KAZHDAN-LUSZTIG POLYNOMIALS AND CHARACTER FORMULAE FOR THE LIE SUPERALGEBRA $\mathfrak{q}(n)$

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1. Introduction

The problem of computing the characters of the finite dimensional irreducible representations for the classical Lie superalgebras was posed originally by Kac in 1977 [K1, K2, K3]. For the family $\mathfrak{q}(n)$, various special cases were treated by Sergeev [S1] (polynomial representations) and Penkov [P1, P3] (typical then generic representations), culminating in the complete solution of Kac' problem for $\mathfrak{q}(n)$ in the work of Penkov and Serganova [PS2, PS3] in 1996. In this article, we will explain a different approach to computing the characters of the irreducible "integrable" representations of $\mathfrak{q}(n)$, i.e. the representations that lift to the supergroup $Q(n)$.

The strategy followed runs parallel to our recent work [B1] on representations of $GL(m|n)$, and is in keeping with the Lascoux-Leclerc-Thibon philosophy [LLT]. We first study the canonical basis of the representation

$$\mathscr{F}^n := \bigwedge^n \mathscr{V},$$

where $\mathscr{V}$ denotes the natural representation of the quantized enveloping algebra $U_q(\mathfrak{b}_\infty)$. This provides a natural Lie theoretic framework for the combinatorics associated to the representation theory of $Q(n)$. The idea that $\mathfrak{b}_\infty$ should be relevant here is already apparent from [LT, BK1]. Our main theorem shows that the transition matrix between the canonical basis and the natural monomial basis of $\mathscr{F}^n$ at $q = 1$ is transpose to the transition matrix between the bases for the Grothendieck group of finite dimensional representations of $Q(n)$ given by certain Euler characteristics and by the irreducible representations.

In order to define the canonical basis of $\mathscr{F}^n$, we must also consider the tensor space

$$\mathscr{T}^n := \bigotimes^n \mathscr{V}.$$

Work of Lusztig [L2, ch. 27] shows how to construct a canonical basis for $\mathscr{T}^n$. We then pass from there to the space $\mathscr{F}^n$, which we realize as a quotient of $\mathscr{T}^n$ following Jing, Misra and Okado [JMO]. The entries of the transition matrix between the canonical basis and the natural monomial basis of $\mathscr{T}^n$ should be viewed as the combinatorial analogues for the Lie superalgebra $\mathfrak{q}(n)$ of the Kazhdan-Lusztig polynomials of [KL]. We conjecture that these polynomials evaluated at $q = 1$ compute the composition multiplicities of the Verma modules in the analogue of category $\mathcal{O}$ for $\mathfrak{q}(n)$, see §4-h for a precise statement.

Work partially supported by the NSF (grant no. DMS-0139019).

We now state our main result precisely. Let $\mathbb{Z}^n_+$ denote the set of all tuples $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ such that $\lambda_1 \geq \dots \geq \lambda_n$ and moreover $\lambda_r = \lambda_{r+1}$ implies $\lambda_r = 0$ for each $r = 1, \dots, n-1$. For $\lambda \in \mathbb{Z}^n_+$, let $z(\lambda)$ denote the number of λ_r ($r = 1, \dots, n$) that equal zero. Also let δ_r denote the n-tuple with rth entry equal to 1 and all other entries equal to zero. Given $\lambda \in \mathbb{Z}^n_+$, there is an irreducible representation $L(\lambda)$ of $Q(n)$ of highest weight λ, unique up to isomorphism. Let L_λ denote the character of $L(\lambda)$, giving us a canonical basis $\{L_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ for the character ring of $Q(n)$, see §4-a. There is another basis denoted $\{E_\mu\}_{\mu \in \mathbb{Z}^n_+}$ which arises naturally from certain Euler characteristics, see §4-b. We can write

$$E_\mu = \sum_{\lambda \in \mathbb{Z}^n_+} d_{\mu,\lambda} L_\lambda$$

for coefficients $d_{\mu,\lambda} \in \mathbb{Z}$, where $d_{\mu,\mu} = 1$ and $d_{\mu,\lambda} = 0$ for $\lambda \not\leq \mu$ in the dominance ordering. The E_μ's are explicitly known: they are multiples of the symmetric functions known as *Schur's P-functions*. So the problem of computing L_λ for each $\lambda \in \mathbb{Z}^n_+$ is equivalent to determining the decomposition numbers $d_{\mu,\lambda}$ for each $\lambda, \mu \in \mathbb{Z}^n_+$.

Main Theorem. *Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n_+$. Choose p to be maximal such that there exist $1 \leq r_1 < \dots < r_p < s_p < \dots < s_1 \leq n$ with $\lambda_{r_q} + \lambda_{s_q} = 0$ for all $q = 1, \dots, p$. Let $I_0 = \{|\lambda_1|, \dots, |\lambda_n|\}$. For $q = 1, \dots, p$, define I_q and k_q inductively according to the following rules:*

(1) *if $\lambda_{r_q} > 0$, let k_q be the smallest positive integer with $\lambda_{r_q} + k_q \notin I_{q-1}$, and set $I_q = I_{q-1} \cup \{\lambda_{r_q} + k_q\}$;*
(2) *if $\lambda_{r_q} = 0$, let k_q and k'_q be the smallest positive integers with $k_q, k'_q \notin I_{q-1}$, $k_q < k'_q$ if $z(\lambda)$ is even and $k_q > k'_q$ if $z(\lambda)$ is odd, and set $I_q = I_{q-1} \cup \{k_q, k'_q\}$.*

Finally, for each $\theta = (\theta_1, \dots, \theta_p) \in \{0,1\}^p$, let $\mathtt{R}_\theta(\lambda)$ denote the unique element of $\mathbb{Z}^n_+$ that is conjugate to $\lambda + \sum_{q=1}^p \theta_q k_q (\delta_{r_q} - \delta_{s_q})$. Then,

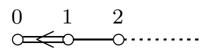
$$d_{\mu,\lambda} = \begin{cases} 2^{(z(\lambda)-z(\mu))/2} & \text{if } \mu = \mathtt{R}_\theta(\lambda) \text{ for some } \theta = (\theta_1, \dots, \theta_p) \in \{0,1\}^p, \\ 0 & \text{otherwise.} \end{cases}$$

The remainder of the article is organized as follows. In §2, we introduce the quantum group of type $\mathfrak{b}_\infty$, and construct the canonical basis of the tensor space $\mathscr{T}^n$. In §3, we pass from $\mathscr{T}^n$ to the quotient $\mathscr{F}^n$, and study its canonical basis. This time it turns out to be quite easy to compute explicitly. Finally in §4, we prove the character formulae. Note there is one difficult place in our proof when we need to appeal to the existence of certain homomorphisms between Verma modules, see Lemma 4.36. For this we appeal to the earlier work of Penkov and Serganova [PS2, Proposition 2.1], which in turn relies upon a special case of Penkov's generic character formula [P3, Corollary 2.2]. It would be nice to find an independent proof of this fact.

Notation. Generally speaking, indices i, j, k will run over the natural numbers $\mathbb{N} = \{0, 1, 2, \dots\}$, indices a, b, c will run over all of $\mathbb{Z}$, and indices r, s, t will run over the set $\{1, 2, \dots, n\}$ where n is a fixed positive integer.

2. Tensor algebra

§2-a. **Quantum group of type $\mathfrak{b}_\infty$.** Let P be the free abelian group on basis $\varepsilon_1, \varepsilon_2, \varepsilon_3, \dots$, equipped with a symmetric bilinear form $(.,.)$ defined by $(\varepsilon_i, \varepsilon_j) = 2\delta_{i,j}$ for all $i, j \geq 1$. Inside P, we have the root system $\{\pm\varepsilon_i, \pm\varepsilon_i \pm \varepsilon_j \,|\, 1 \leq i < j\}$ of type $\mathfrak{b}_\infty$. We use the following labeling for the Dynkin diagram:



We take the simple roots $\alpha_0, \alpha_1, \dots$ defined from

$$\alpha_0 = -\varepsilon_1, \qquad \alpha_i = \varepsilon_i - \varepsilon_{i+1} \quad (i \geq 1).$$

This choice induces a dominance ordering $\leq$ on P: $\beta \leq \gamma$ if $\gamma - \beta$ is a sum of simple roots. The Cartan matrix $A = (a_{i,j})_{i,j \geq 0}$ is defined by $a_{i,j} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$.

We will work over the ground field $\mathbb{Q}(q)$, where q is an indeterminate. Let $q_i = q^{(\alpha_i, \alpha_i)/2}$, i.e. $q_0 = q$ and $q_i = q^2$ for $i > 0$. Define the quantum integer

$$[n]_i = \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and the associated quantum factorial $[n]_i^! = [n]_i[n-1]_i \dots [1]_i$. There is a field automorphism $- : \mathbb{Q}(q) \to \mathbb{Q}(q)$ with $\overline{q} = q^{-1}$. We will call an additive map $f : V \to W$ between $\mathbb{Q}(q)$-vector spaces *antilinear* if $f(cv) = \overline{c} f(v)$ for all $c \in \mathbb{Q}(q), v \in V$.

The quantum group $\mathscr{U} = U_q(\mathfrak{b}_\infty)$ is the $\mathbb{Q}(q)$-algebra generated by elements E_i, F_i, K_i $(i \geq 0)$ subject to the relations

$$K_i K_i^{-1} = K_i^{-1} K_i = 1, \qquad K_i K_j = K_j K_i,$$
$$K_i E_j K_i^{-1} = q^{(\alpha_i, \alpha_j)} E_j, \qquad K_i F_j K_i^{-1} = q^{-(\alpha_i, \alpha_j)} F_j,$$
$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i - K_i^{-1}}{q_i - q_i^{-1}},$$
$$\sum_{k=0}^{1-a_{i,j}} (-1)^k E_i^{(k)} E_j E_i^{(1-a_{i,j}-k)} = \sum_{k=0}^{1-a_{i,j}} (-1)^k F_i^{(k)} F_j F_i^{(1-a_{i,j}-k)} = 0 \qquad (i \neq j),$$

where $E_i^{(r)} := E_i^r/[r]_i^!$ and $F_i^{(r)} := F_i^r/[r]_i^!$. Also let

$$\begin{bmatrix} K_i \\ r \end{bmatrix} = \prod_{s=1}^{r} \frac{K_i q_i^{1-s} - K_i^{-1} q_i^{s-1}}{q_i^s - q_i^{-s}}.$$

for each $r \geq 1$.

We regard $\mathscr{U}$ as a Hopf algebra with comultiplication $\Delta : \mathscr{U} \to \mathscr{U} \otimes \mathscr{U}$ defined on generators by

$$\Delta(E_i) = 1 \otimes E_i + E_i \otimes K_i^{-1},$$
$$\Delta(F_i) = K_i \otimes F_i + F_i \otimes 1,$$
$$\Delta(K_i) = K_i \otimes K_i.$$

This is the comultiplication used by Kashiwara [Ka2], which is different from the one in Lusztig's book [L2].

Let us introduce various (anti)automorphisms of $\mathscr{U}$. First, we have the bar involution $-:\mathscr{U}\to\mathscr{U}$, the unique antilinear algebra automorphism such that

$$\overline{E_i}=E_i,\qquad \overline{F_i}=F_i,\qquad \overline{K_i}=K_i^{-1}. \tag{2.1}$$

We will also need the linear algebra antiautomorphisms $\sigma,\tau:\mathscr{U}\to\mathscr{U}$ and the linear algebra automorphism $\omega:\mathscr{U}\to\mathscr{U}$ defined by

$$\sigma(E_i)=E_i,\qquad \sigma(F_i)=F_i,\qquad \sigma(K_i)=K_i^{-1}, \tag{2.2}$$

$$\tau(E_i)=q_iF_iK_i^{-1},\qquad \tau(F_i)=q_i^{-1}K_iE_i,\qquad \tau(K_i)=K_i, \tag{2.3}$$

$$\omega(E_i)=F_i,\qquad \omega(F_i)=E_i,\qquad \omega(K_i)=K_i^{-1}. \tag{2.4}$$

Lemma 2.5. *The maps τ and $-\circ\sigma$ are coalgebra automorphisms, i.e. we have that $\Delta(\varphi(x))=(\varphi\otimes\varphi)(\Delta(x))$ for all $x\in\mathscr{U}$ and either $\varphi=\tau$ or $\varphi=-\circ\sigma$. The map ω is a coalgebra antiautomorphism, i.e. $\Delta(\omega(x))=P((\omega\otimes\omega)(\Delta(x)))$ where P is the twist $x\otimes y\mapsto y\otimes x$.*

We will occasionally need Lusztig's $\mathbb{Z}[q,q^{-1}]$-form $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$ for $\mathscr{U}$. We recall from [L1, §6] that this is the $\mathbb{Z}[q,q^{-1}]$-subalgebra of $\mathscr{U}$ generated by the elements $E_i^{(r)}, F_i^{(r)}, K_i^{\pm 1}$ and $\left[\begin{matrix}K_i\\ r\end{matrix}\right]$ for all $i\geq 0$, $r\geq 1$. It inherits from $\mathscr{U}$ the structure of a Hopf algebra over $\mathbb{Z}[q,q^{-1}]$.

§2-b. **Tensor space.** The natural representation of $\mathscr{U}$ is the $\mathbb{Q}(q)$-vector space $\mathscr{V}$ on basis $\{v_a\}_{a\in\mathbb{Z}}$ with action defined by

$$E_0v_a=\delta_{a,0}(q+q^{-1})v_{-1}+\delta_{a,1}v_0,\qquad E_iv_a=\delta_{a,i+1}v_i+\delta_{a,-i}v_{-i-1},$$

$$F_0v_a=\delta_{a,0}(q+q^{-1})v_1+\delta_{a,-1}v_0,\qquad F_iv_a=\delta_{a,i}v_{i+1}+\delta_{a,-i-1}v_{-i},$$

$$K_0v_a=q^{2\delta_{a,-1}-2\delta_{a,1}}v_a,\qquad K_iv_a=q^{2\delta_{a,i}-2\delta_{a,i+1}+2\delta_{a,-i-1}-2\delta_{a,-i}}v_a,$$

for all $a\in\mathbb{Z}$ and $i\geq 1$, see for example [J2, §5A.2]. Let $\mathscr{T}=\bigoplus_{n\geq 0}\mathscr{T}^n$ be the tensor algebra of $\mathscr{V}$, so $\mathscr{T}^n=\mathscr{V}\otimes\cdots\otimes\mathscr{V}$ (n times) viewed as a $\mathscr{U}$-module in the natural way.

Let $\mathbb{Z}^n$ denote the set of all n-tuples $\lambda=(\lambda_1,\dots,\lambda_n)$ of integers. The symmetric group S_n acts on $\mathbb{Z}^n$ by the rule $w\lambda=(\lambda_{w^{-1}1},\dots,\lambda_{w^{-1}n})$. We will always denote the longest element of S_n by w_0, so $w_0\lambda=(\lambda_n,\dots,\lambda_1)$. Given $\lambda\in\mathbb{Z}^n$, let

$$N_\lambda=v_{\lambda_1}\otimes\cdots\otimes v_{\lambda_n}\in\mathscr{T}^n. \tag{2.6}$$

The vectors $\{N_\lambda\}_{\lambda\in\mathbb{Z}^n}$ obviously give a basis for $\mathscr{T}^n$. For $\lambda\in\mathbb{Z}^n$, let $z(\lambda)$ denote the number of λ_r ($r=1,\dots,n$) that equal zero. We get another basis $\{M_\lambda\}_{\lambda\in\mathbb{Z}^n}$ for $\mathscr{T}^n$ by defining

$$M_\lambda=\left(q+q^{-1}\right)^{-z(\lambda)}N_\lambda \tag{2.7}$$

for each $\lambda\in\mathbb{Z}^n$.

Let $(.,.)$ be the symmetric bilinear form on $\mathscr{T}^n$ such that

$$(M_\lambda, N_\mu) = q^{z(\lambda)}\delta_{\lambda,\mu} \tag{2.8}$$

for all $\lambda, \mu \in \mathbb{Z}^n$. Also define an antilinear automorphism $\sigma : \mathscr{T}^n \to \mathscr{T}^n$ and a linear automorphism $\omega : \mathscr{T}^n \to \mathscr{T}^n$ by

$$\sigma(N_\lambda) = q^{-z(\lambda)}N_{-\lambda}, \tag{2.9}$$

$$\omega(N_\lambda) = N_{-w_0\lambda} \tag{2.10}$$

for all $\lambda \in \mathbb{Z}^n$. The following lemma is checked by reducing to the case $n = 1$ using Lemma 2.5.

Lemma 2.11. *The following hold for all $x \in \mathscr{U}$ and $u, v \in \mathscr{T}^n$:*
(i) $\sigma(xv) = \tau(\overline{\sigma(x)})\sigma(v)$;
(ii) $(xu, v) = (u, \tau(x)v)$;
(iii) $\omega(xu) = \omega(x)\omega(u)$.

§2-c. **Bruhat ordering.** Recall that a vector v in a $\mathscr{U}$-module is said to be of *weight* $\gamma \in P$ if $K_i v = q^{(\alpha_i,\gamma)}v$ for each $i \geq 0$. The weight of the basis vector v_i of $\mathscr{V}$ is ε_i, where we write $\varepsilon_0 = 0$ and $\varepsilon_{-i} = -\varepsilon_i$. Hence the vectors N_λ and M_λ have weight

$$\mathrm{wt}(\lambda) := \varepsilon_{\lambda_1} + \cdots + \varepsilon_{\lambda_n} \in P. \tag{2.12}$$

More generally, for each $r = 1, \dots, n$, let

$$\mathrm{wt}_r(\lambda) := \varepsilon_{\lambda_r} + \cdots + \varepsilon_{\lambda_n} \in P. \tag{2.13}$$

The *Bruhat ordering* $\preceq$ on the set $\mathbb{Z}^n$ is defined by $\lambda \preceq \mu$ if $\mathrm{wt}_r(\lambda) \leq \mathrm{wt}_r(\mu)$ for each $r = 1, \dots, n$ with equality for $r = 1$. Here is an equivalent formulation of the definition. Write $\lambda \downarrow \mu$ if one of the following holds:

(1) for some $1 \leq r < s \leq n$ such that $\lambda_r > \lambda_s$, we have that $\mu_r = \lambda_s, \mu_s = \lambda_r$ and $\mu_t = \lambda_t$ for all $t \neq r, s$;
(2) for some $1 \leq r < s \leq n$ such that $\lambda_r + \lambda_s = 0$, we have that $\mu_r = \lambda_r - 1, \mu_s = \lambda_s + 1$ and $\mu_t = \lambda_t$ for all $t \neq r, s$.

The following lemma has been checked in many examples, but I still have not found a proof in general. However, it is not used in an essential way in the remainder of the article.

Lemma 2.14. *$\lambda \succeq \mu$ if and only if there exist $\nu_1, \dots, \nu_k \in \mathbb{Z}^n$ such that $\lambda = \nu_1 \downarrow \nu_2 \downarrow \dots \downarrow \nu_k = \mu$.*

The *degree of atypicality* of $\lambda \in \mathbb{Z}^n$ is defined by

$$\#\lambda := n - \frac{1}{2}\sum_{i \geq 1} |(\mathrm{wt}(\lambda), \varepsilon_i)|. \tag{2.15}$$

We will say that $\lambda \in \mathbb{Z}^n$ is *typical* if $\#\lambda \leq 1$. Equivalently, λ is typical if $\lambda_r + \lambda_s \neq 0$ for all $1 \leq r < s \leq n$.

§2-d. **Bar involution.** We next define a bar involution on $\mathscr{T}^n$ (actually on some completion $\widehat{\mathscr{T}^n}$ of $\mathscr{T}^n$) compatible with the bar involution on $\mathscr{U}$, following

[L2, §27.3]. To begin with, let us recall the definition of Lusztig's quasi-R-matrix Θ, translated suitably since we are working with a different comultiplication.

Write $\mathscr{U} = \mathscr{U}^-\mathscr{U}^0\mathscr{U}^+ = \mathscr{U}^+\mathscr{U}^0\mathscr{U}^-$ for the usual triangular decompositions of $\mathscr{U}$, so $\mathscr{U}^-$ is generated by all F_i, $\mathscr{U}^0$ is generated by all $K_i^{\pm 1}$ and $\mathscr{U}^+$ is generated by all E_i. For a weight $\nu \geq 0$, let $\mathscr{U}^+_\nu$ (resp. $\mathscr{U}^-_\nu$) denote the subspace of $\mathscr{U}^+$ (resp. $\mathscr{U}^-$) spanned by all monomials $E_{i_1} \dots E_{i_r}$ (resp $F_{i_1} \dots F_{i_r}$) with $\alpha_{i_1} + \cdots + \alpha_{i_r} = \nu$, so

$$\mathscr{U}^+ = \bigoplus_{\nu \geq 0} \mathscr{U}^+_\nu, \qquad \mathscr{U}^- = \bigoplus_{\nu \geq 0} \mathscr{U}^-_\nu.$$

Define $\mathrm{tr}\,\nu := \sum_{i \geq 0} a_i$ if $\nu = \sum_{i \geq 0} a_i \alpha_i$. Let $(\mathscr{U} \otimes \mathscr{U})^\wedge$ denote the completion of the vector space $\mathscr{U} \otimes \mathscr{U}$ with respect to the descending filtration $((\mathscr{U} \otimes \mathscr{U})_d)_{d \in \mathbb{N}}$ where

$$(\mathscr{U} \otimes \mathscr{U})_d = \sum_{\mathrm{tr}\,\nu \geq d} \left(\mathscr{U}^-\mathscr{U}^0\mathscr{U}^+_\nu \otimes \mathscr{U} + \mathscr{U} \otimes \mathscr{U}^+\mathscr{U}^0\mathscr{U}^-_\nu \right).$$

Exactly as in [L2, §4.1.1], we embed $\mathscr{U} \otimes \mathscr{U}$ into $(\mathscr{U} \otimes \mathscr{U})^\wedge$ in the obvious way, then extend the $\mathbb{Q}(q)$-algebra structure on $\mathscr{U} \otimes \mathscr{U}$ to $(\mathscr{U} \otimes \mathscr{U})^\wedge$ by continuity. The bar involution on $\mathscr{U} \otimes \mathscr{U}$ is defined by $\overline{x \otimes y} := \overline{x} \otimes \overline{y}$, and also extends by continuity to $(\mathscr{U} \otimes \mathscr{U})^\wedge$. Finally, the antiautomorphism $\sigma \otimes \sigma : \mathscr{U} \otimes \mathscr{U} \to \mathscr{U} \otimes \mathscr{U}$ and the automorphism $P \circ (\omega \otimes \omega) : \mathscr{U} \otimes \mathscr{U} \to \mathscr{U} \otimes \mathscr{U}$ (where P is the twist $x \otimes y \mapsto y \otimes x$) extend to the completion too.

Lemma 2.16. *There is a unique family of elements $\Theta_\nu \in \mathscr{U}^+_\nu \otimes \mathscr{U}^-_\nu$ such that $\Theta_0 = 1$ and $\Theta := \sum_\nu \Theta_\nu \in (\mathscr{U} \otimes \mathscr{U})^\wedge$ satisfies $\Delta(u)\Theta = \Theta\overline{\Delta(\overline{u})}$ for all $u \in \mathscr{U}$ (identity in $(\mathscr{U} \otimes \mathscr{U})^\wedge$). Moreover, each Θ_ν belongs to $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$.*

Proof. The first part of the lemma is [L2, Theorem 4.1.2(a)]. The second part about integrality follows using [L2, Corollary 24.1.6]. Actually *loc. cit.* applies only to finite type root systems but one can pass from $\mathfrak{b}_n$ to $\mathfrak{b}_\infty$ by a limiting argument. □

Lemma 2.17. *The following equalities hold in $(\mathscr{U} \otimes \mathscr{U})^\wedge$:*

(i) $\Theta\overline{\Theta} = \overline{\Theta}\Theta = 1 \otimes 1$;
(ii) $(\sigma \otimes \sigma)(\Theta) = \Theta$;
(iii) $(\sigma \otimes \sigma)(\overline{\Theta}) = \overline{\Theta}$;
(iv) $(P \circ (\omega \otimes \omega))(\Theta) = \Theta$.

Proof. (i) This is [L2, Corollary 4.1.3].

(ii) Recall by Lemma 2.5 that the map $- \circ \sigma = \sigma \circ -$ is a coalgebra automorphism. Using this one checks that

$$\Delta(u) = (\sigma \otimes \sigma)(\overline{\Delta(\sigma(\overline{u}))}), \qquad \overline{\Delta(\overline{u})} = (\sigma \otimes \sigma)(\Delta(\sigma(u))).$$

Now apply the antiautomorphism $\sigma \otimes \sigma$ to the equality $\Delta(\sigma(u))\Theta = \Theta\overline{\Delta(\sigma(\overline{u}))}$ to get that $(\sigma \otimes \sigma)(\Theta)\overline{\Delta(\overline{u})} = \Delta(u)(\sigma \otimes \sigma)(\Theta)$. Hence, $(\sigma \otimes \sigma)(\Theta) = \Theta$ by the uniqueness in Lemma 2.16.

(iii) Combine (i) and (ii).

(iv) Follows easily from the uniqueness and Lemma 2.5. □

Let $\widehat{\mathscr{T}^n}$ denote the completion of the vector space $\mathscr{T}^n$ with respect to the descending filtration $(\mathscr{T}^n_d)_{d\in\mathbb{Z}}$, where $\mathscr{T}^n_d$ is the subspace of $\mathscr{T}^n$ spanned by $\{N_\lambda\}$ for $\lambda \in \mathbb{Z}^n$ with $\sum_{i=1}^n i\lambda_i \geq d$. We embed $\mathscr{T}^n$ into $\widehat{\mathscr{T}^n}$ in the natural way. Note that $\widehat{\mathscr{T}^n}$ contains all vectors of the form $N_\lambda + (*)$ where $(*)$ is an infinite linear combination of N_μ's with $\mu \prec \lambda$. The action of $\mathscr{U}$ on $\mathscr{T}^n$ extends by continuity to $\widehat{\mathscr{T}^n}$, as does the map $\omega : \mathscr{T}^n \to \mathscr{T}^n$.

Now we are ready to inductively define the bar involution on $\widehat{\mathscr{T}^n}$. It will turn out to satisfy the following properties:

(1) $- : \widehat{\mathscr{T}^n} \to \widehat{\mathscr{T}^n}$ is a continuous, antilinear involution;
(2) $\overline{xv} = \overline{x}\,\overline{v}$ for all $x \in \mathscr{U}, v \in \widehat{\mathscr{T}^n}$;
(3) $\overline{N_\lambda} \in N_\lambda + \widehat{\sum}_{\mu\prec\lambda}\mathbb{Z}[q,q^{-1}]N_\mu$ and $\overline{M_\lambda} \in M_\lambda + \widehat{\sum}_{\mu\prec\lambda}\mathbb{Z}[q,q^{-1}]M_\mu$ for all $\lambda \in \mathbb{Z}^n$;
(4) $\overline{\omega(v)} = \omega(\overline{v})$ for all $v \in \widehat{\mathscr{T}^n}$.

If $n = 1$, the bar involution is defined by setting $\overline{v_a} = v_a$ for each $a \in \mathbb{Z}$, then extending by continuity. The properties (1)–(4) in this case are easy to check directly. Now suppose that $n > 1$, write $n = n_1 + n_2$ for some $n_1, n_2 \geq 1$, and assume we have already constructed bar involutions on $\widehat{\mathscr{T}^{n_1}}$ and $\widehat{\mathscr{T}^{n_2}}$ satisfying properties (1)–(4). Because of the way the completion $\widehat{\mathscr{T}^n}$ is defined, multiplication by Θ gives a linear map

$$\Theta_{n_1,n_2} : \widehat{\mathscr{T}^{n_1}} \otimes \widehat{\mathscr{T}^{n_2}} \to \widehat{\mathscr{T}^n}. \tag{2.18}$$

Given $v \in \mathscr{T}^{n_1}, w \in \mathscr{T}^{n_2}$, let $\overline{v \otimes w} := \Theta_{n_1,n_2}(\overline{v} \otimes \overline{w})$, defining an antilinear map $- : \mathscr{T}^n \to \widehat{\mathscr{T}^n}$. The explicit form of Θ from Lemma 2.16 combined with the inductive hypothesis implies that

$$\overline{N_\lambda} = N_\lambda + (\text{a possibly infinite } \mathbb{Q}(q)\text{-linear combination of } N_\mu\text{'s with } \mu \prec \lambda),$$
$$\overline{M_\lambda} = M_\lambda + (\text{a possibly infinite } \mathbb{Q}(q)\text{-linear combination of } M_\mu\text{'s with } \mu \prec \lambda).$$

Each Θ_ν belongs to $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$ by Lemma 2.16, and $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$ leaves the $\mathbb{Z}[q,q^{-1}]$-lattices in $\widehat{\mathscr{T}^n}$ generated by either the N_λ's or the M_λ's invariant. Hence, the coefficients actually all lie in $\mathbb{Z}[q,q^{-1}]$, so property (3) holds. Now property (3) immediately implies that bar is continuous, so it extends uniquely to a continuous antilinear map $- : \widehat{\mathscr{T}^n} \to \widehat{\mathscr{T}^n}$ still satisfying (3). The argument in [L2, Lemma 24.1.2] shows that property (2) is satisfied. Lemma 2.17(i) gives that bar is an involution, whence property (1) holds, while property (4) follows from Lemma 2.17(iv). Note finally that as in [L2, §27.3.6], the definition is independent of the initial choices of n_1, n_2.

Example 2.19. The bar involution on $\widehat{\mathscr{T}^2}$ is uniquely determined by the following formulae.

$$\overline{v_a \otimes v_b} = v_a \otimes v_b \qquad (a \leq b, a + b \neq 0)$$
$$\overline{v_a \otimes v_b} = v_a \otimes v_b + (q^2 - q^{-2})v_b \otimes v_a \qquad (a > b, a + b \neq 0)$$

$$\begin{aligned}
\overline{v_0 \otimes v_0} &= v_0 \otimes v_0 + \sum_{b<0} (q+q^{-1})(q^2-q^{-2})(-q^2)^{b+1} v_b \otimes v_{-b} \\
\overline{v_{-a} \otimes v_a} &= v_{-a} \otimes v_a + \sum_{b>a} (q^2-q^{-2})(-q^2)^{a+1-b} v_{-b} \otimes v_b \qquad (a \geq 1) \\
\overline{v_a \otimes v_{-a}} &= v_a \otimes v_{-a} + q^2(q^2-q^{-2}) v_{-a} \otimes v_a \qquad (a \geq 1) \\
&\quad + \sum_{0<b<a} (q^2-q^{-2})(-q^2)^{b+1-a} v_b \otimes v_{-b} \\
&\quad + (q-q^{-1})(-q^2)^{1-a} v_0 \otimes v_0 \\
&\quad + \sum_{b<0} q^2(q^2-q^{-2})(-q^2)^{b+1-a} v_b \otimes v_{-b}
\end{aligned}$$

Now that the bar involution has been defined on $\widehat{\mathscr{T}}^n$, we can define a new bilinear form $\langle .,. \rangle$ on $\widehat{\mathscr{T}}^n$ by setting

$$\langle u, v \rangle = (u, \sigma(\overline{v})) \tag{2.20}$$

for all $u, v \in \widehat{\mathscr{T}}^n$, where $(.,.)$ and σ are as in Lemma 2.11. Note this makes sense even though the map σ and the form $(.,.)$ are not defined on the completion.

Lemma 2.21. *$\langle .,. \rangle$ is a symmetric bilinear form with $\langle xu, v \rangle = \langle u, \sigma(x)v \rangle$ for all $x \in \mathscr{U}, u, v \in \widehat{\mathscr{T}}^n$.*

Proof. For the second part of the lemma, we calculate using Lemma 2.11 to get that

$$\begin{aligned}
\langle xu, v \rangle = (xu, \sigma(\overline{v})) = (u, \tau(x)\sigma(\overline{v})) &= (u, \tau(\overline{\sigma(\overline{\sigma(x)})})\sigma(\overline{v})) \\
= (u, \sigma(\overline{\sigma(x)}\overline{v})) = (u, \sigma(\overline{\sigma(x)v})) &= \langle u, \sigma(x)v \rangle.
\end{aligned}$$

Now let us show by induction on n that $\langle .,. \rangle$ is a symmetric bilinear form, this being obvious in case $n = 1$. For $n > 1$, write $n = n_1 + n_2$ for $n_1, n_2 \geq 1$. Take $u_1 \otimes u_2, v_1 \otimes v_2 \in \mathscr{T}^{n_1} \otimes \mathscr{T}^{n_2}$. Write $\Theta = \sum_{i \in I} x_i \otimes y_i \in (\mathscr{U} \otimes \mathscr{U})^\wedge$. Recall by Lemma 2.17(iii) that

$$\sum_{i \in I} \sigma(\overline{x_i}) \otimes \sigma(\overline{y_i}) = \sum_{i \in I} \overline{x_i} \otimes \overline{y_i}.$$

Combining this with the inductive hypothesis, we calculate from the definition of the bar involution on $\widehat{\mathscr{T}}^n$:

$$\begin{aligned}
\langle u_1 \otimes u_2, v_1 \otimes v_2 \rangle = (u_1 \otimes u_2, \sigma(\Theta_{n_1,n_2}(\overline{v_1} \otimes \overline{v_2}))) &= \sum_{i \in I} (u_1 \otimes u_2, \sigma(x_i \overline{v_1} \otimes y_i \overline{v_2})) \\
= \sum_{i \in I} (u_1, \sigma(x_i \overline{v_1}))(u_2, \sigma(y_i \overline{v_2})) &= \sum_{i \in I} \langle u_1, \overline{x_i} v_1 \rangle \langle u_2, \overline{y_i} v_2 \rangle \\
= \sum_{i \in I} \langle \overline{x_i} v_1, u_1 \rangle \langle \overline{y_i} v_2, u_2 \rangle &= \sum_{i \in I} \langle v_1, \sigma(\overline{x_i}) u_1 \rangle \langle v_2, \sigma(\overline{y_i}) u_2 \rangle \\
= \sum_{i \in I} \langle v_1, \overline{x_i} u_1 \rangle \langle v_2, \overline{y_i} u_2 \rangle &= \langle v_1 \otimes v_2, u_1 \otimes u_2 \rangle.
\end{aligned}$$

Hence $\langle .,. \rangle$ is symmetric. $\square$

§2-e. **Canonical basis of $\widehat{\mathscr{T}^n}$.** Now that we have constructed the bar involution on $\widehat{\mathscr{T}^n}$ satisfying the properties (1)–(4) above, we get the following theorem by general principles, cf. the proof of [B1, Theorem 2.17].

Theorem 2.22. *There exist unique topological bases $\{T_\lambda\}_{\lambda\in\mathbb{Z}^n}$ and $\{L_\lambda\}_{\lambda\in\mathbb{Z}^n}$ for $\widehat{\mathscr{T}^n}$ such that*

(i) $\overline{T_\lambda} = T_\lambda$ *and* $\overline{L_\lambda} = L_\lambda$;

(ii) $\overline{T_\lambda} \in N_\lambda + \widehat{\sum}_{\mu\in\mathbb{Z}^n} q\mathbb{Z}[q]N_\mu$ *and* $\overline{L_\lambda} \in M_\lambda + \widehat{\sum}_{\mu\in\mathbb{Z}^n} q^{-1}\mathbb{Z}[q^{-1}]M_\mu$,

for all $\lambda \in \mathbb{Z}^n$. Actually, we have that $\overline{T_\lambda} \in N_\lambda + \widehat{\sum}_{\mu\prec\lambda} q\mathbb{Z}[q]N_\mu$ and that $\overline{L_\lambda} \in M_\lambda + \widehat{\sum}_{\mu\prec\lambda} q^{-1}\mathbb{Z}[q^{-1}]M_\mu$. Also, $\omega(T_\lambda) = T_{-w_0\lambda}$ and $\omega(L_\lambda) = L_{-w_0\lambda}$.

Example 2.23. Suppose that $n = 2$. Using Example 2.19, one checks:

$$
\begin{aligned}
T_{(a,b)} &= N_{(a,b)} && (a \le b, a+b \ne 0)\\
T_{(a,b)} &= N_{(a,b)} + q^2 N_{(b,a)} && (a > b, a+b \ne 0)\\
T_{(-a,a)} &= N_{(-a,a)} + q^2 N_{(-a-1,a+1)} && (a \ge 1)\\
T_{(a,-a)} &= N_{(a,-a)} + q^2(N_{(a-1,1-a)} + N_{(1-a,a-1)}) + q^4 N_{(-a,a)} && (a \ge 2)\\
T_{(0,0)} &= N_{(0,0)} + (q+q^3)N_{(-1,1)}\\
T_{(1,-1)} &= N_{(1,-1)} + qN_{(0,0)} + q^4 N_{(-1,1)}
\end{aligned}
$$

Note in this example that each T_λ is a finite sum of N_μ's. I conjecture that this is true in general. On the other hand, the L_λ's need not be finite sums of M_μ's even for $n = 2$.

We call the topological basis $\{T_\lambda\}_{\lambda\in\mathbb{Z}^n}$ the *canonical basis* of $\widehat{\mathscr{T}^n}$ and $\{L_\lambda\}_{\lambda\in\mathbb{Z}^n}$ the *dual canonical basis*. Let us introduce notation for the coefficients: let

$$T_\lambda = \sum_{\mu\in\mathbb{Z}^n} t_{\mu,\lambda}(q)N_\mu, \qquad L_\lambda = \sum_{\mu\in\mathbb{Z}^n} l_{\mu,\lambda}(q)M_\mu \tag{2.24}$$

for polynomials $t_{\mu,\lambda}(q) \in \mathbb{Z}[q]$ and $l_{\mu,\lambda}(q) \in \mathbb{Z}[q^{-1}]$. We know that $t_{\mu,\lambda}(q) = l_{\mu,\lambda}(q) = 0$ unless $\mu \preceq \lambda$, and that $t_{\lambda,\lambda}(q) = l_{\lambda,\lambda}(q) = 1$.

Lemma 2.25. *For $\lambda, \mu \in \mathbb{Z}^n$, $\langle L_\lambda, T_{-\mu}\rangle = \delta_{\lambda,\mu}$.*

Proof. A calculation using the definition of $\langle .,. \rangle$ shows that

$$\langle L_\lambda, T_{-\mu}\rangle = \sum_{\mu\preceq\nu\preceq\lambda} l_{\nu,\lambda}(q)t_{-\nu,-\mu}(q^{-1}), \tag{2.26}$$

$$\langle T_{-\mu}, L_\lambda\rangle = \sum_{\mu\preceq\nu\preceq\lambda} l_{\nu,\lambda}(q^{-1})t_{-\nu,-\mu}(q). \tag{2.27}$$

Hence, $\langle L_\lambda, T_{-\mu}\rangle$ equals 1 if $\lambda = \mu$ and belongs to $q^{-1}\mathbb{Z}[q^{-1}]$ if $\lambda \ne \mu$. Similarly, $\langle T_{-\mu}, L_\lambda\rangle$ equals 1 if $\lambda = \mu$ and belongs to $q\mathbb{Z}[q]$ if $\lambda \ne \mu$. But $\langle L_\lambda, T_{-\mu}\rangle = \langle T_{-\mu}, L_\lambda\rangle$ by Lemma 2.21. □

Corollary 2.28. *For $\lambda \in \mathbb{Z}^n$, $M_\lambda = \sum_{\mu\in\mathbb{Z}^n} t_{-\lambda,-\mu}(q^{-1})L_\mu$.*

Proof. By Lemma 2.25, we can write $M_\lambda = \sum_{\mu \in \mathbb{Z}^n} \langle M_\lambda, T_{-\mu} \rangle L_\mu$. Now a calculation from the definition of the form $\langle ., . \rangle$ gives that $\langle M_\lambda, T_{-\mu} \rangle = t_{-\lambda,-\mu}(q^{-1})$. $\square$

Example 2.29. Suppose that $n = 2$. Using Example 2.23, one checks:

$$
\begin{aligned}
M_{(a,b)} &= L_{(a,b)} && (a \leq b, a+b \neq 0)\\
M_{(a,b)} &= L_{(a,b)} + q^{-2} L_{(b,a)} && (a > b, a+b \neq 0)\\
M_{(-a,a)} &= L_{(-a,a)} + q^{-2} L_{(-a-1,a+1)} && (a \geq 1)\\
M_{(a,-a)} &= L_{(a,-a)} + q^{-2}(L_{(a-1,1-a)} + L_{(-a-1,a+1)}) + q^{-4} L_{(-a,a)} && (a \geq 2)\\
M_{(0,0)} &= L_{(0,0)} + q^{-1} L_{(-1,1)}\\
M_{(1,-1)} &= L_{(1,-1)} + (q^{-1} + q^{-3}) L_{(0,0)} + q^{-2} L_{(-2,2)} + q^{-4} L_{(-1,1)}
\end{aligned}
$$

I conjecture for arbitrary n that each M_λ is always a finite linear combination of L_μ's.

§2-f. **Crystal structure.** Now we describe the crystal structure underlying the module $\mathscr{T}^n$. The basic reference followed here is [Ka2]. Let $\mathscr{A}$ be the subring of $\mathbb{Q}(q)$ consisting of rational functions having no pole at $q = 0$. Evaluation at $q = 0$ induces an isomorphism $\mathscr{A}/q\mathscr{A} \to \mathbb{Q}$.

Let $\mathscr{V}_\mathscr{A}$ be the $\mathscr{A}$-lattice in $\mathscr{V}$ spanned by the v_a's. Then, $\mathscr{V}_\mathscr{A}$ together with the basis of the $\mathbb{Q}$-vector space $\mathscr{V}_\mathscr{A}/q\mathscr{V}_\mathscr{A}$ given by the images of the v_a's is a lower crystal base for $\mathscr{V}$ at $q = 0$ in the sense of [Ka2, 4.1]. Write $\tilde{E}_i, \tilde{F}_i$ for the corresponding crystal operators. Rather than view these as operators on the crystal base $\{v_a + q\mathscr{V}_\mathscr{A}\}_{a \in \mathbb{Z}}$, we will view them simply as operators on the set $\mathbb{Z}$ parameterizing the crystal base. Then, the crystal graph is as follows:

$$\cdots \longrightarrow -3 \xrightarrow{\tilde{F}_2} -2 \xrightarrow{\tilde{F}_1} -1 \xrightarrow{\tilde{F}_0} 0 \xrightarrow{\tilde{F}_0} 1 \xrightarrow{\tilde{F}_1} 2 \xrightarrow{\tilde{F}_2} 3 \longrightarrow \cdots.$$

Thus, $\tilde{F}_i(a)$ equals $a+1$ if $a = i$ or $a = -i-1$, $\varnothing$ otherwise, and $\tilde{E}_i(a)$ equals $a-1$ if $a = i+1$ or $a = -i$, $\varnothing$ otherwise. The maps $\varepsilon_i, \varphi_i : \mathbb{Z} \to \mathbb{N}$ defined by

$$\varepsilon_i(a) = \max\{k \geq 0 \mid \tilde{E}_i^k a \neq \varnothing\}, \qquad \varphi_i(a) = \max\{k \geq 0 \mid \tilde{F}_i^k a \neq \varnothing\}$$

only ever take the values $0, 1$ or 2 (the last possibility occurring only if $i = 0$).

Since $\mathscr{T}^n$ is a tensor product of n copies of $\mathscr{V}$, it has an induced crystal structure. The crystal lattice is $\mathscr{T}^n_\mathscr{A}$, namely, the $\mathscr{A}$-span of the basis $\{N_\lambda\}_{\lambda \in \mathbb{Z}^n}$, and the images of the N_λ's in $\mathscr{T}^n_\mathscr{A}/q\mathscr{T}^n_\mathscr{A}$ give the crystal base. Like in the previous paragraph, we will view the crystal operators as maps on the underlying set $\mathbb{Z}^n$ parametrizing the crystal base. However, we will denote them by $\tilde{E}'_i, \tilde{F}'_i$, since we want to reserve the unprimed symbols for something else later on. In order to describe them explicitly, we introduce a little more combinatorial notation. Given $\lambda \in \mathbb{Z}^n$ and $i \geq 0$, let $(\sigma_1, \ldots, \sigma_n)$ be the *i-signature* of λ, namely, the sequence defined by

$$\sigma_r = \begin{cases} + & \text{if } i \neq 0 \text{ and } \lambda_r = i \text{ or } -i-1, \\ - & \text{if } i \neq 0 \text{ and } \lambda_i = i+1 \text{ or } -i, \\ ++ & \text{if } i = 0 \text{ and } \lambda_r = -1, \\ -+ & \text{if } i = 0 \text{ and } \lambda_r = 0, \\ -- & \text{if } i = 0 \text{ and } \lambda_r = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{2.30}$$

Form the *reduced i-signature* by successively replacing subwords of $(\sigma_1, \dots, \sigma_n)$ of the form $+-$ (possibly separated by 0's in between) with 0's until we are left with a sequence of the form $(\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$ in which no $-$ appears after a $+$. For $r = 1, \dots, n$, let δ_r be the n-tuple with a 1 in the rth position and 0's elsewhere. Then,

$$\tilde{E}'_i(\lambda) = \begin{cases} \varnothing & \text{if there are no } -\text{'s in the reduced } i\text{-signature,} \\ \lambda - \delta_r & \text{otherwise, where the rightmost } - \text{ occurs in } \tilde{\sigma}_r, \end{cases}$$

$$\tilde{F}'_i(\lambda) = \begin{cases} \varnothing & \text{if there are no } +\text{'s in the reduced } i\text{-signature,} \\ \lambda + \delta_r & \text{otherwise, where the leftmost } + \text{ occurs in } \tilde{\sigma}_r, \end{cases}$$

$$\varepsilon'_i(\lambda) = \text{the total number of } -\text{'s in the reduced } i\text{-signature,}$$

$$\varphi'_i(\lambda) = \text{the total number of } +\text{'s in the reduced } i\text{-signature.}$$

We have now described the crystal $(\mathbb{Z}^n, \tilde{E}'_i, \tilde{F}'_i, \varepsilon'_i, \varphi'_i, \mathrm{wt})$ associated to the module $\mathscr{T}^n$ purely combinatorially.

Example 2.31. Consider $\lambda = (1, 2, 0, -3, -2, -1, 0, 1) \in \mathbb{Z}^8$. The 1-signature is $(+, -, 0, 0, +, -, 0, +)$. Hence the reduced 1-signature is $(0, 0, 0, 0, 0, 0, 0, +)$, so we get that $\tilde{E}'_1\lambda = \varnothing$ and $\tilde{F}'_1\lambda = \lambda + \delta_8$. On the other hand, the 0-signature is $(--, 0, -+, 0, 0, ++, -+, --)$, which reduces to $(--, 0, -+, 0, 0, 0, 0, 0)$, so $\tilde{E}'_0\lambda = \lambda - \delta_3$, $\tilde{F}'_0\lambda = \lambda + \delta_3$.

The following lemma is a general property of canonical bases/lower global crystal bases. It follows ultimately from [Ka1, Proposition 5.3.1]. See [B1, Theorem 2.31] for a similar situation.

Lemma 2.32. *Let $\lambda \in \mathbb{Z}^n$ and $i \geq 0$.*

(i) $E_i T_\lambda = [\varphi'_i(\lambda) + 1]_i T_{\tilde{E}'_i(\lambda)} + \sum_{\mu \in \mathbb{Z}^n} u^i_{\mu,\lambda} T_\mu$ *where* $u^i_{\mu,\lambda} \in q q_i^{1-\varphi'_i(\mu)}\mathbb{Z}[q]$ *is zero unless* $\varepsilon'_j(\mu) \geq \varepsilon'_j(\lambda)$ *for all* $j \geq 0$.

(ii) $F_i T_\lambda = [\varepsilon'_i(\lambda) + 1]_i T_{\tilde{F}'_i(\lambda)} + \sum_{\mu \in \mathbb{Z}^n} v^i_{\mu,\lambda} T_\mu$ *where* $v^i_{\mu,\lambda} \in q q_i^{1-\varepsilon'_i(\mu)}\mathbb{Z}[q]$ *is zero unless* $\varphi'_j(\mu) \geq \varphi'_j(\lambda)$ *for all* $j \geq 0$.

(In (i) resp. (ii), the first term on the right hand side should be omitted if $\tilde{E}'_i(\lambda)$ resp. $\tilde{F}'_i(\lambda)$ is $\varnothing$.)

Motivated by Lemmas 2.21 and 2.25, we also introduce the *dual crystal operators* defined by

$$\tilde{E}^*_i(\lambda) = -\tilde{F}'_i(-\lambda), \qquad \tilde{F}^*_i(\lambda) = -\tilde{E}'_i(-\lambda), \tag{2.33}$$

$$\varepsilon^*_i(\lambda) = \varphi'_i(-\lambda), \qquad \varphi^*_i(\lambda) = \varepsilon'_i(-\lambda). \tag{2.34}$$

These can be described explicitly in a similar way to the above: for fixed $i \geq 0$ and $\lambda \in \mathbb{Z}^n$, let $(\sigma_1, \dots, \sigma_n)$ be the i-signature of λ defined according to (2.30). First, replace all σ_r that equal $-+$ with $+-$. Now form the dual reduced i-signature $(\tilde{\sigma}_1, \dots, \tilde{\sigma}_n)$ from this by repeatedly replacing subwords of $(\sigma_1, \dots, \sigma_n)$ of the form $-+$ (possibly separated by 0's) by 0's, until no $+$ appears after a $-$. Finally,

$$\tilde{E}_i^*(\lambda) = \begin{cases} 0 & \text{if there are no } -\text{'s in the dual reduced } i\text{-signature,} \\ \lambda - \delta_r & \text{otherwise, where the leftmost } - \text{ occurs in } \tilde{\sigma}_r, \end{cases}$$

$$\tilde{F}_i^*(f) = \begin{cases} 0 & \text{if there are no } +\text{'s in the dual reduced } a\text{-signature,} \\ \lambda + \delta_r & \text{otherwise, where the rightmost } + \text{ occurs in } \tilde{\sigma}_r, \end{cases}$$

$$\varepsilon_i^*(\lambda) = \text{the total number of } -\text{'s in the dual reduced } i\text{-signature,}$$

$$\varphi_i^*(\lambda) = \text{the total number of } +\text{'s in the dual reduced } i\text{-signature.}$$

In this way, we obtain the dual crystal structure $(\mathbb{Z}^n, \tilde{E}_i^*, \tilde{F}_i^*, \varepsilon_i^*, \varphi_i^*, \mathrm{wt})$.

Lemma 2.35. *Let $\lambda \in \mathbb{Z}^n$ and $i \geq 0$.*

(i) $E_i L_\lambda = [\varepsilon_i^*(\lambda)]_i L_{\tilde{E}_i^*(\lambda)} + \sum_{\mu \in \mathbb{Z}^n} w_{\mu,\lambda}^i L_\mu$ *where* $w_{\mu,\lambda}^i \in q q_i^{1-\varepsilon_i^*(\lambda)} \mathbb{Z}[q]$ *is zero unless* $\varphi_j^*(\mu) \leq \varphi_j^*(\lambda)$ *for all* $j \geq 0$.

(ii) $F_i L_\lambda = [\varphi_i^*(\lambda)]_i L_{\tilde{F}_i^*(\lambda)} + \sum_{\mu \in \mathbb{Z}^n} x_{\mu,\lambda}^i L_\mu$ *where* $x_{\mu,\lambda}^i \in q q_i^{1-\varphi_i^*(\lambda)} \mathbb{Z}[q]$ *is zero unless* $\varepsilon_j^*(\mu) \leq \varepsilon_j^*(\lambda)$ *for all* $j \geq 0$.

Proof. Dualize Lemma 2.32 using Lemmas 2.21 and 2.25. □

Remark 2.36. (i) Suppose we are given $\varepsilon_i, \varphi_i \in \mathbb{N}$ for all $i \geq 0$. One can show from the combinatorial description of the maps $\varepsilon_i^*, \varphi_i^*$ above that there exist only finitely many $\lambda \in \mathbb{Z}^n$ with $\varepsilon_i^*(\lambda) = \varepsilon_i$ and $\varphi_i^*(\lambda) = \varphi_i$ for all $i \geq 0$.

(ii) Using (i), one deduces easily that all but finitely many terms of the sums occurring in Lemma 2.35 are zero, i.e. both $E_i L_\lambda$ and $F_i L_\lambda$ are finite $\mathbb{Z}[q, q^{-1}]$-linear combinations of L_μ's. We will not make us of this observation in the remainder of the article.

(iii) If the finiteness conjecture made in Example 2.23 holds, then it is also the case that all but finitely many terms of the sums occurring in Lemma 2.32 are zero, i.e. both $E_i T_\lambda$ and $F_i T_\lambda$ are finite $\mathbb{Z}[q, q^{-1}]$-linear combinations of T_μ's, cf. the argument in the last paragraph of the proof of Lemma 3.30 below.

3. Exterior algebra

§3-a. **Exterior powers.** Define $\mathscr{K}$ to be the two-sided ideal of the tensor algebra $\mathscr{T}$ generated by the vectors

$$\begin{aligned} &v_a \otimes v_a && (a \neq 0) \\ &v_a \otimes v_b + q^2 v_b \otimes v_a && (a > b, a + b \neq 0) \\ &v_a \otimes v_{-a} + q^2(v_{a-1} \otimes v_{1-a} + v_{1-a} \otimes v_{a-1}) + q^4 v_{-a} \otimes v_a && (a \geq 2) \\ &v_1 \otimes v_{-1} + q v_0 \otimes v_0 + q^4 v_{-1} \otimes v_1, \end{aligned}$$

for all admissible $a, b \in \mathbb{Z}$. These relations are a limiting case of the relations in [JMO, Proposition 2.3] (with q replaced by q^2), hence $\mathscr{K}$ is invariant under the action of $\mathscr{U}$. Let $\mathscr{F} := \mathscr{T}/\mathscr{K}$. Since $\mathscr{K} = \bigoplus_{n\geq 0}\mathscr{K}^n$ is a homogeneous ideal of $\mathscr{T}$, $\mathscr{F}$ is also graded as $\mathscr{F} = \bigoplus_{n\geq 0}\mathscr{F}^n$, where $\mathscr{F}^n = \mathscr{T}^n/\mathscr{K}^n$. We view the space $\mathscr{F}^n$ as a quantum analogue of the exterior power $\bigwedge^n \mathscr{V}$ in type $\mathfrak{b}_\infty$. As usual, we will write $u_1 \wedge \cdots \wedge u_n$ for the image of $u_1 \otimes \cdots \otimes u_n \in \mathscr{T}^n$ under the quotient map $\pi : \mathscr{T}^n \to \mathscr{F}^n$.

As in the introduction, let $\mathbb{Z}^n_+$ denote the set of all tuples $\lambda \in \mathbb{Z}^n$ such that $\lambda_r > \lambda_{r+1}$ if $\lambda_r \neq 0$, $\lambda_r \geq \lambda_{r+1}$ if $\lambda_r = 0$, for each $r = 1, \dots, n-1$. For $\lambda \in \mathbb{Z}^n_+$, let

$$F_\lambda = \pi(N_{w_0\lambda}) = v_{\lambda_n} \wedge \cdots \wedge v_{\lambda_1} \in \mathscr{F}^n. \tag{3.1}$$

The following lemma follows from the defining relations for $\mathscr{K}^n$.

Lemma 3.2. *For $\lambda \in \mathbb{Z}^n$, $\pi(N_{w_0\lambda})$ equals F_λ if $\lambda \in \mathbb{Z}_n^+$, otherwise $\pi(N_{w_0\lambda})$ is a $q\mathbb{Z}[q]$-linear combination of F_μ's for $\mu \in \mathbb{Z}^n_+$ with $\mu \succeq \lambda$.*

This shows that the elements $\{F_\lambda\}_{\lambda\in\mathbb{Z}^n_+}$ span $\mathscr{F}^n$. In fact, one can check routinely using Bergman's diamond lemma [Bg, 1.2] that:

Lemma 3.3. *The vectors $\{F_\lambda\}_{\lambda\in\mathbb{Z}^n_+}$ give a basis for $\mathscr{F}^n$.*

Let $\widehat{\mathscr{K}}^n$ be the closure of $\mathscr{K}^n$ in $\widehat{\mathscr{T}}^n$ and $\widehat{\mathscr{F}}^n := \widehat{\mathscr{T}}^n/\widehat{\mathscr{K}}^n$, giving a completion of the vector space $\mathscr{F}^n$. The vectors $\{F_\lambda\}_{\lambda\in\mathbb{Z}^n_+}$ give a topological basis for $\widehat{\mathscr{F}}^n$. Note that $\sigma : \mathscr{T}^n \to \mathscr{T}^n$ leaves $\mathscr{K}^n$ invariant, hence induces $\sigma : \mathscr{F}^n \to \mathscr{F}^n$. Similarly the continuous automorphism $\omega : \widehat{\mathscr{T}}^n \to \widehat{\mathscr{T}}^n$ leaves $\widehat{\mathscr{K}}^n$ invariant, so induces $\omega : \widehat{\mathscr{F}}^n \to \widehat{\mathscr{F}}^n$ with $\omega(F_\lambda) = F_{-w_0\lambda}$.

§3-b. **Canonical basis of $\widehat{\mathscr{F}}^n$.** Now we construct the canonical basis of $\widehat{\mathscr{F}}^n$. To start with, we need a bar involution.

Lemma 3.4. *The bar involution on $\widehat{\mathscr{T}}^n$ leaves $\widehat{\mathscr{K}}^n$ invariant, hence induces a continuous antilinear involution $- : \widehat{\mathscr{F}}^n \to \widehat{\mathscr{F}}^n$ such that*

(1) *$\overline{xv} = \overline{x}\,\overline{v}$ for all $x \in \mathscr{U}, v \in \widehat{\mathscr{F}}^n$;*
(2) *$\overline{F_\lambda} \in F_\lambda + \sum_{\mu\succ\lambda} \mathbb{Z}[q,q^{-1}]F_\mu$ for all $\lambda \in \mathbb{Z}^n_+$;*
(3) *$\overline{\omega(v)} = \omega(\overline{v})$ for all $v \in \widehat{\mathscr{F}}^n$.*

Proof. In the case $n = 2$, all the generators of $\mathscr{K}^2$ are bar invariant by Example 2.23, hence $\mathscr{K}^2$ is bar invariant. In general, $\mathscr{K}^n$ is spanned by vectors of the form $v \otimes k \otimes w$ for $v \in \mathscr{T}^{n_1}, k \in \mathscr{K}^2, w \in T^{n_2}$ and some $n_1, n_2 \geq 0$ with $n_1 + n_2 + 2 = n$. By the definition of the bar involution,

$$\overline{v \otimes k \otimes w} = \Theta_{n_1+2,n_2}(\overline{v} \otimes \Theta_{2,n_2}(\overline{k} \otimes \overline{w})).$$

We have already shown that $\overline{k} \in \mathscr{K}^2$, and $\mathscr{K}^2$ is $\mathscr{U}$-invariant, hence this belongs to $\widehat{\mathscr{K}}^n$. This shows that $\overline{\mathscr{K}^n} \subset \widehat{\mathscr{K}}^n$, hence $\widehat{\mathscr{K}}^n$ itself is bar invariant by continuity. Now properties (1) and (3) are immediate from the analogous properties of the bar involution on $\widehat{\mathscr{T}}^n$. For property (2), take $\lambda \in \mathbb{Z}^n_+$. We know $\overline{N_{w_0\lambda}}$ equals $N_{w_0\lambda}$ plus a $\mathbb{Z}[q,q^{-1}]$-linear combination of $N_{w_0\mu}$'s with $w_0\mu \prec w_0\lambda$, or equivalently, $\mu \succ \lambda$. By Lemma 3.2, $\pi(N_{w_0\mu})$ is a $\mathbb{Z}[q,q^{-1}]$-linear combination of F_ν's with $\nu \succeq \mu$. Hence (2) holds. □

We get the following theorem by general principles, just as in Theorem 2.22 before.

Theorem 3.5. *There exists a unique topological basis* $\{U_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ *for* $\widehat{\mathscr{F}}^n$ *such that* $\overline{U_\lambda} = U_\lambda$ *and* $U_\lambda \in F_\lambda + \widehat{\sum}_{\mu \in \mathbb{Z}^n_+} q\mathbb{Z}[q] F_\mu$*, for all* $\lambda \in \mathbb{Z}^n_+$*. Actually, we have that* $U_\lambda \in F_\lambda + \widehat{\sum}_{\mu \succ \lambda} q\mathbb{Z}[q] F_\mu$*. Also,* $\omega(U_\lambda) = U_{-w_0\lambda}$*.*

Example 3.6. For $n = 2$, one deduces from Example 2.23 and Lemma 3.8 below that:

$$\begin{aligned} U_{(a,b)} &= F_{(a,b)} && (a > b, a + b \neq 0) \\ U_{(a,-a)} &= F_{(a,-a)} + q^2 F_{(a+1,-a-1)} && (a \geq 1) \\ U_{(0,0)} &= F_{(0,0)} + (q + q^3) F_{(1,-1)} \end{aligned}$$

We call the topological basis $\{U_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ the *canonical basis* of $\widehat{\mathscr{F}}^n$. Write

$$U_\lambda = \sum_{\mu \in \mathbb{Z}^n_+} u_{\mu,\lambda}(q) F_\mu \tag{3.7}$$

for polynomials $u_{\mu,\lambda}(q) \in \mathbb{Z}[q]$. We know that $u_{\mu,\lambda}(q) = 0$ unless $\mu \succeq \lambda$, and that $u_{\lambda,\lambda}(q) = 1$. The following lemma explains the relationship between U_λ and the canonical basis element T_λ of $\widehat{\mathscr{T}}^n$ constructed earlier.

Lemma 3.8. *For* $\lambda \in \mathbb{Z}^n$*, we have that*

$$\pi(T_{w_0\lambda}) = \begin{cases} U_\lambda & \text{if } \lambda \in \mathbb{Z}^n_+, \\ 0 & \text{if } \lambda \notin \mathbb{Z}^n_+. \end{cases}$$

Proof. Suppose first that $\lambda \in \mathbb{Z}^n_+$. We know that $T_{w_0\lambda}$ equals $N_{w_0\lambda}$ plus a $q\mathbb{Z}[q]$-linear combination of $N_{w_0\mu}$'s with $\mu \succ \lambda$. Moreover, by Lemma 3.2, $\pi(N_{w_0\mu})$ is a $\mathbb{Z}[q]$-linear combination of F_ν's with $\nu \succeq \mu$. Hence,

$$\pi(T_{w_0\lambda}) = F_\lambda + (\text{a } q\mathbb{Z}[q]\text{-linear combination of } F_\mu\text{'s with } \mu \succ \lambda).$$

Since it is automatically bar invariant, it must equal U_λ by the uniqueness in Theorem 3.5. An entirely similar argument in case $\lambda \notin \mathbb{Z}^n_+$ shows that

$$\begin{aligned} \pi(T_{w_0\lambda}) &= (\text{a } q\mathbb{Z}[q]\text{-linear combination of } F_\mu\text{'s with } \mu \succ \lambda) \\ &= (\text{a } q\mathbb{Z}[q]\text{-linear combination of } U_\mu\text{'s with } \mu \succ \lambda). \end{aligned}$$

Since it is bar invariant, it must be zero. □

Corollary 3.9. *The vectors* $\{T_{w_0\lambda}\}_{\lambda \notin \mathbb{Z}^n_+}$ *form a topological basis for* $\widehat{\mathscr{K}}^n$*.*

§3-c. **Dual canonical basis.** For $\lambda \in \mathbb{Z}^n_+$, define

$$E_\lambda = \sum_{\mu \in \mathbb{Z}^n_+} u_{-w_0\lambda,-w_0\mu}(q^{-1}) L_\mu. \tag{3.10}$$

Let $\mathscr{E}^n$ be the subspace of $\widehat{\mathscr{T}}^n$ spanned by the $\{E_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$. Since there are only finitely many $\mu \in \mathbb{Z}^n_+$ with $\mu \preceq \lambda$, we see that E_λ is a finite linear combination of L_μ's, and vice versa. So the vectors $\{L_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ also form a basis for $\mathscr{E}^n$.

Example 3.11. For $n = 2$, we have by Example 3.6 that:

$$\begin{aligned} E_{(a,b)} &= L_{(a,b)} && (a > b, a+b \neq 0)\\ E_{(a,-a)} &= L_{(a,-a)} + q^{-2} L_{(a-1,-a+1)} && (a \geq 2)\\ E_{(1,-1)} &= L_{(1,-1)} + (q^{-1} + q^{-3}) L_{(0,0)}\\ E_{(0,0)} &= L_{(0,0)} \end{aligned}$$

Let $\widehat{\mathscr{E}}^n$ be the closure of $\mathscr{E}^n$ in $\widehat{\mathscr{T}}^n$. By Corollary 3.9 and Lemma 2.25, $\widehat{\mathscr{E}}^n$ and $\widehat{\mathscr{K}}^n$ are orthogonal with respect to the bilinear form $\langle .,. \rangle$. Hence we get induced a well-defined pairing $\langle .,. \rangle$ between $\widehat{\mathscr{E}}^n$ and $\widehat{\mathscr{F}}^n$. By Lemmas 3.8 and 2.25, we have that

$$\langle L_{-w_0\lambda}, U_\mu \rangle = \delta_{\lambda,\mu} \tag{3.12}$$

for all $\lambda, \mu \in \mathbb{Z}^n_+$.

Lemma 3.13. *$\widehat{\mathscr{E}}^n$ is a $\mathscr{U}$-submodule of $\widehat{\mathscr{T}}^n$.*

Proof. It suffices to show that $E_i L_\lambda$ and $F_i L_\lambda$ both belong to $\widehat{\mathscr{E}}^n$ for each $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$. Write $E_i L_\lambda = \sum_{\mu \in \mathbb{Z}^m} b_{\mu,\lambda}(q) L_\mu$. Apply $\langle ., T_{-\mu} \rangle$ to both sides and use Lemma 2.25 to get $b_{\mu,\lambda}(q) = \langle L_\lambda, E_i T_{-\mu} \rangle$. For $\mu \notin \mathbb{Z}^n_+$, $T_{-\mu}$ belongs to $\widehat{\mathscr{K}}^n$. But $\widehat{\mathscr{K}}^n$ is $\mathscr{U}$-invariant, hence $E_i T_{-\mu}$ belongs to $\widehat{\mathscr{K}}^n$. So we get that $b_{\mu,\lambda}(q) = 0$ for all $\mu \notin \mathbb{Z}^n_+$ by Corollary 3.9. Hence $E_i L_\lambda$ belongs to $\widehat{\mathscr{E}}^n$ still, and similarly for $F_i L_\lambda$. □

The bilinear form $(.,.)$ on $\mathscr{T}^n$ also induces a pairing $(.,.)$ between $\mathscr{E}^n$ and $\mathscr{F}^n$, actually by (2.20) we have that

$$(u, v) = \langle u, \overline{\sigma(v)} \rangle \tag{3.14}$$

for each $u \in \mathscr{E}^n, v \in \mathscr{F}^n$.

Lemma 3.15. *For all $\lambda, \mu \in \mathbb{Z}^n_+$, we have that $(E_\lambda, N_\mu) = q^{z(\lambda)} \delta_{\lambda,\mu}$.*

Proof. Let $F_\mu = \sum_{\gamma \in \mathbb{Z}^n_+} v_{\gamma,\mu}(q) U_\gamma$, so $\sum_{\gamma \in \mathbb{Z}^n_+} u_{\mu,\gamma}(q) v_{\gamma,\nu}(q) = \delta_{\mu,\nu}$. Now calculate:

$$\begin{aligned} (E_{-w_0\lambda}, N_{-w_0\mu}) &= \langle E_{-w_0\lambda}, \overline{\sigma(N_{-w_0\mu})} \rangle\\ &= q^{z(\mu)} \langle E_{-w_0\lambda}, \overline{N_{w_0\mu}} \rangle = q^{z(\mu)} \langle E_{-w_0\lambda}, \overline{F_\mu} \rangle\\ &= q^{z(\mu)} \sum_{\nu,\gamma \in \mathbb{Z}^n_+} u_{\lambda,\nu}(q^{-1}) v_{\gamma,\mu}(q^{-1}) \langle L_{-w_0\nu}, U_\gamma \rangle = q^{z(\mu)} \delta_{\lambda,\mu}. \end{aligned}$$

□

Now we get the following characterization of the basis $\{L_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ in terms of the restriction of the bar involution to $\widehat{\mathscr{E}}^n$.

Theorem 3.16. *For $\lambda \in \mathbb{Z}^n_+$, L_λ is the unique element of $\widehat{\mathscr{E}}^n$ such that $\overline{L_\lambda} = L_\lambda$ and $L_\lambda \in E_\lambda + \widehat{\sum}_{\mu \in \mathbb{Z}^n_+} q^{-1}\mathbb{Z}[q^{-1}]E_\mu$. Moreover,*

$$L_\lambda = \sum_{\mu \in \mathbb{Z}^n_+} l_{\mu,\lambda}(q) E_\mu, \tag{3.17}$$

where $l_{\mu,\lambda}(q)$ is as in (2.24).

Proof. For each $\mu \in \mathbb{Z}^n_+$, $(L_\lambda, N_\mu) = q^{z(\mu)} l_{\mu,\lambda}(q)$. Now apply Lemma 3.15 to deduce that $L_\lambda = \sum_{\mu \in \mathbb{Z}^n_+} l_{\mu,\lambda}(q) E_\mu$. Finally, if $L'_\lambda \in E_\lambda + \sum_{\mu \in \mathbb{Z}^n_+} q^{-1}\mathbb{Z}[q^{-1}]E_\mu$ is another bar invariant element of $\widehat{\mathscr{E}}^n$, then $L_\lambda - L'_\lambda$ is bar invariant and can be expressed as a $q^{-1}\mathbb{Z}[q^{-1}]$-linear combination of L_ν's. Hence it must be zero. □

To state the next lemma, we define polynomials $a_{\lambda,\mu}(q) \in \mathbb{Z}[q]$ for each $\lambda \in \mathbb{Z}^n_+, \mu \in \mathbb{Z}^n$ by

$$\pi(N_{w_0\mu}) = \sum_{\lambda \in \mathbb{Z}^n_+} a_{\lambda,\mu}(q) F_\lambda. \tag{3.18}$$

Recalling Lemma 3.2, we have that $a_{\lambda,\mu}(q) = 0$ unless $\lambda \succeq \mu$.

Lemma 3.19. *For each $\lambda \in \mathbb{Z}^n_+$, $E_\lambda = \sum_{\mu \in \mathbb{Z}^n} a_{-w_0\lambda,-w_0\mu}(q^{-1}) M_\mu$.*

Proof. Applying the antilinear map σ to (3.18) gives that

$$\pi(N_\mu) = \sum_{\lambda \in \mathbb{Z}^n_+} q^{z(\mu)-z(\lambda)} a_{-w_0\lambda,-w_0\mu}(q^{-1}) \pi(N_\lambda)$$

for each $\mu \in \mathbb{Z}^n$. Hence, invoking Lemma 3.15, $(E_\lambda, N_\mu) = q^{z(\mu)} a_{-w_0\lambda,-w_0\mu}(q^{-1})$ for each $\mu \in \mathbb{Z}^n$. The lemma follows. □

Finally in this subsection, we describe the action of $\mathscr{U}$ on the basis $\{E_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ of $\mathscr{E}^n$ explicitly. In particular, this shows that $\mathscr{E}^n$ itself is a $\mathscr{U}$-submodule of $\widehat{\mathscr{T}}^n$, as could also be proved using Lemma 3.13 and Remark 2.36(ii).

Lemma 3.20. *Let $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$, and let $(\sigma_1, \dots, \sigma_n)$ be the i-signature of λ defined according to (2.30). Then,*

$$E_i E_\lambda = q^{-(\alpha_i, \varepsilon_{\lambda_{r+1}} + \dots + \varepsilon_{\lambda_n})} \sum_{\substack{r \text{ with } \lambda - \delta_r \in \mathbb{Z}^n_+, \\ \sigma_r = -, -+ \text{ or } --}} c_{\lambda,r}(q) E_{\lambda - \delta_r},$$

$$F_i E_\lambda = q^{(\alpha_i, \varepsilon_{\lambda_1} + \dots + \varepsilon_{\lambda_{r-1}})} \sum_{\substack{r \text{ with } \lambda + \delta_r \in \mathbb{Z}^n_+, \\ \sigma_r = +, -+ \text{ or } ++}} c_{\lambda,r}(q) E_{\lambda + \delta_r},$$

where $c_{\lambda,r}(q) = \begin{cases} (q+q^{-1}) \sum_{s=0}^{z(\lambda)} (-q^{-2})^s & \text{if } \sigma_r = -- \text{ or } ++, \\ 1 & \text{otherwise.} \end{cases}$

Proof. We sketch the proof for E_i. By Lemmas 3.13, 3.15 and 2.11(ii), we may write $E_i E_\lambda = \sum_{\mu \in \mathbb{Z}^n_+} c_{\mu,\lambda}(q) E_\mu$ where

$$c_{\mu,\lambda}(q) = q^{-z(\mu)}(E_i E_\lambda, N_\mu) = q^{-z(\mu)}(E_\lambda, q_i F_i K_i^{-1} N_\mu).$$

The right hand side is computed using Lemma 3.15 and the fact that E_λ is orthogonal to $\mathscr{K}^n$. □

§3-d. **Crystal structure.** The crystal structure underlying the canonical basis $\{U_\lambda\}_{\lambda\in\mathbb{Z}^n_+}$ of $\widehat{\mathscr{F}}^n$ is easily deduced from the results of §2-f and Lemma 3.8. Let us denote the resulting crystal operators on the index set $\mathbb{Z}^n_+$ parameterizing the bases of $\widehat{\mathscr{F}}^n$ by $\tilde{E}_i, \tilde{F}_i, \varepsilon_i, \varphi_i$. By definition,

$$\tilde{E}_i(\lambda) = w_0\tilde{E}'_i(w_0\lambda), \qquad \tilde{F}_i(\lambda) = w_0\tilde{F}'_i(w_0\lambda), \tag{3.21}$$

$$\varepsilon_i(\lambda) = \varepsilon'_i(w_0\lambda), \qquad \varphi_i(\lambda) = \varphi'_i(w_0\lambda), \tag{3.22}$$

where $\tilde{E}'_i, \tilde{F}'_i, \varepsilon'_i$ and φ'_i are as in §2-f. By properties of the automorphism ω (or by directly checking all of the cases listed below), the operators $\tilde{E}_i, \tilde{F}_i, \varepsilon_i, \varphi_i$ are *the same as* the restrictions to $\mathbb{Z}^n_+$ of the dual crystal operators $\tilde{E}^*_i, \tilde{F}^*_i, \varepsilon^*_i, \varphi^*_i$ defined in §2-f, so we will not need the latter notation again.

In fact, there are now so few possibilities that we can describe the crystal graph explicitly. First suppose that $i = 0$. Then the possible i-strings in the crystal graph are as follows:

(1) $(\cdots)$;

(2) $(\cdots, 0^r, -1, \cdots) \xrightarrow{\tilde{F}_0} (\cdots, 0^{r+1}, \cdots) \xrightarrow{\tilde{F}_0} (\cdots, 1, 0^r, \cdots)$;

(3) $(\cdots, 1, 0^r, -1, \cdots)$.

Here, $\cdots$ denotes some fixed entries different from $1, 0, -1$ and $r \geq 0$. Similarly for $i > 0$, the possible i-strings in the crystal are as follows:

(1) $(\cdots)$;

(2) $(\cdots, i, \cdots) \xrightarrow{\tilde{F}_i} (\cdots, i+1, \cdots)$;

(3) $(\cdots, -i-1, \cdots) \xrightarrow{\tilde{F}_i} (\cdots, -i, \cdots)$;

(4) $(\cdots, i, \cdots, -i-1, \cdots) \xrightarrow{\tilde{F}_i} (\cdots, i, \cdots, -i, \cdots)$
$\xrightarrow{\tilde{F}_i} (\cdots, i+1, \cdots, -i, \cdots)$;

(5) $(\cdots, i+1, \cdots, -i-1, \cdots)$;

(6) $(\cdots, -i, -i-1, \cdots)$;

(7) $(\cdots, i+1, i, \cdots)$;

(8) $(\cdots, i+1, i, \cdots, -i-1, \cdots) \xrightarrow{\tilde{F}_i} (\cdots, i+1, i, \cdots, -i, \cdots)$;

(9) $(\cdots, i, \cdots, -i, -i-1, \cdots) \xrightarrow{\tilde{F}_i} (\cdots, i+1, \cdots, -i, -i-1, \cdots)$;

(10) $(\cdots, i+1, i, \cdots, -i, -i-1, \cdots)$,

where again $\cdots$ denotes fixed entries different from $i, i+1, -i, -i-1$. A crucial observation deduced from this analysis is that *all* i-strings are of length ≤ 2.

Lemma 3.23. *Let $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$.*

(i) $E_i L_\lambda = [\varepsilon_i(\lambda)]_i L_{\tilde{E}_i(\lambda)} + \sum_{\mu\in\mathbb{Z}^n_+} w^i_{\mu,\lambda} L_\mu$ *where* $w^i_{\mu,\lambda} \in qq_i^{1-\varepsilon_i(\lambda)}\mathbb{Z}[q]$ *is zero unless* $\varphi_j(\mu) \leq \varphi_j(\lambda)$ *for all* $j \geq 0$.

(ii) $F_i L_\lambda = [\varphi_i(\lambda)]_i L_{\tilde{F}_i(\lambda)} + \sum_{\mu\in\mathbb{Z}^n_+} x^i_{\mu,\lambda} L_\mu$ *where* $x^i_{\mu,\lambda} \in qq_i^{1-\varphi_i(\lambda)}\mathbb{Z}[q]$ *is zero unless* $\varepsilon_j(\mu) \leq \varepsilon_j(\lambda)$ *for all* $j \geq 0$.

Proof. This is a special case of Lemma 2.35, since $\tilde{E}_i = \tilde{E}_i^*$, $\tilde{F}_i = \tilde{F}_i^*, \dots$. □

Lemma 3.24. *Let $\lambda \in \mathbb{Z}_+^n$ and $i \geq 0$.*

(i) $E_i U_\lambda = [\varphi_i(\lambda)+1]_i U_{\tilde{E}_i(\lambda)} + \sum_{\mu \in \mathbb{Z}_+^n} y^i_{\mu,\lambda} U_\mu$ *where* $y^i_{\mu,\lambda} \in q q_i^{1-\varphi_i(\mu)} \mathbb{Z}[q]$ *is zero unless* $\varepsilon_j(\mu) \geq \varepsilon_j(\lambda)$ *for all* $j \geq 0$.

(ii) $F_i U_\lambda = [\varepsilon_i(\lambda)+1]_i U_{\tilde{F}_i(\lambda)} + \sum_{\mu \in \mathbb{Z}_+^n} z^i_{\mu,\lambda} U_\mu$ *where* $z^i_{\mu,\lambda} \in q q_i^{1-\varepsilon_i(\mu)} \mathbb{Z}[q]$ *is zero unless* $\varphi_j(\mu) \geq \varphi_j(\lambda)$ *for all* $j \geq 0$.

(In (i) resp. (ii), the first term on the right hand side should be omitted if $\tilde{E}_i(\lambda)$ resp. $\tilde{F}_i(\lambda)$ is $\varnothing$.)

Proof. Dualize Lemma 3.23. □

Corollary 3.25. *Let $\lambda \in \mathbb{Z}_+^n$ and $i \geq 0$.*

(i) *If $\varepsilon_i(\lambda) > 0$ then $E_i U_\lambda = [\varphi_i(\lambda)+1]_i U_{\tilde{E}_i(\lambda)}$.*

(ii) *If $\varphi_i(\lambda) > 0$ then $F_i U_\lambda = [\varepsilon_i(\lambda)+1]_i U_{\tilde{F}_i(\lambda)}$.*

Proof. We prove (i), (ii) being similar. Lemma 3.24 gives us that $E_i U_\lambda = [\varphi_i(\lambda)+1]_i U_{\tilde{E}_i(\lambda)} + \sum_{\mu \in \mathbb{Z}_+^n} y^i_{\mu,\lambda} U_\mu$ where $y^i_{\mu,\lambda}$ belongs to $q q_i^{1-\varphi_i(\mu)} \mathbb{Z}[q]$ and is zero unless $\varepsilon_j(\mu) \geq \varepsilon_j(\mu)$ for all $j \geq 0$. Suppose that $y^i_{\mu,\lambda} \neq 0$ for some μ. By assumption, $\varepsilon_i(\mu) \geq \varepsilon_i(\lambda) \geq 1$, so $\varphi_i(\mu) \leq 1$ since all i-strings are of length ≤ 2. So $0 \neq y^i_{\mu,\lambda} \in q\mathbb{Z}[q]$. But $y^i_{\mu,\lambda}$ is bar invariant, so this is a contradiction. □

§3-e. **Computation of U_λ's.** Now we explain a simple algorithm to compute U_λ. Recall the definition of the degree of atypicality of λ from (2.15). We will need the following:

Procedure 3.26. Suppose we are given $\lambda \in \mathbb{Z}_+^n$ with $\#\lambda \geq 2$. Compute $\mu \in \mathbb{Z}_+^n$ and an operator $X_i \in \{E_i, F_i\}_{i \geq 0}$ by following the instructions below starting at step (0).

(0) Choose the minimal $r \in \{1, \dots, n\}$ such that $\lambda_r + \lambda_s = 0$ for some $s > r$. Go to step (1).

(1) If $r > 1$ and $\lambda_r = \lambda_{r-1} - 1$, replace r by $(r-1)$ and repeat step (1). Otherwise, go to step (2).

(2) If $\lambda_r + \lambda_s + 1 = 0$ for some (necessarily unique) $s \in \{1, \dots, n\}$ go to step (1)′. Otherwise, set $X_i = E_{\lambda_r}$ and $\mu = \lambda + \delta_r$. Stop.

(1)′ If $s < n$ and $\lambda_s = \lambda_{s+1} + 1$, replace s by $(s+1)$ and repeat step (1)′. Otherwise, go to step (2)′.

(2)′ If $\lambda_r + \lambda_s - 1 = 0$ for some (necessarily unique) $r \in \{1, \dots, n\}$ go to step (1). Otherwise, set $X_i = F_{-\lambda_s}$ and $\mu = \lambda - \delta_s$. Stop.

The following lemma follows immediately from the nature of the above procedure and Corollary 3.25.

Lemma 3.27. *Take $\lambda \in \mathbb{Z}_+^n$ with $\#\lambda \geq 2$. Apply Procedure 3.26 to get $\mu \in \mathbb{Z}_+^n$ and $X_i \in \{E_i, F_i\}_{i \geq 0}$. Then, $\#\mu \leq \#\lambda$ and $X_i U_\mu = U_\lambda$. Moreover, after at most $(n-1)$ repetitions of the procedure, the atypicality must get strictly smaller. Hence after finitely many recursions, the procedure reduces λ to a typical weight.*

Lemma 3.27 implies the following algorithm for computing U_λ. If λ is typical then $U_\lambda = F_\lambda$, since such λ's are maximal in the Bruhat ordering. Otherwise, apply Procedure 3.26 to get $\mu \in \mathbb{Z}^n_+$ and $X_i \in \{E_i, F_i\}_{i\geq 0}$. Since the procedure always reduces λ to a typical weight in finitely many steps, we may assume U_μ is known recursively. Then $U_\lambda = X_i U_\mu$.

Example 3.28. Let us compute $U_{(5,3,2,1,0,0,-1,-4,-6)}$. Apply Procedure 3.26 repeatedly to get the following sequence of weights:

$$\begin{aligned}
&(5,3,2,1,0,0,-1,-4,-7), && (6,3,2,1,0,0,-1,-4,-7),\\
&(6,3,2,1,0,0,-1,-5,-7), && (6,4,2,1,0,0,-1,-5,-7),\\
&(6,4,3,1,0,0,-1,-5,-7), && (6,4,3,2,0,0,-1,-5,-8),\\
&(7,4,3,2,0,0,-1,-6,-8), && (7,5,3,2,0,0,-1,-6,-8),\\
&(7,5,4,2,0,0,-1,-6,-8), && (7,5,4,3,0,0,-1,-6,-8),\\
&(7,5,4,3,0,0,-2,-6,-8), && (7,5,4,3,1,0,-2,-6,-8).
\end{aligned}$$

Hence,

$$\begin{aligned}
U_{(5,3,2,1,0,0,-1,-4,-6)} &= F_6E_5F_4E_3E_2E_1F_7E_6F_5E_4E_3E_2F_1E_0F_{(7,5,4,3,1,0,-2,-6,-8)}\\
&= F_{(5,3,2,1,0,0,-1,-4,-6)} + q^2F_{(7,5,3,2,0,0,-4,-6,-7)}\\
&\quad + (q+q^3)F_{(8,5,3,2,1,-1,-4,-6,-8)}\\
&\quad + (q^3+q^5)F_{(8,7,5,3,2,-4,-6,-7,-8)}.
\end{aligned}$$

Example 3.29. Using the algorithm, one computes for $n = 3$ that:

$$\begin{aligned}
U_{(a,b,c)} &= F_{(a,b,c)} && (a+b, a+c, b+c \neq 0)\\
U_{(a,b,-b)} &= F_{(a,b,-b)} + q^2F_{(a,b+1,-b-1)} && (a \geq b+2, b \neq 0)\\
U_{(b+1,b,-b)} &= F_{(b+1,b,-b)} + q^2F_{(b+2,b+1,-b-2)} && (b \neq 0)\\
U_{(a,-a,-b)} &= F_{(a,-a,-b)} + q^2F_{(a+1,-a-1,-b)} && (b \geq a+2, a \neq 0)\\
U_{(a,-a,-a-1)} &= F_{(a,-a,-a-1)} + q^2F_{(a+2,-a-1,-a-2)} && (a \neq 0)\\
U_{(a,b,-a)} &= F_{(a,b,-a)} + q^2F_{(a+1,b,-a-1)} && (a > b > -a)\\
U_{(a,0,0)} &= F_{(a,0,0)} + (q+q^3)F_{(a,1,-1)} && (a \geq 2)\\
U_{(1,0,0)} &= F_{(1,0,0)} + (q+q^3)F_{(2,1,-2)}\\
U_{(0,0,-b)} &= F_{(0,0,-b)} + (q+q^3)F_{(1,-1,-b)} && (b \geq 2)\\
U_{(0,0,-1)} &= F_{(0,0,-1)} + (q+q^3)F_{(2,-1,-2)}\\
U_{(0,0,0)} &= F_{(0,0,0)} + (q-q^5)F_{(1,0,-1)} + (q+q^3)F_{(2,0,-2)}
\end{aligned}$$

§3-f. **Specialization.** When we specialize at $q = 1$, various things become simpler to compute. To formalize the specialization process, we will work with the $\mathbb{Z}[q,q^{-1}]$-lattices

$$\mathscr{E}^n_{\mathbb{Z}[q,q^{-1}]} = \sum_{\lambda \in \mathbb{Z}^n_+} \mathbb{Z}[q,q^{-1}] L_\lambda = \sum_{\lambda \in \mathbb{Z}^n_+} \mathbb{Z}[q,q^{-1}] E_\lambda,$$

$$\mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]} = \sum_{\lambda \in \mathbb{Z}^n_+} \mathbb{Z}[q,q^{-1}] U_\lambda \subset \sum_{\lambda \in \mathbb{Z}^n_+} \mathbb{Z}[q,q^{-1}] F_\lambda.$$

Also let $\widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]}$ denote the completion of $\mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]}$, i.e. its closure in $\widehat{\mathscr{F}}^n$. We need this because the element $F_\lambda \in \mathscr{F}^n$ need not belong to the lattice $\mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]}$, though it always belongs to $\widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]}$. Indeed, the $\{F_\lambda\}_{\lambda \in \mathbb{Z}^n_+}$ form a topological basis for $\widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]}$.

Lemma 3.30. *$\mathscr{E}^n_{\mathbb{Z}[q,q^{-1}]}, \mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]}$ and $\widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]}$ are modules over $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$.*

Proof. The $\mathbb{Z}[q,q^{-1}]$-lattice in $\widehat{\mathscr{T}}^n$ generated by $\{M_\lambda\}_{\lambda \in \mathbb{Z}^n}$ is invariant under $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$. Combining this with Corollary 2.28 and Lemma 3.19, we deduce that each $E_i^{(r)} E_\lambda$ can be expressed as a (possibly infinite) $\mathbb{Z}[q,q^{-1}]$-linear combination of E_μ's. Actually, it is a finite linear combination by Lemma 3.20. Similarly, each $F_i^{(r)} E_\lambda$ is a finite $\mathbb{Z}[q,q^{-1}]$-linear combination of E_μ's. So $\mathscr{E}^n_{\mathbb{Z}[q,q^{-1}]}$ is a $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$-module.

Considering instead the $\mathbb{Z}[q,q^{-1}]$-lattice in $\widehat{\mathscr{T}}^n$ generated by $\{N_\lambda\}_{\lambda \in \mathbb{Z}^n}$ and passing to the quotient $\widehat{\mathscr{F}}^n$, we also get easily that each $E_i^{(r)} F_\lambda$ and each $F_i^{(r)} F_\lambda$ is a finite $\mathbb{Z}[q,q^{-1}]$-linear combination of F_μ's, hence $\widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]}$ is a $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$-module. This means in particular that each $E_i^{(r)} U_\lambda$ and each $F_i^{(r)} U_\lambda$ is a (possibly infinite) $\mathbb{Z}[q,q^{-1}]$-linear combination of U_μ's. Finally to show that $\mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]}$ is a $\mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$-module we need to show $E_i^{(r)} U_\lambda$ and $F_i^{(r)} U_\lambda$ are actually finite $\mathbb{Z}[q,q^{-1}]$-linear combination of U_μ's. We explain the argument for $E_i^{(r)} U_\lambda$ only.

By the algorithm described in the previous subsection, U_λ is a finite linear combination of F_μ's, hence $E_i^{(r)} U_\lambda$ is a bar invariant, finite linear combination of F_μ's. Say

$$E_i^{(r)} U_\lambda = \sum_{\mu \in \mathbb{Z}^n} f_\mu(q) F_\mu$$

for polynomials $f_\mu(q) \in \mathbb{Z}[q,q^{-1}]$. Let d be minimal such that all $f_\mu(q)$ belong to $q^{-d}\mathbb{Z}[q]$. If $d < 0$ then the right hand side is a $q\mathbb{Z}[q]$-linear combination of F_μ's, hence since it also a bar invariant combination of U_μ's it must be zero. Otherwise, define $g_\mu(q)$ to be the unique bar invariant polynomial with $g_\mu(q) \equiv f_\mu(q) \pmod{q\mathbb{Z}[q]}$ and subtract $\sum_\mu g_\mu(q) U_\mu$ from the right hand side, summing over the finitely many μ such that $\deg g_\mu(q) = d$. The result is a bar invariant, finite $q^{1-d}\mathbb{Z}[q]$-linear combination of F_μ's. Now repeat the process to reduce the expression to zero after finitely many steps. $\square$

Define

$$\mathscr{E}^n_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{E}^n_{\mathbb{Z}[q,q^{-1}]},$$
$$\mathscr{F}^n_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{F}^n_{\mathbb{Z}[q,q^{-1}]},$$
$$\widehat{\mathscr{F}}^n_{\mathbb{Z}} = \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \widehat{\mathscr{F}}^n_{\mathbb{Z}[q,q^{-1}]},$$

where we are viewing $\mathbb{Z}$ as a $\mathbb{Z}[q,q^{-1}]$-module so that q acts as 1. For $\lambda \in \mathbb{Z}^n_+$, we will denote the elements $1 \otimes E_\lambda \in \mathscr{E}^n_{\mathbb{Z}}, 1 \otimes L_\lambda \in \mathscr{E}^n_{\mathbb{Z}}$, $1 \otimes F_\lambda \in \widehat{\mathscr{F}}^n_{\mathbb{Z}}$ and $1 \otimes U_\lambda \in \mathscr{F}^n_{\mathbb{Z}}$ by $E_\lambda(1)$, $L_\lambda(1)$, $F_\lambda(1)$ and $U_\lambda(1)$, respectively. By Lemma 3.30, $\mathscr{E}^n_{\mathbb{Z}}, \mathscr{F}^n_{\mathbb{Z}}$ and $\widehat{\mathscr{F}}^n_{\mathbb{Z}}$ are modules over $\mathscr{U}_{\mathbb{Z}} := \mathbb{Z} \otimes_{\mathbb{Z}[q,q^{-1}]} \mathscr{U}_{\mathbb{Z}[q,q^{-1}]}$. In their action on these lattices, the elements

$$E_i^{(r)} = 1 \otimes E_i^{(r)}, \quad F_i^{(r)} = 1 \otimes F_i^{(r)}, \quad \binom{H_i}{r} = 1 \otimes \begin{bmatrix} K_i \\ r \end{bmatrix} \tag{3.31}$$

of $\mathscr{U}_{\mathbb{Z}}$ satisfy the defining relations of the usual generators of the Kostant $\mathbb{Z}$-form for the universal enveloping algebra of type $\mathfrak{b}_\infty$.

We note that at $q = 1$, the defining relations for $\mathscr{F}^n_{\mathbb{Z}}$ simplify to the following:

$$\begin{aligned} v_a \wedge v_a &= 0 && (a \neq 0) \\ v_a \wedge v_b &= -v_b \wedge v_a && (a > b, a + b \neq 0) \\ v_a \wedge v_{-a} &= -v_{-a} \wedge v_a + (-1)^a v_0 \wedge v_0 && (a \geq 1) \end{aligned}$$

The action of $\mathscr{U}_{\mathbb{Z}}$ on $\widehat{\mathscr{F}}^n_{\mathbb{Z}}$ is given explicitly by the formulae

$$E_i F_\lambda(1) = \sum_{\substack{r \text{ with } \lambda - \delta_r \in \mathbb{Z}^n_+ \\ \sigma_r = -, -+ \text{ or } --}} b_{\lambda,r} F_{\lambda - \delta_r}(1), \tag{3.32}$$

$$F_i F_\lambda(1) = \sum_{\substack{r \text{ with } \lambda + \delta_r \in \mathbb{Z}^n_+ \\ \sigma_r = +, -+ \text{ or } ++}} b_{\lambda,r} F_{\lambda + \delta_r}(1), \tag{3.33}$$

where $(\sigma_1, \dots, \sigma_n)$ is the i-signature of $\lambda \in \mathbb{Z}^n_+$ defined according to (2.30) and

$$b_{\lambda,r} = \begin{cases} 1 + (-1)^{z(\lambda)} & \text{if } \sigma_r = -+, \\ 1 & \text{otherwise.} \end{cases}$$

The pairing $\langle ., . \rangle$ between $\widehat{\mathscr{E}}^n$ and $\widehat{\mathscr{F}}^n$ defined earlier induces a pairing at $q = 1$ between $\mathscr{E}^n_{\mathbb{Z}}$ and $\widehat{\mathscr{F}}^n_{\mathbb{Z}}$ with

$$\langle L_{-w_0 \lambda}(1), U_\mu(1) \rangle = \delta_{\lambda,\mu} \tag{3.34}$$

for all $\lambda, \mu \in \mathbb{Z}^n_+$. Alternatively, as follows immediately from (3.7) and (3.10) taken at $q = 1$, we have that

$$\langle E_{-w_0 \lambda}(1), F_\mu(1) \rangle = \delta_{\lambda,\mu} \tag{3.35}$$

for all $\lambda, \mu \in \mathbb{Z}^n_+$.

The final theorem of the section gives an explicit formula for the coefficients $u_{\mu,\lambda}(q)$ at $q = 1$:

Theorem 3.36. *Let $\lambda \in \mathbb{Z}^n_+$ and $p := \lfloor \#\lambda/2 \rfloor$. Choose $1 \leq r_1 < \cdots < r_p < s_p < \cdots < s_1 \leq n$ such that $\lambda_{r_q} + \lambda_{s_q} = 0$ for all $q = 1, \dots, p$. Let $I_0 = \{|\lambda_1|, \dots, |\lambda_n|\}$. For $q = 1, \dots, p$, define I_q and k_q inductively according to the following rules:*

(1) *if $\lambda_{r_q} > 0$, let k_q be the smallest positive integer with $\lambda_{r_q} + k_q \notin I_{q-1}$, and set $I_q = I_{q-1} \cup \{\lambda_{r_q} + k_q\}$;*
(2) *if $\lambda_{r_q} = 0$, let k_q and k'_q be the smallest positive integers with $k_q, k'_q \notin I_{q-1}$, $k_q < k'_q$ if $z(\lambda)$ is even and $k_q > k'_q$ if $z(\lambda)$ is odd, and set $I_q = I_{q-1} \cup \{k_q, k'_q\}$.*

Finally, for each $\theta = (\theta_1, \dots, \theta_p) \in \{0,1\}^p$, let $\mathtt{R}_\theta(\lambda)$ denote the unique element of $\mathbb{Z}^n_+$ lying in the same S_n-orbit as the weight $\lambda + \sum_{q=1}^p \theta_q k_q (\delta_{r_q} - \delta_{s_q})$. Then,

$$U_\lambda(1) = \sum_\theta 2^{(z(\lambda) - z(\mathtt{R}_\theta(\lambda)))/2} F_{\mathtt{R}_\theta(\lambda)}(1),$$

summing over all $\theta = (\theta_1, \dots, \theta_p) \in \{0,1\}^p$.

Proof. Use the algorithm explained in §3-e, (3.32)–(3.33) and induction. See [B1, Theorem 3.34(i)] for a similar argument. □

Corollary 3.37. *For all $\lambda \in \mathbb{Z}^n_+$, $\sum_{\mu \in \mathbb{Z}^n_+} u_{\mu,\lambda}(1) = 2^{(\#\lambda - z(\lambda))/2} 3^{\lfloor z(\lambda)/2 \rfloor}$.*

Example 3.38. Let us compute $U_{(5,3,2,1,0,0,-1,-4,-6)}(1)$ using the theorem, recall Example 3.28. We have that $p = 2$ and $(i_1, i_2, j_2, j_1) = (4,5,6,7)$, then get $k_1 = 6$ (hence the entries $(1,-1)$ change to $(7,-7)$ and $(k_2, k'_2) = (8,9)$ (hence the entries $(0,0)$ change to $(8,-8)$). Hence

$$\begin{aligned} U_{(5,3,2,1,0,0,-1,-4,-6)}(1) &= F_{(5,3,2,1,0,0,-1,-4,-6)}(1) + F_{(7,5,3,2,0,0,-4,-6,-7)}(1) \\ &\quad + 2F_{(8,5,3,2,1,-1,-4,-6,-8)}(1) + 2F_{(8,7,5,3,2,-4,-6,-7,-8)}(1). \end{aligned}$$

For comparison,

$$\begin{aligned} U_{(5,3,2,1,0,0,0,-1,-4,-6)}(1) &= F_{(5,3,2,1,0,0,0,-1,-4,-6)}(1) + F_{(7,5,3,2,0,0,0,-4,-6,-7)}(1) \\ &\quad + 2F_{(9,5,3,2,1,0,-1,-4,-6,-9)}(1) + 2F_{(9,7,5,3,2,0,-4,-6,-7,-9)}(1). \end{aligned}$$

Example 3.39. We have by the theorem that:

$$\begin{aligned} U_{(0,0)}(1) &= F_{(0,0)}(1) + 2F_{(1,-1)}(1), \\ U_{(0,0,0)}(1) &= F_{(0,0,0)}(1) + 2F_{(2,0,-2)}(1), \\ U_{(0,0,0,0)}(1) &= F_{(0,0,0,0)}(1) + 2F_{(1,0,0,-1)}(1) + 2F_{(3,0,0,-3)}(1) + 4F_{(3,1,-1,-3)}(1), \\ U_{(1,-1)}(1) &= F_{(1,-1)}(1) + F_{(2,-2)}(1), \\ U_{(1,0,-1)}(1) &= F_{(1,0,-1)}(1) + F_{(2,0,-2)}(1), \\ U_{(1,0,0,-1)}(1) &= F_{(1,0,0,-1)}(1) + F_{(2,0,0,-2)}(1) + 2F_{(3,1,-1,-3)}(1) + 2F_{(3,2,-2,-3)}(1). \end{aligned}$$

Hence by (3.10),

$$
\begin{aligned}
E_{(1,-1)}(1) &= L_{(1,-1)}(1) + 2L_{(0,0)}(1),\\
E_{(1,0,-1)}(1) &= L_{(1,0,-1)}(1),\\
E_{(1,0,0,-1)}(1) &= L_{(1,0,0,-1)}(1) + 2L_{(0,0,0,0)}(1),\\
E_{(2,-2)}(1) &= L_{(2,-2)}(1),\\
E_{(2,0,-2)}(1) &= L_{(2,0,-2)}(1) + L_{(1,0,-1)}(1) + 2L_{(0,0,0)}(1),\\
E_{(2,0,0,-2)}(1) &= L_{(2,0,0,-2)}(1) + L_{(1,0,0,-1)}(1).
\end{aligned}
$$

4. Character formulae

§4-a. **Representations of $Q(n)$.** Now we are ready to introduce the supergroup $G = Q(n)$ into the picture. For the remainder of the article, we will work over the ground field $\mathbb{C}$, and all objects (superalgebras, superschemes, ...) will be defined over $\mathbb{C}$ without further mention. We refer the reader to [BK2, B2] for a fuller account of the basic results concerning the representation theory of G summarized here, most of which were proved originally by Penkov in [P1].

By definition [BK2, §3], G is the functor from the category of commutative superalgebras to the category of groups defined on a superalgebra A so that $G(A)$ is the group of all invertible $2n \times 2n$ matrices of the form

$$g = \left(\begin{array}{c|c} S & S' \\ \hline -S' & S \end{array}\right) \tag{4.1}$$

where S is an $n \times n$ matrix with entries in $A_{\bar{0}}$ and S' is an $n \times n$ matrix with entries in $A_{\bar{1}}$. The underlying purely even group G_{ev} of G is by definition the functor from superalgebras to groups with $G_{\text{ev}}(A) := G(A_{\bar{0}})$. In our case, $G_{\text{ev}}(A)$ consists of all matrices in $G(A)$ of the form (4.1) with $S' = 0$, so $G_{\text{ev}} \cong GL(n)$. We also need the *Cartan subgroup* H of G defined on a commutative superalgebra A so that $H(A)$ consists of all matrices in $G(A)$ with S, S' being diagonal matrices, and the *negative Borel subgroup* B of G defined so that $B(A)$ consists of all matrices in $G(A)$ with S, S' being lower triangular.

Let $T = H_{\text{ev}}$ be the usual maximal torus of G_{ev} consisting of diagonal matrices. The character group $X(T) = \text{Hom}(T, \mathbb{G}_m)$ is the free abelian group on generators $\delta_1, \dots, \delta_n$, where δ_r picks out the rth diagonal entry of a diagonal matrix. We will always identify $X(T)$ with $\mathbb{Z}^n$, the tuple $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ corresponding to the character $\sum_{r=1}^n \lambda_r \delta_r \in X(T)$. The root system associated to G_{ev} is denoted $R = R^+ \cup (-R^+)$, where $R^+ = \{\delta_r - \delta_s \,|\, 1 \le r < s \le n\} \subset \mathbb{Z}^n$. The dominance ordering on $\mathbb{Z}^n$ is defined by $\lambda \le \mu$ if and only if $\mu - \lambda$ is a sum of positive roots. This is not the same as the Bruhat ordering $\preceq$ defined in §2-c, though we have by Lemma 2.14 that $\lambda \preceq \mu$ implies $\lambda \le \mu$.

A representation of G means a natural transformation $\rho : G \to GL(M)$ for some vector superspace M, where $GL(M)$ is the supergroup with $GL(M, A)$ being equal to the group of all even automorphisms of the A-supermodule $M \otimes A$, for each commutative superalgebra A. Equivalently, as with group schemes [J1, I.2.8], M is a right $k[G]$-comodule, i.e. there is an even structure map

$\eta : M \to M \otimes k[G]$ satisfying the usual comodule axioms. We will usually refer to such an M as a *G-supermodule*. Let $\mathcal{C}_n$ denote the category of all finite dimensional G-supermodules. Note we allow arbitrary (not necessarily homogeneous) morphisms in $\mathcal{C}_n$. We write Π for the parity change functor, and denote the dual of a finite dimensional G-supermodule M by M^*. There is another natural duality $M \mapsto M^\tau$ on $\mathcal{C}_n$, see e.g. [BK2, §10] for the definition.

There are two sorts of irreducible G-supermodule: either *type* $\mathtt{M}$ if $\mathrm{End}_G(M)$ is one dimensional or *type* $\mathtt{Q}$ if $\mathrm{End}_G(M)$ is two dimensional. For example, we have the *natural representation* V of G, the vector superspace on basis $v_1, \dots, v_n, v_1', \dots, v_n'$, where v_r is even and v_r' is odd. For a superalgebra A, we identify elements of $V \otimes A$ with column vectors

$$\sum_{r=1}^n (v_r \otimes a_r + v_r' \otimes a_r') \longleftrightarrow \begin{pmatrix} a_1 \\ \vdots \\ a_n \\ a_1' \\ \vdots \\ a_n' \end{pmatrix}.$$

Then, the action of $G(A)$ on $V \otimes A$ defining the supermodule structure is the obvious action on column vectors by left multiplication. The map

$$J : V \to V, \qquad v_r \mapsto v_r', v_r' \mapsto -v_r \tag{4.2}$$

is an odd automorphism of V as a G-supermodule. Hence, V is irreducible of type $\mathtt{Q}$.

For $\lambda \in \mathbb{Z}^n$, we write $x^\lambda = x_1^{\lambda_1} \dots x_n^{\lambda_n} \in \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Given an H-supermodule M, we let M_λ denote its λ-weight space with respect to the torus T. Identifying the Weyl group associated to G_{ev} with the symmetric group S_n, the character

$$\mathrm{ch}\, M := \sum_{\lambda \in \mathbb{Z}^n} (\dim M_\lambda) x^\lambda$$

of a finite dimensional G-supermodule M is naturally S_n-invariant, so is an element of the ring $\mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}$ of symmetric functions.

For every $\lambda \in \mathbb{Z}^n$, there is by [BK2, Lemma 6.4] a unique irreducible H-supermodule denoted $\mathfrak{u}(\lambda)$ with character $2^{\lfloor (h(\lambda)+1)/2 \rfloor} x^\lambda$, where $h(\lambda) = n - z(\lambda)$ denotes the number of $r = 1, \dots, n$ for which $\lambda_r \neq 0$. It is an irreducible H-supermodule of type $\mathtt{M}$ if $h(\lambda)$ is even, type $\mathtt{Q}$ otherwise. We will often regard $\mathfrak{u}(\lambda)$ instead as an irreducible B-supermodule via the obvious epimorphism $B \to H$. Introduce the induced supermodule

$$H^0(\lambda) := \mathrm{ind}_B^G \mathfrak{u}(\lambda) = H^0(G/B, \mathscr{L}(\mathfrak{u}(\lambda))), \tag{4.3}$$

see [BK2, §6] and [B2, §2] for the detailed construction. The following theorem is due to Penkov [P1, Theorem 4], see also [BK2, Theorem 6.11].

Theorem 4.4. *For $\lambda \in \mathbb{Z}^n$, $H^0(\lambda)$ is non-zero if and only if $\lambda \in \mathbb{Z}^n_+$. In that case $H^0(\lambda)$ has a unique irreducible submodule denoted $L(\lambda)$. Moreover,*

(i) $\{L(\lambda)\}_{\lambda \in \mathbb{Z}^n_+}$ *is a complete set of pairwise non-isomorphic irreducible G-supermodules;*
(ii) $L(\lambda)$ *is of the same type as* $\mathfrak{u}(\lambda)$, *i.e. type* $\mathtt{M}$ *if* $h(\lambda)$ *is even, type* $\mathtt{Q}$ *otherwise;*
(iii) $\mathrm{ch}\, L(\lambda) = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} x^\lambda + (*)$ *where* $(*)$ *is a linear combination of* x^μ *for* $\mu < \lambda$;
(iv) $L(\lambda)^* \cong L(-w_0\lambda)$ *and* $L(\lambda)^\tau \cong L(\lambda)$.

§4-b. **Euler characteristics.** Let $K(\mathcal{C}_n)$ denote the Grothendieck group of the superadditive category $\mathcal{C}_n$, in the sense of [BK1, §2-c]. So by Theorem 4.4(i), $K(\mathcal{C}_n)$ is the free abelian group on basis $\{[L(\lambda)]\}_{\lambda \in \mathbb{Z}^n_+}$. Since ch is additive on short exact sequences, there is an induced map

$$\mathrm{ch}\, : K(\mathcal{C}_n) \to \mathbb{Z}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]^{S_n}, \qquad [M] \mapsto \mathrm{ch}\, M.$$

Theorem 4.4(iii) implies that the irreducible characters are linearly independent, hence this map is injective. By Theorem 4.4(iv), the duality $*$ on finite dimensional G-supermodules induces an involution $* : K(\mathcal{C}_n) \to K(\mathcal{C}_n)$ with $[L(\lambda)]^* = [L(-w_0\lambda)]$. On the other hand, the duality τ leaves characters invariant, so gives the identity map at the level of Grothendieck groups.

For each $\lambda \in \mathbb{Z}^n$, we have the higher cohomology supermodules

$$H^i(\lambda) := R^i \mathrm{ind}_P^G \mathfrak{u}(\lambda) = H^i(G/P, \mathscr{L}(\mathfrak{u}(\lambda))) \tag{4.5}$$

where P is the largest parabolic subgroup of G to which the B-supermodule $\mathfrak{u}(\lambda)$ can be lifted, we refer to [B2, §4] for details. In particular, it is known that each $H^i(\lambda)$ is finite dimensional and is zero for $i \gg 0$, so the Euler characteristic

$$[E(\lambda)] := \sum_{i \geq 0} (-1)^i [H^i(\lambda)] \tag{4.6}$$

is a well-defined element of the Grothendieck group $K(\mathcal{C}_n)$. Its character

$$\mathrm{ch}\, E(\lambda) := \sum_{i \geq 0} (-1)^i \mathrm{ch}\, H^i(\lambda) \tag{4.7}$$

can be computed explicitly by a method going back at least to Penkov [P2, §2.3]. We need to recall the definition of *Schur's P-function* p_λ for $\lambda \in \mathbb{Z}^n_+$:

$$p_\lambda = \sum_{w \in S_n/S_\lambda} w\left(x^\lambda \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i > \lambda_j}} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right), \tag{4.8}$$

where S_n/S_λ denotes the set of minimal length coset representatives for the stabilizer S_λ of λ in S_n. Note that p_λ is equal to the Hall-Littlewood symmetric function

$$p_\lambda(t) = \sum_{w \in S_n/S_\lambda} w\left(x^\lambda \prod_{\substack{1 \leq i < j \leq n \\ \lambda_i > \lambda_j}} \frac{x_i - t x_j}{x_i - x_j} \right), \tag{4.9}$$

evaluated at $t = -1$, see [M, III(2.2)] and [M, III.8] (actually, Macdonald only describes the case when all $\lambda_i \geq 0$, but everything easily extends to $\lambda_i \in \mathbb{Z}$). We note the following combinatorial lemma.

Lemma 4.10. *$(x_1 + \cdots + x_n)p_\lambda = \sum_r p_{\lambda+\delta_r}$, where the sum is over all $r = 1, \dots, n$ such that $\lambda + \delta_r \in \mathbb{Z}^n_+$ and moreover $\lambda_r \neq -1$ if $z(\lambda)$ is odd.*

Proof. Let $\lambda \in \mathbb{Z}^n$ with $\lambda_1 \geq \cdots \geq \lambda_n$. Following the proof of [M, III(3.2)], one shows that

$$(x_1 + \cdots + x_n)p_\lambda(t) = \sum_r (1 + t + \cdots + t^{n_r})p_{\lambda+\delta_r}(t),$$

where the sum is over all $r = 1, \dots, n$ with $\lambda_r < \lambda_{r-1}$ if $r > 1$, and n_r denotes $\#\{s \mid 1 \leq s \leq n, \lambda_s = \lambda_r + 1\}$. The lemma follows on setting $t = -1$. □

The following theorem is [PS1, Proposition 1], see [B2, Theorem 4.7] for a more detailed exposition.

Theorem 4.11. *For each $\lambda \in \mathbb{Z}^n_+$, $\operatorname{ch} E(\lambda) = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} p_\lambda$.*

This shows in particular that $\operatorname{ch} E(\lambda)$ equals $2^{\lfloor (h(\lambda)+1)/2 \rfloor} x^\lambda + (*)$ where $(*)$ is a linear combination of x^μ for $\mu < \lambda$. Comparing with Theorem 4.4(iii), we deduce that for each $\lambda \in \mathbb{Z}^n_+$, the *decomposition numbers* $d_{\mu,\lambda}$ defined from

$$[E(\mu)] = \sum_{\lambda \in \mathbb{Z}^n_+} d_{\mu,\lambda}[L(\lambda)] \tag{4.12}$$

satisfy $d_{\mu,\mu} = 1$ and $d_{\mu,\lambda} = 0$ for $\lambda \not\leq \mu$. Hence, $\{[E(\lambda)]\}_{\lambda \in \mathbb{Z}^n_+}$ gives another natural basis for the Grothendieck group $K(\mathcal{C}_n)$. Also define the *inverse decomposition numbers* $d^{-1}_{\mu,\lambda}$ from

$$[L(\mu)] = \sum_{\lambda \in \mathbb{Z}^n_+} d^{-1}_{\mu,\lambda}[E(\lambda)]. \tag{4.13}$$

Again we have that $d^{-1}_{\mu,\mu} = 1$ and $d^{-1}_{\mu,\lambda} = 0$ for $\lambda \not\leq \mu$.

§4-c. **Category $\mathcal{O}_n$.** The Lie superalgebra $\mathfrak{g}$ of G can be identified with the Lie superalgebra $\mathfrak{q}(n)$ of all matrices of the form

$$x = \left(\begin{array}{c|c} S & S' \\ \hline S' & S \end{array}\right) \tag{4.14}$$

under the supercommutator $[.,.]$, where S and S' are $n \times n$ matrices over $\mathbb{C}$ and such a matrix is even if $S' = 0$ or odd if $S = 0$, see [BK2, §4]. We will let $e_{r,s}$ resp. $e'_{r,s}$ denote the even resp. odd matrix unit, i.e. the matrix of the form (4.14) with the rs-entry of S resp. S' equal to 1 and all other entries equal to zero. We will abbreviate $h_r := e_{r,r}, h'_r := e'_{r,r}$. Let $\mathfrak{h}$ be the *Cartan subalgebra* of $\mathfrak{g}$ spanned by $\{h_r, h'_r \mid 1 \leq r \leq n\}$, and let $\mathfrak{b}$ be the *positive Borel subalgebra* spanned by $\{e_{r,s}, e'_{r,s} \mid 1 \leq r \leq s \leq n\}$. Note $\mathfrak{h}$ is the Lie superalgebra of the subgroup H, but perversely $\mathfrak{b}$ is not the Lie superalgebra of B since that consisted of lower triangular matrices!

For any $\lambda \in \mathbb{Z}^n$ and an $\mathfrak{h}$-supermodule M, we define the λ-weight space M_λ of M by

$$M_\lambda = \{m \in M \mid h_i m = \lambda_i m \text{ for each } i = 1, \dots, n\}.$$

When M is an H-supermodule viewed as an $\mathfrak{h}$-supermodule in the canonical way, the notion of weight space defined here agrees with the earlier one. Let $\mathcal{O}_n$ denote the category of all finitely generated $\mathfrak{g}$-supermodules M that are locally finite dimensional as $\mathfrak{b}$-supermodules and satisfy

$$M = \bigoplus_{\lambda \in \mathbb{Z}^n} M_\lambda,$$

cf. [BGG]. By [BK2, Corollary 5.7], we can identify the category $\mathcal{C}_n$ with the full subcategory of $\mathcal{O}_n$ consisting of all the finite dimensional objects.

Lemma 4.15. *Every $M \in \mathcal{O}_n$ has a composition series.*

Proof. Since the universal enveloping superalgebra $U(\mathfrak{g})$ of $\mathfrak{g}$ is Noetherian, every finitely generated $\mathfrak{g}$-supermodule M admits a descending filtration $M = M_0 > M_1 > \dots$ with each M_{i-1}/M_i simple. We need to show this filtration is of finite length in case $M \in \mathcal{O}_n$. It suffices for this to show that the restriction of M to $\mathfrak{g}_{\bar{0}} \cong \mathfrak{gl}(n)$ has a composition series. Since $U(\mathfrak{g})$ is a free left $U(\mathfrak{g}_{\bar{0}})$-supermodule of finite rank, M is still finitely generated when viewed as a $\mathfrak{g}_{\bar{0}}$-module, and clearly it is locally finite dimensional over $\mathfrak{b}_{\bar{0}}$. Hence the restriction of M to $\mathfrak{g}_{\bar{0}}$ belongs to the analogue of category $\mathcal{O}_n$ for $\mathfrak{g}_{\bar{0}}$. It is well-known all such $\mathfrak{g}_{\bar{0}}$-modules have a composition series. □

For each $\lambda \in \mathbb{Z}^n$, we define the *Verma supermodule*

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathfrak{u}(\lambda) \in \mathcal{O}_n, \tag{4.16}$$

where $\mathfrak{u}(\lambda)$ is as defined in the previous subsection but viewed now as a $\mathfrak{b}$-supermodule by inflation from $\mathfrak{h}$. By the PBW theorem, we have that

$$\operatorname{ch} M(\lambda) = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} x^\lambda \prod_{1 \leq i < j \leq n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j}. \tag{4.17}$$

Also note by its definition as an induced supermodule that $M(\lambda)$ is universal amongst all $\mathfrak{g}$-supermodules generated by a $\mathfrak{b}$-stable submodule isomorphism to $\mathfrak{u}(\lambda)$. The following theorem is quite standard and parallels Theorem 4.4 above.

Theorem 4.18. *For every $\lambda \in \mathbb{Z}^n$, $M(\lambda)$ has a unique irreducible quotient denoted $L(\lambda)$. Moreover,*

- (i) *$\{L(\lambda)\}_{\lambda \in \mathbb{Z}^n}$ is a complete set of pairwise non-isomorphic irreducibles in category $\mathcal{O}_n$;*
- (ii) *$L(\lambda)$ is of the same type as $\mathfrak{u}(\lambda)$, i.e. type* `M` *if $h(\lambda)$ is even, type* `Q` *otherwise;*
- (iii) *$\operatorname{ch} L(\lambda) = 2^{\lfloor (h(\lambda)+1)/2 \rfloor} x^\lambda + (*)$ where $(*)$ is a linear combination of x^μ for $\mu < \lambda$;*
- (iv) *$L(\lambda)$ is finite dimensional if and only if $\lambda \in \mathbb{Z}^n_+$, in which case it is the same as the supermodule denoted $L(\lambda)$ before.*

Let $K(\mathcal{O}_n)$ be the Grothendieck group of the category $\mathcal{O}_n$. By Lemma 4.15 and Theorem 4.18(i), $K(\mathcal{O}_n)$ is the free abelian group on basis $\{[L(\lambda)]\}_{\lambda\in\mathbb{Z}^n}$. However the $[M(\lambda)]$'s do not form a basis for $K(\mathcal{O}_n)$. To get round this problem, let $\widehat{K}(\mathcal{O}_n)$ be the completion of $K(\mathcal{O}_n)$ with respect to the descending filtration $(K_d(\mathcal{O}_n))_{d\in\mathbb{Z}}$, where $K_d(\mathcal{O}_n)$ is the subgroup of $K(\mathcal{O}_n)$ generated by $\{[L(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n$ with $\sum_{i=1}^n i\lambda_i \geq d$. Taking characters induces a well-defined map

$$\mathrm{ch} : \widehat{K}(\mathcal{O}_n) \to \mathbb{Z}[[x_1^{\pm 1}, \dots, x_n^{\pm 1}]], \qquad [M] \mapsto \mathrm{ch}\, M.$$

By Theorem 4.18(iii), the characters of the irreducible supermodules in $\mathcal{O}_n$ are linearly independent, hence the map ch is injective. Moreover, $[M(\lambda)]$ equals $[L(\lambda)] + (*)$ where $(*)$ is a finite linear combination of $[L(\mu)]$'s for $\mu < \lambda$. So working in $\widehat{K}(\mathcal{O}_n)$, we can also write $[L(\lambda)]$ as $[M(\lambda)] + (\dagger)$ where $(\dagger)$ is a possibly infinite linear combination of $[M(\mu)]$'s for $\mu < \lambda$. This shows that the elements $\{[M(\lambda)]\}_{\lambda\in\mathbb{Z}^n}$ form a topological basis for $\widehat{K}(\mathcal{O}_n)$.

§4-d. **Central characters.** Let Z denote the even center of the universal enveloping superalgebra $U(\mathfrak{g})$. Sergeev [S2] has constructed an explicit set of generators of Z. For each $\lambda \in \mathbb{Z}^n$, let χ_λ be the central character afforded by the Verma supermodule $M(\lambda)$, so $z \in Z$ acts on $M(\lambda)$ as scalar multiplication by $\chi_\lambda(z)$. Also recall the definition of $\mathrm{wt}(\lambda)$ from (2.12), which is an element of the weight lattice P of type $\mathfrak{b}_\infty$ as defined in §2-a. As a consequence of Sergeev's description of Z, see e.g. [BK2, Lemma 8.9(ii),(iv)], we have the following fundamental fact:

Theorem 4.19. *For $\lambda, \mu \in \mathbb{Z}^n$, $\chi_\lambda = \chi_\mu$ if and only if $\mathrm{wt}(\lambda) = \mathrm{wt}(\mu)$.*

For each $\gamma \in P$, let $\mathcal{O}_\gamma$ denote the full subcategory of $\mathcal{O}_n$ consisting of the objects all of whose irreducible subquotients are of the form $L(\lambda)$ for $\lambda \in \mathbb{Z}^n$ with $\mathrm{wt}(\lambda) = \gamma$. As a consequence of Theorem 4.19, we have the *block decomposition*

$$\mathcal{O}_n = \bigoplus_{\gamma \in P} \mathcal{O}_\gamma. \tag{4.20}$$

Moreover, $\mathcal{O}_\gamma$ is non-zero if and only if γ is a non-trivial weight of the tensor space $\mathscr{T}^n$ of §2-b. Note that given $\lambda \in \mathbb{Z}^n$ with $\mathrm{wt}(\lambda) = \gamma$, $h(\lambda) \equiv (\gamma,\gamma)/2 \pmod 2$. Hence, recalling Theorem 4.18(ii), all the irreducible supermodules belonging to the block $\mathcal{O}_\gamma$ are of the same type. We refer to this as the *type* of the block $\mathcal{O}_\gamma$: type $\mathtt{M}$ if $(\gamma,\gamma)/2$ is even, type $\mathtt{Q}$ if $(\gamma,\gamma)/2$ is odd.

The block decomposition (4.20) induces the block decomposition

$$K(\mathcal{O}_n) = \bigoplus_{\gamma \in P} K(\mathcal{O}_\gamma) \tag{4.21}$$

of the Grothendieck group, so here $K(\mathcal{O}_\gamma)$ is the Grothendieck group of the category $\mathcal{O}_\gamma$. Let $\widehat{K}(\mathcal{O}_\gamma)$ be the closure of $K(\mathcal{O}_\gamma)$ in $\widehat{K}(\mathcal{O}_n)$. The elements $\{[M(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n$ with $\mathrm{wt}(\lambda) = \gamma$ form a topological basis for $\widehat{K}(\mathcal{O}_\gamma)$.

In a similar fashion, we have the block decomposition of the category $\mathcal{C}_n$ of finite dimensional G-supermodules:

$$\mathcal{C}_n = \bigoplus_{\gamma \in P} \mathcal{C}_\gamma. \tag{4.22}$$

This time, $\mathcal{C}_\gamma$ is non-zero if and only if γ is a non-trivial weight of the dual exterior power $\mathscr{E}^n$ from §3-c. Note moreover that the natural embedding of $\mathcal{C}_n$ into $\mathcal{O}_n$ embeds $\mathcal{C}_\gamma$ into $\mathcal{O}_\gamma$. We also get the block decomposition of the Grothendieck group:

$$K(\mathcal{C}_n) = \bigoplus_{\gamma \in P} K(\mathcal{C}_\gamma). \tag{4.23}$$

The elements $\{[L(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n_+$ with $\mathrm{wt}(\lambda) = \gamma$ form a basis for $K(\mathcal{C}_\gamma)$.

Lemma 4.24. $\prod_{1 \le i < j \le n} \frac{1 - x_i^{-1} x_j}{1 + x_i^{-1} x_j} \in \mathbb{Z}[[x_i^{-1} x_j \mid 1 \le i < j \le n]]$ *can be expressed as an infinite linear combination of* x^μ*'s for* $\mu \le 0$ *with* $\mathrm{wt}(\mu) = 0$*.*

Proof. Since the $\{[M(\mu)]\}_{\mu \in \mathbb{Z}^n, \mathrm{wt}(\mu) = 0}$ form a topological basis for $\widehat{K}(\mathcal{O}_0)$, we can write $[L(0)] = \sum_{\mu \le 0, \mathrm{wt}(\mu) = 0} a_\mu [M(\mu)]$ for some coefficients $a_\mu \in \mathbb{Z}$. Taking characters using (4.17) gives

$$1 = \sum_{\substack{\mu \le 0 \\ \mathrm{wt}(\mu) = 0}} \left(2^{\lfloor (h(\mu)+1)/2 \rfloor} a_\mu x^\mu \prod_{1 \le i < j \le n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right).$$

The lemma follows. □

Theorem 4.25. *For* $\lambda \in \mathbb{Z}^n_+$ *with* $\mathrm{wt}(\lambda) = \gamma$*, the Euler characteristic* $[E(\lambda)]$ *belongs to the block* $K(\mathcal{C}_\gamma)$*.*

Proof. Take $\lambda \in \mathbb{Z}^n_+$. By Theorem 4.11 and (4.8), we have that

$$\begin{aligned} \mathrm{ch}\, E(\lambda) &= 2^{\lfloor (h(\lambda)+1)/2 \rfloor} \sum_{w \in S_n / S_\lambda} w \left(x^\lambda \prod_{\substack{1 \le i < j \le n \\ \lambda_i = \lambda_j}} \frac{1 - x_i^{-1} x_j}{1 + x_i^{-1} x_j} \prod_{1 \le i < j \le n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j} \right) \\ &= 2^{\lfloor (h(\lambda)+1)/2 \rfloor} \sum_{w \in S_n / S_\lambda} (-1)^{\ell(w)} x^{w\lambda} \prod_{\substack{1 \le i < j \le n \\ \lambda_i = \lambda_j}} \frac{1 - x_{wi}^{-1} x_{wj}}{1 + x_{wi}^{-1} x_{wj}} \prod_{1 \le i < j \le n} \frac{1 + x_i^{-1} x_j}{1 - x_i^{-1} x_j}. \end{aligned}$$

By Lemma 4.24, $x^{w\lambda} \prod_{1 \le i < j \le n, \lambda_i = \lambda_j} \frac{1 - x_{wi}^{-1} x_{wj}}{1 + x_{wi}^{-1} x_{wj}}$ is a (possibly infinite) linear combination of x^μ's for $\mu \le w\lambda$ with $\mathrm{wt}(\mu) = \mathrm{wt}(w\lambda) = \gamma$. Hence recalling (4.17), $\mathrm{ch}\, E(\lambda)$ is a (possibly infinite) linear combination of $\mathrm{ch}\, M(\mu)$'s for $\mu \in \mathbb{Z}^n$ with $\mathrm{wt}(\mu) = \gamma$. Working instead in $\widehat{K}(\mathcal{O}_n)$, we have shown that we can write $[E(\lambda)]$ as a (possibly infinite) linear combination of $[M(\mu)]$'s for $\mu \in \mathbb{Z}^n$ with $\mathrm{wt}(\mu) = \gamma$, i.e. $[E(\lambda)] \in \widehat{K}(\mathcal{O}_\gamma)$. The theorem follows. □

The theorem immediately implies that the decomposition numbers $d_{\mu,\lambda}$ defined in (4.12) and the inverse decomposition numbers $d^{-1}_{\mu,\lambda}$ defined in (4.13) are zero whenever $\mathrm{wt}(\mu) \neq \mathrm{wt}(\lambda)$. In particular:

Corollary 4.26. *The elements* $\{[E(\lambda)]\}$ *for* $\lambda \in \mathbb{Z}^n_+$ *with* $\mathrm{wt}(\lambda) = \gamma$ *form a basis for* $K(\mathcal{C}_\gamma)$.

§4-e. **Translation functors.** Now we are ready to link the Grothendieck group $K(\mathcal{C}_n)$ with the $\mathscr{U}_\mathbb{Z}$-module $\mathscr{E}^n_\mathbb{Z}$ constructed in §3-f. We define an isomorphism of abelian groups

$$\iota : K(\mathcal{C}_n) \to \mathscr{E}^n_\mathbb{Z}, \qquad [E(\lambda)] \mapsto E_\lambda(1) \qquad (\lambda \in \mathbb{Z}^+_n). \tag{4.27}$$

Using ι, we lift the actions of the generators of $\mathscr{U}_\mathbb{Z}$ on $\mathscr{E}^n_\mathbb{Z}$ from (3.31) to define an action of $\mathscr{U}_\mathbb{Z}$ directly on the Grothendieck group $K(\mathcal{C}_n)$. Note Corollary 4.26 shows that the block decomposition (4.23) coincides with the usual weight space decomposition of $K(\mathcal{C}_n)$ as a $\mathscr{U}_\mathbb{Z}$-module. Using Lemma 3.20 specialized at $q = 1$, we can write down the action of the operators E_i, F_i on $K(\mathcal{C}_n)$ explicitly:

$$E_i[E(\lambda)] = \sum_{\substack{r \text{ with } \lambda-\delta_r \in \mathbb{Z}^n_+,\\ \sigma_r = -,-+ \text{ or } --}} c_{\lambda,r}[E(\lambda - \delta_r)], \tag{4.28}$$

$$F_i[E(\lambda)] = \sum_{\substack{r \text{ with } \lambda+\delta_r \in \mathbb{Z}^n_+,\\ \sigma_r = +,-+ \text{ or } ++}} c_{\lambda,r}[E(\lambda + \delta_r)], \tag{4.29}$$

for $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$. Here, $(\sigma_1, \dots, \sigma_n)$ is the i-signature of λ defined according to (2.30) and

$$c_{\lambda,r} = \begin{cases} 1 + (-1)^{z(\lambda)} & \text{if } \sigma_r = -- \text{ or } ++,\\ 1 & \text{otherwise.} \end{cases}$$

The goal in the remainder of this subsection is to give a representation theoretic interpretation of the operators E_i and F_i.

We need certain "translation functors"

$$\mathrm{Tr}^i, \mathrm{Tr}_i : \mathcal{O}_n \to \mathcal{O}_n,$$

one for each $i \geq 0$, similar to the functors defined by Penkov and Serganova in [PS2, §2.3]. Recalling the decomposition (4.22), it suffices to define the functors $\mathrm{Tr}^i, \mathrm{Tr}_i$ on the subcategory $\mathcal{O}_\gamma$ for some fixed $\gamma \in P$, since one can then extend additively to get the definition on $\mathcal{O}_n$ itself. We will write $\mathrm{pr}_\gamma : \mathcal{O}_n \to \mathcal{O}_\gamma$ for the natural projection functor. Given $M \in \mathcal{O}_\gamma$, let

$$\mathrm{Tr}^i M = \mathrm{pr}_{\gamma-\alpha_i}(M \otimes V), \qquad \mathrm{Tr}_i M = \mathrm{pr}_{\gamma+\alpha_i}(M \otimes V^*). \tag{4.30}$$

On a morphism $f : M \to N$ in $\mathcal{O}_\gamma$, $\mathrm{Tr}^i f$ and $\mathrm{Tr}_i f$ are the restrictions of the maps $f \otimes \mathrm{id}$. Obviously these functors send finite dimensional supermodules to finite dimensional supermodules, so also give us functors

$$\mathrm{Tr}^i, \mathrm{Tr}_i : \mathcal{C}_n \to \mathcal{C}_n$$

by restriction.

Lemma 4.31. *For each $i \geq 0$, the functors Tr^i and Tr_i (on either of the categories $\mathcal{O}_n$ or $\mathcal{C}_n$) are both left and right adjoint to each other, hence both are exact functors. Moreover, there are natural isomorphisms*

$$(\mathrm{Tr}^i M)^* \cong \mathrm{Tr}_i(M^*), \qquad (\mathrm{Tr}_i M)^* \cong \mathrm{Tr}^i(M^*), \tag{4.32}$$

$$(\mathrm{Tr}^i M)^\tau \cong \mathrm{Tr}^i(M^\tau) \qquad (\mathrm{Tr}_i M)^\tau \cong \mathrm{Tr}_i(M^\tau) \tag{4.33}$$

for each $M \in \mathcal{C}_n$.

Proof. The first part is a well-known fact about translation functors. For the second part, we obviously have natural isomorphisms $(M \otimes V)^* \cong M^* \otimes V^*$, $(M \otimes V)^\tau \cong M^\tau \otimes V$. So it suffices to show that $*$-duality maps $\mathcal{C}_\gamma$ into $\mathcal{C}_{-\gamma}$. and that τ-duality maps $\mathcal{C}_\gamma$ into itself. This is immediate from Theorem 4.4(iv). □

Lemma 4.34. *For $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$, and a finite dimensional G-supermodule M belonging to the block $\mathcal{C}_\gamma$ for some $\gamma \in P$, we have that*

$$[\mathrm{Tr}_i M] = \begin{cases} E_i[M] & \text{if } i = 0 \text{ and } \mathcal{C}_\gamma \text{ is of type } \mathtt{M}, \\ 2E_i[M] & \text{if } i \neq 0 \text{ or } \mathcal{C}_\gamma \text{ is of type } \mathtt{Q}; \end{cases}$$

$$[\mathrm{Tr}^i M] = \begin{cases} F_i[M] & \text{if } i = 0 \text{ and } \mathcal{C}_\gamma \text{ is of type } \mathtt{M}, \\ 2F_i[M] & \text{if } i \neq 0 \text{ or } \mathcal{C}\gamma \text{ is of type } \mathtt{Q}. \end{cases}$$

Proof. We explain the argument for Tr^i, the case of Tr_i being similar. Since the functor Tr^i is exact, it induces an additive map also denoted Tr^i on $K(\mathcal{C}_n)$. By Corollary 4.26, the $\{[E(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n_+$ with $\mathrm{wt}(\lambda) = \gamma$ form a basis for $K(\mathcal{C}_\gamma)$. Therefore it suffices to consider $\mathrm{Tr}^i[E(\lambda)]$ for such a λ.

Observe that $\mathrm{ch}\, V = 2(x_1 + \cdots + x_n)$ and that the functor $? \otimes V$ is isomorphic to the functor $\bigoplus_{i \geq 0} \mathrm{Tr}^i$. So a calculation using Theorem 4.11 and Lemma 4.10 gives that

$$\bigoplus_{i \geq 0} \mathrm{Tr}^i[E_\lambda] = \sum_{r \text{ with } \lambda + \delta_r \in \mathbb{Z}^n_+} d_{\lambda,r}[E_{\lambda+\delta_r}]$$

where

$$d_{\lambda,r} = \begin{cases} 2 & \text{if } \lambda_r \neq -1, 0, \\ 1 & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ is even}, \\ 2 & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ is odd}, \\ 0 & \text{if } \lambda_r = -1 \text{ and } z(\lambda) \text{ is odd}, \\ 2 & \text{if } \lambda_r = -1,\ z(\lambda) \text{ is even and } h(\lambda) \text{ is even}, \\ 4 & \text{if } \lambda_r = -1,\ z(\lambda) \text{ is even and } h(\lambda) \text{ is odd}. \end{cases}$$

Now apply $\mathrm{pr}_{\mathrm{wt}(\lambda)-\alpha_i}$ to both sides and use Theorem 4.25 and (4.29) to deduce that

$$\mathrm{Tr}^i[E(\lambda)] = \begin{cases} F_i[E(\lambda)] & \text{if } i = 0 \text{ and } h(\lambda) \text{ is even}, \\ 2F_i[E(\lambda)] & \text{if } i \neq 0 \text{ or } h(\lambda) \text{ is odd}. \end{cases}$$

This completes the proof. □

§4-f. **Crystal structure.** In this section we relate the structure of the supermodules $\mathrm{Tr}^i L(\lambda)$ and $\mathrm{Tr}_i L(\lambda)$ for certain weights λ to the crystal operators defined in §3-d. The methods employed here are essentially the same as those of Penkov and Serganova in [PS2, §2.3]. Throughout the subsection we will use the following notation: for supermodules X and Y, $\frac{X}{Y}$ will denote some extension of X by Y, and $[M : L]$ will denote the composition multiplicity of an irreducible L in M.

Lemma 4.35. *Let* $\lambda = (1, 0^{n-2}, -1)$. *Then* $[M(\lambda) : L(0)] \geq 2$.

Proof. Pick a basis v, v' for $M(\lambda)_\lambda$ such that $h'_1 v = v', h'_1 v' = v, h'_n v = v', h'_n v' = -v$ and $h'_r v = h'_r v' = 0$ for $r = 2, \dots, n-1$. Let

$$x = e'_{2,1}(e'_{3,2} + e_{3,2}h'_3) \dots (e'_{n,n-1} + e_{n,n-1}h'_n)v, \qquad y = h'_1 x.$$

We claim that $\langle x, y \rangle$ is a two dimensional indecomposable $\mathfrak{b}$-submodule of $M(\lambda)$. The lemma follows immediately form this, since both x and y are of weight 0. To prove the claim, we proceed by induction on n. The base case $n = 2$ follows on checking that $x = e'_{2,1}v$ is annihilated by $e_{1,2}$ and $e'_{1,2}$ and that $y = h'_1 x = h'_2 x$. For $n > 2$, we set

$$w = (e'_{n,n-1} + e_{n,n-1}h'_n)v = e'_{n,n-1}v + e_{n,n-1}v',$$
$$w' = -(e'_{n,n-1} + e_{n,n-1}h'_n)v' = e_{n,n-1}v - e'_{n,n-1}v'.$$

Now one checks that $h'_1 w = w', h'_1 w' = w, h'_{n-1} w = w', h'_{n-1} w' = -w$ and that $h'_r w = h'_r w' = 0$ for each $r = 2, \dots, n-2, n$. Moreover, w, w' are annihilated by $e_{r,r+1}, e'_{r,r+1}$ for $r = 1, \dots, n-1$. Now the result follows from the induction hypothesis. □

Lemma 4.36. *Let* $\lambda \in \mathbb{Z}^n_+$ *with* $\lambda_r > 0$ *and* $\lambda_r + \lambda_s = 0$ *for some* $1 \leq r < s \leq n$. *Then,*

$$[M(\lambda) : L(\lambda - \delta_r + \delta_s)] = \begin{cases} 1 & \text{if } \lambda_r > 1, \\ 2 & \text{if } \lambda_r = 1. \end{cases}$$

Proof. We will actually only prove here that the left hand side is $\geq$ the right hand side, since that is all we really need later on. The fact that the left hand side actually *equals* the right hand side can easily be extracted from the proof of Lemma 4.39 below. First suppose that $\lambda_r > 1$. By [PS2, Proposition 2.1],

$$\mathrm{Hom}_{\mathfrak{g}}(M(\lambda - \delta_r + \delta_s), M(\lambda)) \neq 0.$$

This immediately implies that $[M(\lambda) : L(\lambda - \delta_r + \delta_s)] \geq 1$. Now assume that $\lambda_r = 1$, when we need to show that in fact $[M(\lambda) : L(\lambda - \delta_r + \delta_s)] \geq 2$. Let $\mathfrak{p}$ be the upper triangular parabolic subalgebra of $\mathfrak{g}$ of type $(1^{r-1}, s - r + 1, 1^{n-s})$. Consider the Verma supermodule $M_{\mathfrak{p}}(\lambda) = U(\mathfrak{p}) \otimes_{U(\mathfrak{b})} \mathfrak{u}(\lambda)$ over $\mathfrak{p}$. By Lemma 4.35, it contains $L_{\mathfrak{p}}(\lambda - \delta_r + \delta_s)$ with multiplicity at least two. Since $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} M_{\mathfrak{p}}(\lambda) \cong M(\lambda)$ and $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} L_{\mathfrak{p}}(\lambda - \delta_r + \delta_s) \twoheadrightarrow L(\lambda - \delta_r + \delta_s)$, the lemma follows. □

Lemma 4.37. *Let $\lambda \in \mathbb{Z}^n$. Then $M := M(\lambda) \otimes V$ has a filtration $0 = M_0 < M_1 < \cdots < M_n = M$ such that*

$$M_r/M_{r-1} \cong \begin{cases} M(\lambda+\delta_r) \oplus \Pi M(\lambda+\delta_r) & \text{if } \lambda_r \neq -1, 0 \\ M(\lambda+\delta_r) & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ even,} \\ M(\lambda+\delta_r) \oplus \Pi M(\lambda+\delta_r) & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ odd,} \\ \frac{M(\lambda+\delta_r)}{M(\lambda+\delta_r)} & \text{if } \lambda_r = -1 \text{ and } h(\lambda) \text{ even,} \\ \frac{\Pi M(\lambda+\delta_r)}{M(\lambda+\delta_r)} \oplus \frac{M(\lambda+\delta_r)}{\Pi M(\lambda+\delta_r)} & \text{if } \lambda_r = -1 \text{ and } h(\lambda) \text{ odd,} \end{cases}$$

for each $r = 1, \dots, n$.

Proof. Note $V \cong \mathfrak{u}(\delta_1) \oplus \cdots \oplus \mathfrak{u}(\delta_n)$ as an $\mathfrak{h}$-supermodule. Now the lemma follows by a standard construction, see e.g. [B1, Lemma 4.24] for a similar situation, from the observation that

$$\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r) \cong \begin{cases} \mathfrak{u}(\lambda+\delta_r) \oplus \Pi\mathfrak{u}(\lambda+\delta_r) & \text{if } \lambda_r \neq -1, 0 \\ \mathfrak{u}(\lambda+\delta_r) & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ even,} \\ \mathfrak{u}(\lambda+\delta_r) \oplus \Pi\mathfrak{u}(\lambda+\delta_r) & \text{if } \lambda_r = 0 \text{ and } h(\lambda) \text{ odd,} \\ \frac{\mathfrak{u}(\lambda+\delta_r)}{\mathfrak{u}(\lambda+\delta_r)} & \text{if } \lambda_r = -1 \text{ and } h(\lambda) \text{ even,} \\ \frac{\Pi\mathfrak{u}(\lambda+\delta_r)}{\mathfrak{u}(\lambda+\delta_r)} \oplus \frac{\mathfrak{u}(\lambda+\delta_r)}{\Pi\mathfrak{u}(\lambda+\delta_r)} & \text{if } \lambda_r = -1 \text{ and } h(\lambda) \text{ odd} \end{cases}$$

as $\mathfrak{h}$-supermodules. We prove this in just two of the situations, the rest being similar.

First suppose $\lambda_r = -1$ and $h(\lambda)$ is odd. By character considerations $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$ has just four composition factors, all isomorphic to $\mathfrak{u}(\lambda+\delta_r)$. Moreover, both $\mathfrak{u}(\lambda)$ and $\mathfrak{u}(\delta_r)$ possess odd automorphisms J_1, J_2 respectively with $J_i^2 = -1$. The $\pm\sqrt{-1}$-eigenspaces of the even automorphism $J_1 \otimes J_2$ of $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$ decompose it into a direct sum of two factors, and the map $J_1 \otimes 1$ gives an odd isomorphism between the factors. Hence $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$ has the structure given.

Second suppose $\lambda_r \neq -1, 0$ and $h(\lambda)$ is even. In that case, $\mathfrak{u}(\lambda)$ is of type `M`, and by character considerations $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$ has just two composition factors, both isomorphic to $\mathfrak{u}(\lambda+\delta_r)$ which also has type `M`. Let J be an odd automorphism of $\mathfrak{u}(\delta_r)$. Then, $1 \otimes J$ is an odd automorphism of $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$. If $\mathfrak{u}(\lambda) \otimes \mathfrak{u}(\delta_r)$ was to be a non-split extension of $\mathfrak{u}(\lambda+\delta_r)$ and $\Pi\mathfrak{u}(\lambda+\delta_r)$, it would possess a nilpotent odd endomorphism, but no odd automorphism. Hence it must split as a direct sum as required. □

Note on applying the exact functor $\mathrm{pr}_{\mathrm{wt}(\lambda)-\alpha_i}$ to the filtration constructed in the lemma, we obtain a filtration of $\mathrm{Tr}^i M(\lambda)$. Moreover, if $\lambda \in \mathbb{Z}^n_+$, $L(\lambda)$ is a finite dimensional quotient of $M(\lambda)$, so $\mathrm{Tr}^i L(\lambda)$ is a finite dimensional quotient of $\mathrm{Tr}^i M(\lambda)$. Hence we also get induced a filtration of $\mathrm{Tr}^i L(\lambda)$ whose the factors are finite dimensional quotients of the factors of the filtration of $\mathrm{Tr}^i M(\lambda)$. We refer to this as the *canonical filtration* of $\mathrm{Tr}^i L(\lambda)$. Its properties are used repeatedly in the proofs of the next two lemmas.

Lemma 4.38. *Suppose $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$ are such that $\varphi_i(\lambda) = 1$. Then, $F_i[L(\lambda)] = [L(\tilde{F}_i(\lambda))]$.*

Proof. We actually prove the slightly stronger statement that

$$\mathrm{Tr}^i L(\lambda) \cong \begin{cases} L(\mu) & \text{if } i = 0 \text{ and } h(\lambda) \text{ is even,} \\ L(\mu) \oplus \Pi L(\mu) & \text{if } i > 0 \text{ or } h(\lambda) \text{ is odd,} \end{cases}$$

where $\mu := \tilde{F}_i(\lambda)$. The lemma follows from this and Lemma 4.34 above. The possible cases for λ, μ are listed explicitly in §3-d, in particular we have that $\mu = \lambda + \delta_s$ for some $1 \leq s \leq n$. By considering the canonical filtration in each of the cases, $\mathrm{Tr}^i M(\lambda) \cong M(\mu)$ if $i = 0$ and $h(\lambda)$ is even, $\mathrm{Tr}^i M(\lambda) \cong M(\mu) \oplus \Pi M(\mu)$ otherwise. In particular, $[\mathrm{Tr}^i M(\lambda) : L(\mu)] = 1$ if $i = 0$ and $h(\lambda)$ is even, $[\mathrm{Tr}^i M(\lambda) : L(\mu)] = 2$ otherwise. Now, $\mathrm{Tr}^i L(\lambda)$ is a quotient of $\mathrm{Tr}^i M(\lambda)$, and moreover it is self-dual under the duality τ by Theorem 4.4(iv) and (4.33). Hence, $\mathrm{Tr}^i L(\lambda)$ is necessarily a quotient of $L(\mu)$ if $i = 0$ and $h(\lambda)$ is even, $L(\mu) \oplus \Pi(\mu)$ otherwise. So to complete the proof, we just need to show that $[\mathrm{Tr}^i L(\lambda) : L(\mu)] = [\mathrm{Tr}^i M(\lambda) : L(\mu)]$.

Suppose for a contradiction that $[\mathrm{Tr}^i L(\lambda) : L(\mu)] < [\mathrm{Tr}^i M(\lambda) : L(\mu)]$. Then there must be some composition factor $L(\lambda') \not\cong L(\lambda)$ of $M(\lambda)$ such that $[\mathrm{Tr}^i L(\lambda') : L(\mu)] > 0$. Hence, $[\mathrm{Tr}^i M(\lambda') : L(\mu)] > 0$ for some $\lambda' < \lambda$ with $\mathrm{wt}(\lambda') = \mathrm{wt}(\lambda)$. Considering the canonical filtration of $\mathrm{Tr}^i M(\lambda')$, there must exist some $1 \leq r \leq n$ such that $\mu = \lambda + \delta_s \leq \lambda' + \delta_r$. But then $\lambda - \delta_r + \delta_s \leq \lambda' < \lambda$ and $\mathrm{wt}(\lambda') = \mathrm{wt}(\lambda)$. It is easy to see in each case that there is no such λ', giving the desired contradiction. □

Lemma 4.39. *Suppose $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$ are such that $\varphi_i(\lambda) = 0$. Then, $F_i[L(\lambda)] = 0$.*

Proof. The possibilities for λ are listed explictly in §3-d. In almost all of the configurations, we get that $\mathrm{Tr}^i L(\lambda) = 0$, hence $F_i[L(\lambda)] = 0$, immediately by looking at the canonical filtration. There are just two difficult cases in which we need to argue further.

In the first case, $i = 0$ and $\lambda = (\cdots, 1, 0^{r-1}, -1, \cdots)$ for some $r \geq 1$, where $\cdots$ denote entries different from $-1, 0, 1$. We let $\mu = (\cdots, 0^{r+1}, \cdots)$ and $\nu = (\cdots, 1, 0^r, \cdots)$, where the $\cdots$ are the same entries as in λ. Also let $c = 1$ if $h(\lambda)$ is even, $c = 2$ if $h(\lambda)$ is odd. By considering the canonical filtration, we see here that $[\mathrm{Tr}^i M(\lambda) : L(\nu)] = 2c$. By Lemma 4.36, $[M(\lambda) : L(\mu)] \geq 2$, hence $[\mathrm{Tr}^i M(\lambda) : L(\nu)] \geq [\mathrm{Tr}^i L(\lambda) : L(\nu)] + 2[\mathrm{Tr}^i L(\mu) : L(\nu)]$. By Lemma 4.38, $[\mathrm{Tr}^i L(\mu) : L(\nu)] = c$. This shows that $[\mathrm{Tr}^i L(\lambda) : L(\nu)] = 0$ (and also that $[M(\lambda) : L(\mu)] = 2$, completing the proof of Lemma 4.36 in this case). But by the canonical filtration, $\mathrm{Tr}^i L(\lambda)$ has a filtration where all the factors are quotients of $M(\nu)$, so it has to be zero.

In the second case, $i > 0$ and $\lambda = (\dots, i+1, \dots, -i-1, \dots)$, where $\dots$ denote entries different from $-i-1, -i, i, i+1$. We let $\mu = (\cdots, i, \cdots, -i, \cdots)$ and $\nu = (\cdots, i+1, \cdots, -i, \cdots)$, where the $\cdots$ are the same entries as in λ. By the canonical filtration, $\mathrm{Tr}^i M(\lambda) \cong M(\nu) \oplus \Pi M(\nu)$, hence $[\mathrm{Tr}^i M(\lambda) : L(\nu)] = 2$. By Lemma 4.36, $[M(\lambda) : L(\mu)] \geq 1$, hence $[\mathrm{Tr}^i M(\lambda) : L(\nu)] \geq [\mathrm{Tr}^i L(\lambda) : L(\nu)] + [\mathrm{Tr}^i L(\mu) : L(\nu)]$. By Lemma 4.38, $[\mathrm{Tr}^i L(\mu) : L(\nu)] = 2$. This shows that $[\mathrm{Tr}^i L(\lambda) : L(\nu)] = 0$ (and also completes the proof of Lemma 4.36 in this case). Hence, since $\mathrm{Tr}^i L(\lambda)$ is a quotient of $M(\nu) \oplus \Pi M(\nu)$, it has to be zero. $\square$

§4-g. **Injective supermodules.** We refer at this point to [J1, I.3] for the general facts about injective modules over a group scheme, all of which generalize to supergroups. In particular, for every G-supermodule M, there is an injective G-supermodule U, unique up to isomorphism, such that $\mathrm{soc}_G M \cong \mathrm{soc}_G U$. We call U the *injective hull* of M. For $\lambda \in \mathbb{Z}^n_+$, let $U(\lambda)$ denote the injective hull of $L(\lambda)$. Any injective G-supermodule M is isomorphic to a direct sum of $U(\lambda)$'s, the number of summands isomorphic to $U(\lambda)$ being equal to the multiplicity of $L(\lambda)$ in $\mathrm{soc}_G M$.

Lemma 4.40. *Each $U(\lambda)$ is finite dimensional.*

Proof. Consider the functor $\mathrm{ind}^G_{G_{\mathrm{ev}}} := \mathrm{Hom}_{U(\mathfrak{g}_{\bar{0}})}(U(\mathfrak{g}), ?)$. Since $U(\mathfrak{g})$ is a free right $U(\mathfrak{g}_{\bar{0}})$-supermodule of finite rank, this is an exact functor mapping finite dimensional G_{ev}-supermodules to finite dimensional G-supermodules. It is right adjoint to the restriction functor $\mathrm{res}^G_{G_{\mathrm{ev}}}$ from the category of G-supermodules to the category of G_{ev}-supermodules, so sends injectives to injectives. Now take $\lambda \in \mathbb{Z}^n_+$. Since *every* G_{ev}-module is injective, we get that $\mathrm{ind}^G_{G_{\mathrm{ev}}} \mathrm{res}^G_{G_{\mathrm{ev}}} L(\lambda)$ is a finite dimensional injective G-supermodule. Moreover, the unit of the adjunction gives an embedding of $L(\lambda)$ into $\mathrm{ind}^G_{G_{\mathrm{ev}}} \mathrm{res}^G_{G_{\mathrm{ev}}} L(\lambda)$. Hence the injective hull of $L(\lambda)$ is finite dimensional. $\square$

Let $\mathcal{C}^*_n$ be the category of all finite dimensional injective G-supermodules. The block decomposition (4.22) of $\mathcal{C}_n$ induces an analogous block decomposition of the subcategory $\mathcal{C}^*_n$

$$\mathcal{C}^*_n = \bigoplus_{\gamma \in P} \mathcal{C}^*_\gamma. \tag{4.41}$$

Let $K(\mathcal{C}^*_n)$ (resp. $K(\mathcal{C}^*_\gamma)$) be the Grothendieck group of the category $\mathcal{C}^*_n$ (resp. $\mathcal{C}^*_\gamma$). By Lemma 4.40, $K(\mathcal{C}^*_n)$ is the free abelian group on basis $\{[U(\lambda)]\}_{\lambda \in \mathbb{Z}^n_+}$, and $K(\mathcal{C}^*_\gamma)$ is the subgroup generated by the $\{[U(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n_+$ with $\mathrm{wt}(\lambda) = \gamma$.

Form the completion $\widehat{K}(\mathcal{C}^*_n)$ of the Grothendieck group $K(\mathcal{C}^*_n)$ with respect to the descending filtration $(K_d(\mathcal{C}^*_n))_{d \in \mathbb{Z}}$ where $K_d(\mathcal{C}^*_n)$ is the subgroup generated by $\{[U(\lambda)]\}$ for $\lambda \in \mathbb{Z}^n_+$ with $\sum_{i=1}^n i\lambda_{n+1-i} \geq d$. The important thing in this definition is that vectors of the form $[U(\lambda)] + (*)$ make sense whenever $(*)$ is an infinite linear combination of $[U(\mu)]$'s for $\mu > \lambda$. In particular, the following are well-defined elements of $\widehat{K}(\mathcal{C}^*_n)$ for each $\lambda \in \mathbb{Z}^n_+$:

$$[F(\lambda)] := \sum_{\mu \in \mathbb{Z}^n_+} d^{-1}_{-w_0\mu, -w_0\lambda} [U(\mu)], \tag{4.42}$$

recall (4.13). By the unitriangularity of the inverse decomposition numbers, the elements $\{[F(\lambda)]\}_{\lambda \in \mathbb{Z}^n_+}$ give a topological basis for $\widehat{K}(\mathcal{C}^*_n)$. Note $[F(\lambda)]$ does not in general belong to $K(\mathcal{C}^*_n)$ itself.

We define a pairing $\langle .,. \rangle$ between the Grothendieck groups $K(\mathcal{C}_n)$ and $K(\mathcal{C}_n^*)$ by letting

$$\langle [L(-w_0\lambda)], [U(\mu)] \rangle = \delta_{\lambda,\mu} \tag{4.43}$$

for each $\lambda, \mu \in \mathbb{Z}_+^n$. The pairing $\langle .,. \rangle$ extends by continuity to give a pairing also denoted $\langle .,. \rangle$ between $K(\mathcal{C}_n)$ and $\widehat{K}(\mathcal{C}_n^*)$. In that case, by the definitions (4.12), (4.42) and (4.43), we have that

$$\langle [E(-w_0\lambda)], [F(\mu)] \rangle = \delta_{\lambda,\mu} \tag{4.44}$$

for each $\lambda, \mu \in \mathbb{Z}_+^n$. We record the following lemma which follows from a standard property of injective hulls, see [J1, I.3.17(3)].

Lemma 4.45. *Suppose we are given $M \in \mathcal{C}_n$ and $U \in \mathcal{C}_\gamma^*$ for some $\gamma \in P$. Then,*

$$\langle [M], [U] \rangle = \begin{cases} \dim \mathrm{Hom}_G(M^*, U) & \text{if } \mathcal{C}_\gamma \text{ is of type } \mathtt{M}, \\ \frac{1}{2} \dim \mathrm{Hom}_G(M^*, U) & \text{if } \mathcal{C}_\gamma \text{ is of type } \mathtt{Q}. \end{cases}$$

Now recall the definition of the $\mathscr{U}_\mathbb{Z}$-module $\widehat{\mathscr{F}}_\mathbb{Z}^n$ from §3-f. We define a continuous isomorphism

$$\iota^* : \widehat{\mathscr{F}}_\mathbb{Z}^n \to \widehat{K}(\mathcal{C}_n^*), \qquad F_\lambda(1) \mapsto [F(\lambda)] \qquad (\lambda \in \mathbb{Z}_n^+). \tag{4.46}$$

The following lemma shows that this map ι^* is the dual map to ι from (4.27) with respect to the pairings $\langle .,. \rangle$:

Lemma 4.47. *$\langle \iota([M]), v \rangle = \langle [M], \iota^*(v) \rangle$ for all $[M] \in K(\mathcal{C}_n)$ and $v \in \widehat{\mathscr{F}}_\mathbb{Z}^n$.*

Proof. It suffices to check this for $[M] = [E(-w_0\lambda)]$ and $v = F_\mu(1)$ for $\lambda, \mu \in \mathbb{Z}_+^n$, when it follows immediately from (3.35) and (4.44). $\square$

As in §4-e, we lift the action of $\mathscr{U}_\mathbb{Z}$ on $\widehat{\mathscr{F}}_\mathbb{Z}^n$ to the completed Grothendieck group $\widehat{K}(\mathcal{C}_n^*)$ through the isomorphism ι^*. By Lemmas 4.47 and 2.21, we have at once that

$$\langle E_i[L], [U] \rangle = \langle [L], E_i[U] \rangle, \qquad \langle F_i[L], [U] \rangle = \langle [L], F_i[U] \rangle, \tag{4.48}$$

for each $i \geq 0$ and all $[L] \in K(\mathcal{C}_n), [U] \in \widehat{K}(\mathcal{C}_n^*)$. The next lemma gives a purely representation theoretic interpretation of the operators E_i and F_i on $\widehat{K}(\mathcal{C}_n^*)$, cf. Lemma 4.34. For the statement, recall by Lemma 4.31 that the functors Tr_i and Tr^i send injectives to injectives.

Lemma 4.49. *For $\lambda \in \mathbb{Z}_+^n$ and $i \geq 0$, there exist injective G-supermodules $E_iU(\lambda)$ and $F_iU(\lambda)$, unique up to isomorphism, characterized by*

$$\mathrm{Tr}_i U(\lambda) \cong \begin{cases} E_iU(\lambda) & \text{if } i = 0 \text{ and } h(\lambda) \text{ is even}, \\ E_iU(\lambda) \oplus \Pi E_iU(\lambda) & \text{if } i \neq 0 \text{ or } h(\lambda) \text{ is odd}; \end{cases}$$

$$\mathrm{Tr}^i U(\lambda) \cong \begin{cases} F_iU(\lambda) & \text{if } i = 0 \text{ and } h(\lambda) \text{ is even}, \\ F_iU(\lambda) \oplus \Pi F_iU(\lambda) & \text{if } i \neq 0 \text{ or } h(\lambda) \text{ is odd}. \end{cases}$$

Moreover, $E_i[U(\lambda)] = [E_iU(\lambda)]$ and $F_i[U(\lambda)] = [F_iU(\lambda)]$.

Proof. We just explain the proof for F_i. Take $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$. Note uniqueness of $F_i U(\lambda)$ is immediate by Krull-Schmidt. For existence, we consider three separate cases.

Case one. If $i = 0$ and $h(\lambda)$ is even, let $F_i U(\lambda) := \mathrm{Tr}^i U(\lambda)$.

Case two. If $h(\lambda)$ is odd, $L(\lambda)$ is of type $\mathtt{Q}$, so possesses an odd automorphism J_1 with $J_1^2 = -1$. This induces an odd automorphism also denoted J_1 of the injective hull $U(\lambda)$. Also the natural representation V possesses the odd automorphism J defined earlier (4.2). The map $J_1 \otimes J$ induces an even automorphism of the summand $\mathrm{Tr}^i U(\lambda)$ of $U(\lambda) \otimes V$. Its $\pm\sqrt{-1}$-eigenspaces decompose $\mathrm{Tr}^i U(\lambda)$ into a direct sum of two G-supermodules, and the map $1 \otimes J$ is an odd isomorphism between them. Let $F_i U(\lambda)$ be the $\sqrt{-1}$-eigenspace (say), then $\mathrm{Tr}^i U(\lambda) \cong F_i U(\lambda) \oplus \Pi F_i U(\lambda)$ as required.

Case three. If $h(\lambda)$ is even and $i > 0$, then the map $1 \otimes J$ induces an odd automorphism of $\mathrm{Tr}^i U(\lambda)$, hence also of the socle S of $\mathrm{Tr}^i U(\lambda)$. All constituents of S are of type $\mathtt{M}$, so S must decompose as a G-supermodule as $S_- \oplus S_+$ with $(1 \otimes J) S_\pm = S_\mp$. This decomposition of the socle induces a decomposition of the injective supermodule $\mathrm{Tr}^i U(\lambda)$, say $\mathrm{Tr}^i U(\lambda) = M_- \oplus M_+$ where $\mathrm{soc}_G M_\pm = S_\pm$. In this case we let $F_i U(\lambda) = M_+$ (say).

It remains to show that $F_i[U(\lambda)] = [F_i U(\lambda)]$. To do this, it suffices to prove that $\langle [L(\mu)], F_i[U(\lambda)] \rangle = \langle [L(\mu)], [F_i U(\lambda)] \rangle$ for all $\mu \in \mathbb{Z}^n_+$. This is done using Lemmas 4.45, 4.34, (4.48) and the adjointness of $\mathrm{Tr}^i, \mathrm{Tr}_i$ from Lemma 4.31. We just explain the argument in the case that $h(\lambda)$ is even and $i = 0$, the other situations being entirely similar. First note by weight considerations that both sides of the identity we are trying to verify are zero unless $h(\mu)$ is odd. Now compute:

$$\begin{aligned}\langle [L(\mu)], F_i[U(\lambda)] \rangle = \langle F_i[L(\mu)], [U(\lambda)] \rangle &= \frac{1}{2} \langle [\mathrm{Tr}^i L(\mu)], [U(\lambda)] \rangle \\ &= \frac{1}{2} \dim \mathrm{Hom}_G((\mathrm{Tr}^i L(\mu))^*, U(\lambda)) \\ &= \frac{1}{2} \dim \mathrm{Hom}_G(\mathrm{Tr}_i(L(\mu)^*), U(\lambda)) \\ &= \frac{1}{2} \dim \mathrm{Hom}_G(L(\mu)^*, \mathrm{Tr}^i U(\lambda)) \\ &= \langle L(\mu), [\mathrm{Tr}^i U(\lambda)] \rangle = \langle L(\mu), [F_i U(\lambda)] \rangle.\end{aligned}$$

This completes the proof. ∎

Lemma 4.50. *Let $\lambda \in \mathbb{Z}^n_+$ and $i \geq 0$.*

(i) *If $\varphi_i(\lambda) = 0$ and $\varepsilon_i(\lambda) > 0$ then $E_i U(\lambda) \cong U(\tilde{E}_i(\lambda))$.*
(ii) *If $\varepsilon_i(\lambda) = 0$ and $\varphi_i(\lambda) > 0$ then $F_i U(\lambda) = U(\tilde{F}_i(\lambda))$.*

Proof. We just prove (ii), since (i) follows by applying $*$. We know that $F_i U(\lambda)$ is a direct sum of injective indecomposables. To compute the multiplicity of $U(\mu)$ for a given $\mu \in \mathbb{Z}^n_+$, it suffices to compute

$$\langle [L(\mu)], [F_i U(\lambda)] \rangle = \langle [L(\mu)], F_i[U(\lambda)] \rangle = \langle F_i[L(\mu)], [U(\lambda)] \rangle.$$

By block considerations and Lemmas 4.38 and 4.39, that is zero unless $\mu = -w_0\tilde{F}_i(\lambda)$, in which case it is one. Hence $F_iU(\lambda) = U(\tilde{F}_i(\lambda))$. $\square$

Now we can construct the injective supermodules $U(\lambda)$. If λ is typical, there are no problems:

Lemma 4.51. *Suppose that $\lambda \in \mathbb{Z}^n_+$ is typical. Then $U(\lambda) = H^0(\lambda) = L(\lambda)$.*

Proof. If λ is typical then there are no other $\mu \in \mathbb{Z}^n_+$ with $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda)$. So by the linkage principle [BK2, Theorem 8.10], the induced module $H^0(\lambda)$ is actually equal to $L(\lambda)$ in this case. Using this and the observation that $\mathfrak{u}(\lambda)$ is an injective H-supermodule for typical λ, we get from [BK2, Theorem 7.5] that $\mathrm{Ext}^1_G(L(\lambda), L(\mu)) = \mathrm{Ext}^1_G(L(\mu), L(\lambda)) = 0$ for all $\mu \in \mathbb{Z}^n_+$. In particular, $L(\lambda)$ is injective, hence $L(\lambda) = U(\lambda)$. $\square$

Note in particular that the lemma shows that $U(\lambda)$ is self-dual with respect to the duality τ in the case that λ is typical. Now suppose that $\lambda \in \mathbb{Z}^n_+$ is not typical. Apply Procedure 3.26 to get $\mu \in \mathbb{Z}^n_+$ and an operator $X_i \in \{E_i, F_i\}$. Since this process reduces λ to a typical weight in finitely many steps, we may assume inductively that $U(\mu)$ has already been constructed, and that $U(\mu) \cong U(\mu)^\tau$. Just like in Lemma 3.27, but applying Lemma 4.50 in place of Lemma 3.25, we have that $U(\lambda) \cong X_iU(\mu)$. Moreover,

$$U(\lambda)^\tau \cong (X_iU(\mu))^\tau \cong X_i(U(\mu)^\tau) \cong X_iU(\mu) \cong U(\lambda),$$

hence $U(\lambda)$ is also self-dual. We obtain in this way an explicit algorithm to construct all the injective indecomposables. As a by-product we see that each $U(\lambda)$ is actually self-dual with respect to the duality τ, hence is isomorphic to the *projective cover* of $L(\lambda)$.

Now we can prove the main result of the article. It shows that the map ι^* from (4.46) maps the canonical basis of $\mathscr{F}^n_{\mathbb{Z}}$ to the canonical basis of $K(\mathcal{C}_n^*)$ given by the injective indecomposables, and that the map ι from (4.27) maps the canonical basis of $K(\mathcal{C}_n)$ given by the irreducible supermodules to the dual canonical basis of $\mathscr{E}^n_{\mathbb{Z}}$.

Theorem 4.52. *For each $\lambda \in \mathbb{Z}^n_+$, $\iota^*(U_\lambda(1)) = [U(\lambda)]$ and $\iota([L(\lambda)]) = L_\lambda(1)$.*

Proof. In view of Lemma 4.47 and the facts that $[L(\lambda)]$ is dual to $[U(-w_0\lambda)]$ and $L_\lambda(1)$ is dual to $U_{-w_0\lambda}(1)$, it suffices to prove just that $\iota^*(U_\lambda(1)) = [U(\lambda)]$. If λ is typical, then $U_\lambda(1) = F_\lambda(1)$ and $[U(\lambda)] = [F(\lambda)]$, since there are no other $\mu \in \mathbb{Z}^n_+$ with $\mathrm{wt}(\mu) = \mathrm{wt}(\lambda)$. So the result holds for typical weights. The result follows in general because by definition the map ι^* commutes with the operators E_i, F_i and the algorithm for constructing the $U(\lambda)$'s explained above exactly parallels the algorithm for constructing the $U_\lambda(1)$'s from §3-e. $\square$

We get by the theorem and (3.7), (3.10) and (3.17) respectively that

$$[E(\lambda)] = \sum_{\mu \in \mathbb{Z}^n_+} u_{-w_0\lambda,-w_0\mu}(1)[L(\mu)], \tag{4.53}$$

$$[L(\lambda)] = \sum_{\mu \in \mathbb{Z}^n_+} l_{\mu,\lambda}(1)[E(\mu)], \tag{4.54}$$

$$[U(\lambda)] = \sum_{\mu \in \mathbb{Z}^n_+} u_{\mu,\lambda}(1)[F(\mu)], \tag{4.55}$$

for each $\lambda \in \mathbb{Z}^n_+$. Note finally that $u_{-w_0\lambda,-w_0\mu}(q) = u_{\lambda,\mu}(q)$, since $\omega(F_\mu) = F_{-w_0\mu}$ and $\omega(U_\mu) = U_{-w_0\mu}$ by Theorem 3.5. So comparing (4.53) with (4.12) gives that $d_{\mu,\lambda} = u_{\mu,\lambda}(1)$. The Main Theorem stated in the introduction follows from this statement and Theorem 3.36.

§4-h. **Conjectures.** To conclude the article, we make the following conjecture: for each $\lambda \in \mathbb{Z}^n$, we have that

$$[M(\lambda)] = \sum_{\mu \in \mathbb{Z}^n} t_{-\lambda,-\mu}(1)[L(\mu)]. \tag{4.56}$$

Comparing (2.24) and Corollary 2.28, this conjecture is equivalent to the statement that

$$[L(\lambda)] = \sum_{\mu \in \mathbb{Z}^n} l_{\mu,\lambda}(1)[M(\mu)], \tag{4.57}$$

equality in $\widehat{K}(\mathcal{O}_n)$. The N_λ's in section 2 correspond to the supermodules $N(\lambda)$ defined by $N(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \tilde{u}(\lambda)$, where $\tilde{u}(\lambda)$ denotes the projective cover of $\mathfrak{u}(\lambda)$ in the category of $\mathfrak{h}$-supermodules that are semisimple over $\mathfrak{h}_{\bar{0}}$. Note at least that $[N(\lambda)] = 2^{z(\lambda)}[M(\lambda)]$ in the Grothendieck group, cf. (2.7) and [BK2, (7.1)]. The T_λ's in section 2 should correspond to the *indecomposable tilting modules* $T(\lambda)$ in category $\mathcal{O}_n$, cf. [B3, Example 7.10]. We recall that for $\lambda \in \mathbb{Z}^n$, $T(\lambda)$ is the supermodule characterized uniquely up to isomorphism by the properties:

(1) $T(\lambda) \in \mathcal{O}_n$ is indecomposable;
(2) $\mathrm{Ext}^1_{\mathcal{O}_n}(N(\mu), T(\lambda)) = 0$ for all $\mu \in \mathbb{Z}^n$;
(3) $T(\lambda)$ has a filtration where the subquotients are of the form $N(\mu)$ for $\mu \in \mathbb{Z}^n$, starting with $N(\lambda)$ at the bottom.

Conjecture (4.56) is equivalent to the statement

$$[T(\lambda)] = \sum_{\mu \in \mathbb{Z}^n} t_{\mu,\lambda}(1)[N(\mu)], \tag{4.58}$$

as follows by [B3, (7.12)].

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