$ in this case, and by Theorem 3.19(ii), $f(F_{i,n}.z_\lambda)$ must therefore be non-zero (this also follows from the explicit construction of f in Corollary 3.20).

The previous paragraph now implies that $w \in \ker f$ if and only if the coefficient of $\hat{F}_{i,n}.z_\lambda$ is zero when w is written in terms of the standard basis. But by Lemma 3.13 and the definition of $S_{i,j}(A)$, this coefficient is $H_{i,j}(A, \varnothing) = \prod_{t \in A} c_{i,t}(\lambda)$, which is non-zero by definition of $C_{i,n}(\lambda)$.

(ii)$\Rightarrow$(iii). This is immediate from Lemma 5.2.

(iii)$\Rightarrow$(i). Suppose $B_{i,n}(\lambda) \downarrow C_{i,n}(\lambda)$. By (2.2), we can find a subset $C \subseteq C_{i,n}(\lambda)$ such that $|C| = |B_{i,n}(\lambda)|$ and $B_{i,n}(\lambda) \downarrow C$. Let $A = (i..n) \setminus C$ and $w = S_{i,n}(A).z_\lambda$. We first show that $w \notin \operatorname{rad} W$. By definition of $\downarrow$, we can find a weakly increasing bijection $\theta : C \to B_{i,n}(\lambda)$. In particular, this implies that $i \notin B_{i,n}(\lambda)$. Note also that $b_{d,\theta d}(\lambda) = b_{i,\theta d}(\lambda) - c_{i,d}(\lambda) = 0$. So, by Lemma 4.13(ii),

$$K_{i,n}(A).z_\lambda = \prod_{t \in [i..n) \setminus B_{i,n}(\lambda)} b_{i,t}(\lambda).z_\lambda$$

which is non-zero by definition of $B_{i,n}(\lambda)$. Hence, $E_i \dots E_{n-1} S_{i,n}(A).z_\lambda$ is a non-zero scalar multiple of z_λ by Lemma 4.15, so $w = S_{i,n}(A).z_\lambda \notin \operatorname{rad} W$.

To complete the proof, we show that $E_l^{(r)}.w \in \operatorname{rad} W$ for all $1 \leq l < n-1$ and all $r \geq 1$. This suffices, for then, $w + \operatorname{rad} W$ is a non-zero $U(n-1)$-high weight vector in $W / \operatorname{rad} W$ of weight $\lambda(i)$, hence $\operatorname{Hom}_{U(n-1)}(\Delta_{n-1}(\lambda(i)), L_n(\lambda) \downarrow_{U(n-1)}) \neq 0$ by the universal property of standard modules.

First, note that by considering weights $E_l^{(r)}.w = 0$ for all $r > 1$. So it remains to show that $E_l.w \in \operatorname{rad} W$ for all $1 \leq l < n-1$. If $l+1 \in A$, or $l \notin \{i\} \cup A, l+1 \notin A$, then $E_l.w = E_l S_{i,n}(A).z_\lambda = 0$ by Lemma 4.11(i). So suppose that $l \in \{i\} \cup A$ and $l+1 \notin A$. Then, by Lemma 4.11(ii), $E_l.w = -\bar{v} S_{i,l}(A_{i..l}) S_{l+1,n}(A_{l+1..n}).z_\lambda$. It suffices to show that either $S_{i,l}(A_{i..l}).z_\lambda$ or $S_{l+1,n}(A_{l+1..n}).z_\lambda$ lies in $\operatorname{rad} W$, since these two operators commute. Note the assumptions on l imply that $c_{i,l+1}(\lambda) = 0$, so $b_{i,l}(\lambda) \neq 0$, $C_{l+1..n} \subseteq C_{l+1,n}(\lambda)$ and $B_{i,n}(\lambda) = B_{i,l}(\lambda) \cup B_{l+1,n}(\lambda)$.

Suppose that $S_{i,l}(A_{i..l}).z_\lambda \notin \operatorname{rad} W$. By Lemma 5.2, $B_{i,l}(\lambda) \downarrow C_{i..l}$ hence $|B_{i,l}(\lambda)| \leq |C_{i..l}|$. Hence, $|B_{l+1,n}(\lambda)| \geq |C_{l..n}|$, so $|B_{l+1,n}(\lambda)| > |C_{l+1..n}|$. But this implies that $B_{l+1,n}(\lambda) \downarrow C_{l+1..n}$ is false. Hence, $S_{l+1,n}(A_{l+1..n}).z_\lambda \in \operatorname{rad} W$ by Lemma 5.2 again, as required.

The argument in the next theorem is almost identical to Kleshchev's original proof in [23, Theorem 4.11].

5.4 Theorem (Criterion for good nodes) *Let $i \in R(\lambda)$. Then, i is good if and only if i is normal and $c_{h,i}(\lambda) \neq 0$ for all normal $h < i$.*

Proof. Let $i \in R(\lambda)$ be normal. Then, we can find a vector $v(i) \in L_n(\lambda)$, unique up to scalars, such that $v(i)$ is a high weight vector for $U(n-1)$ of weight $\lambda(i)$. We shall prove that $U(n-1).v(i)$ is reducible if and only if $c_{h,i}(\lambda) = 0$ for some normal $h < i$. The theorem follows easily from this by definition of good.

($\Leftarrow$). Let $h < i$ be normal with $c_{h,i}(\lambda) = 0$. Then,

$$C_{h,n}(\lambda) = C_{h,i}(\lambda) \cup \{i\} \cup C_{i,n}(\lambda),$$
$$B_{h,n}(\lambda) = B_{h,i}(\lambda) \cup B_{i,n}(\lambda).$$

By Theorem 5.3 and the definition of $\downarrow$, we can find a weakly decreasing injection $\theta : B_{h,n}(\lambda) \hookrightarrow C_{h,n}(\lambda)$ such that $\theta(B_{i,n}(\lambda)) \subseteq C_{i,n}(\lambda)$. Let $C = \operatorname{im}\theta$. Then, $i \notin C$ and $|C_{h..i}| = |B_{h,i}(\lambda)|, |C_{i..n}| = |B_{i,n}(\lambda)|$.

Let $A = (h..n) \setminus C$ and $k = \max[h..i) \setminus A$. Note $c_{k,i}(\lambda) = 0$ so by Lemma 4.9,

$$S_{h,n}(A).z_\lambda = \delta_{k \neq i} S_{h,n}(\{k\} \cup A \setminus \{i\}).z_\lambda + S_{h,i}(A_{h..i}) S_{i,n}(A_{i..n}).z_\lambda.$$

If $k = i$, the first term on the right hand side is zero. If $k \neq i$ then $|B_{h,i}(\lambda)|$ is greater than $|C_{h..i} \setminus \{k\}|$ so that $B_{h,n}(\lambda) \downarrow \{i\} \cup C \setminus \{k\}$ is false. So by Lemma 5.2, the first term on the right hand side lies in $\operatorname{rad} W$.

By the proof of Theorem 5.3, $v(h) := S_{h,n}(A).z_\lambda + \operatorname{rad} W$ and $v(i) := S_{i,n}(A_{i..n}).z_\lambda + \operatorname{rad} W$ are $U(n-1)$-high weight vectors in $L_n(\lambda)$ of weights $\lambda(h), \lambda(i)$ respectively. We have just shown that $v(h) = S_{h,i}(A_{h..i}).v(i)$. Hence, $v(h) \in U(n-1).v(i)$ and $U(n-1).v(i) \subseteq L_n(\lambda)$ is reducible.

($\Rightarrow$). Suppose that $U(n-1).v(i)$ is reducible. Then for some weight μ,

$$\operatorname{Hom}_{U(n-1)}(L_{n-1}(\mu), U(n-1).v(i))$$

is non-zero for some $\mu < \lambda(i)$ in the dominance order. By Theorem 5.3, we must have that $\mu = \lambda(h)$ for some normal h, with $h < i$ since $\lambda(h) < \lambda(i)$. This implies that $L_{n-1}(\lambda(h))$ is a composition factor of $\Delta_{n-1}(\lambda(i))$, hence that $\lambda(h)$ and $\lambda(i)$ are in the same block.

Now we appeal to the block structure of the q-Schur algebra $S(n-1, r-1)$ from [10, 6.7] (if $r - 1 \leq n - 1$) or [7] (in the general case), to deduce that, as $\lambda(i), \lambda(h)$ are in the same block, $\lambda(i)$ and $\lambda(h)$ have the same residue content. Since the diagrams only differ by one node, this implies that $\mathrm{res}_\ell(i, \lambda_i) \equiv \mathrm{res}_\ell(h, \lambda_h) \pmod{\ell}$. Hence, $c_{h,i}(\lambda) = 0$, as required.

5.5 It just remains to deduce Theorem 2.5 and Theorem 2.6 from these two theorems. The argument here is due to Kleshchev in [22], and generalizes easily to the quantum case. The argument depends on two functors defined by J. A. Green in [14, Chapter 6]; all the facts about these functors that we use below are well-known generalizations of this classical case.

Fix $\lambda \vdash r$ as in (2.1), and choose n so that $n \geq r$. Let $S(n, r)$ denote the q-Schur algebra over $\mathbb{F}$. Let $e \in S(n, r)$ be the idempotent corresponding to the partition (1^r). Let $S(n, r)$-mod and $\mathcal{H}(r)$-mod denote the categories of (left) finite dimensional $S(n, r)$-modules and $\mathcal{H}(r)$-modules respectively. It is known that $eS(n, r)e \cong \mathcal{H}(r)$. So, we can define functors $f_{n,r} : S(n, r)\text{-mod} \to \mathcal{H}(r)\text{-mod}$ and $h_{n,r} : \mathcal{H}(r)\text{-mod} \to S(n, r)\text{-mod}$ by $f_{n,r} : V \mapsto eV$ and $h_{n,r} : W \mapsto S(n, r)e \otimes_{eS(n,r)e} W$, for $V \in S(n, r)\text{-mod}, W \in \mathcal{H}(r)\text{-mod}$ respectively, with the obvious definitions on morphisms. The functor $f_{n,r}$ is exact, and $h_{n,r}$ is left adjoint to $f_{n,r}$.

For any $S(n, r)$-module V, define $V_{(e)}$ to be the sum of all $S(n, r)$-submodules of V annihilated by e and $V^{(e)}$ to be the intersection of all $S(n, r)$-submodules with quotient annihilated by e.

5.6 Lemma (i) *Given $V \in S(n, r)$-mod, the natural map $h_{n,r} \circ f_{n,r}(V) \to V$ has image $V^{(e)}$ and kernel contained in $(h_{n,r} \circ f_{n,r}(V))_{(e)}$.*

(ii) *If $V, W \in S(n, r)$-mod are such that $W^{(e)} = W, V_{(e)} = 0$, then*

$$\mathrm{Hom}_{S(n,r)}(W, V) \cong \mathrm{Hom}_{\mathcal{H}(r)}(f_{n,r}W, f_{n,r}V).$$

Proof. (i) is [20, 2.11(ii)]. For (ii), note that by left adjointness,

$$\mathrm{Hom}_{\mathcal{H}(r)}(f_{n,r}W, f_{n,r}V) = \mathrm{Hom}_{eS(n,r)e}(eW, eV) \cong \mathrm{Hom}_{S(n,r)}(S(n, r)e \otimes_{eS(n,r)e} eW, V).$$

By part (i) and the fact that $W^{(e)} = W$, the module $\bar{W} = S(n, r)e \otimes_{eS(n,r)e} eW$ is an extension of some $S(n, r)$-module K with $eK = 0$ by W. The assumption that $V_{(e)} = 0$ implies that any $S(n, r)$-homomorphism from $\bar{W}$ to V annihilates K, so factors through W. So, $\mathrm{Hom}_{S(n,r)}(\bar{W}, V) \cong \mathrm{Hom}_{S(n,r)}(W, V)$ and the lemma follows.

We also need the following observation, which also has a more conceptual proof; see [21, Theorem B].

5.7 Lemma *Let $V = L_n(\lambda)$, and let V^1 denote the first level of V. If λ is ℓ-restricted, then the socle of the restriction of V^1 to $U(n - 1)$ is ℓ-restricted.*

Proof. By Theorem 5.4, the socle is precisely $\bigoplus_{i \in R_{\mathrm{good}}(\lambda)} L_{n-1}(\lambda(i))$. So, it suffices to show that if λ is ℓ-restricted and $i \in R_{\mathrm{good}}(\lambda)$, then $\lambda(i)$ is also ℓ-restricted. Suppose not, and choose an ℓ-restricted λ and $i \in R_{\mathrm{good}}(\lambda)$ such that $\lambda(i)$ is not ℓ-restricted. Then, $i > 1$ and $\lambda_{i-1} - \lambda_i = \ell - 1$. But this implies that $\mathrm{res}_\ell(i-1, \lambda_{i-1}) = \mathrm{res}_\ell(i, \lambda_i)$, hence that $i-1$ is also normal. But now this contradicts the minimality of i in the definition of good.

Now we can deduce Theorem 2.5 from Theorem 5.3. Let λ, μ be as in Theorem 2.5. Let $W = \Delta_{n-1}(\mu), V = L_n(\lambda)$. Let V^1 denote the first level of V, and note that V^1 is naturally a module for $S(n-1, r-1)$. We use the known facts that $f_{n,r}(\Delta_n(\lambda)) \cong S^*_{\lambda'} \cong (S_\lambda)^\#$ and $f_{n,r}(L_n(\lambda)) \cong (D_\lambda)^\#$, where $\#$ is the involution of (2.9). An argument involving weights shows that $f_{n-1,r-1}(V^1) = f_{n,r}(V) \downarrow_{\mathcal{H}(r-1)}$. Hence, $f_{n-1,r-1}(V^1) \cong (D_\lambda)^\# \downarrow_{\mathcal{H}(r-1)}$. So,

$$\begin{aligned} \mathrm{Hom}_{\mathcal{H}(r-1)}(S_\mu, D_\lambda \downarrow_{\mathcal{H}(r-1)}) &\cong \mathrm{Hom}_{\mathcal{H}(r-1)}((S_\mu)^\#, (D_\lambda)^\# \downarrow_{\mathcal{H}(r-1)}) \\ &\cong \mathrm{Hom}_{\mathcal{H}(r-1)}(f_{n-1,r-1}(W), f_{n-1,r-1}(V^1)) \\ &\cong \mathrm{Hom}_{S(n-1,r-1)}(W, V^1) \\ &\cong \mathrm{Hom}_{U(n-1)}(W, V \downarrow_{U(n-1)}), \end{aligned}$$

using Lemma 5.6 and the fact that the head of W and the socle of V^1 are both ℓ-restricted (the latter being Lemma 5.7). Theorem 2.5 now follows directly from Theorem 5.3. The deduction of Theorem 2.6 from Theorem 5.4 is entirely similar. This completes the proof of the branching rules.

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