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Lowering operators for $GL(n)$ and quantum $GL(n)$

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Abstract. We describe some developments in the representation theory of $GL(n)$ which depend on certain *lowering operators* recently discovered by Kleshchev. We give a simple new definition of these lowering operators and explain the relationship between these and operators which have previously appeared in the work of Carter-Lusztig and others. Our approach simplifies two important applications: the construction of orthogonal bases for Weyl modules over $\mathbb{C}$ and Kleshchev's modular branching rules for symmetric groups. We also describe previously unknown analogues of these two results in the quantum case.

1. Introduction

At the Arcata conference in 1987, Carter [**5**] defined a certain *lowering operator*, denoted $S_{i,j}$ for integers $1 \leq i < j \leq n$, in the universal enveloping algebra $U_{\mathbb{C}}(n)$ of the Lie algebra $\mathfrak{gl}(n, \mathbb{C})$. This operator is defined over $\mathbb{Z}$, so lies in Kostant's $\mathbb{Z}$-form $U_{\mathbb{Z}}(n)$ for $U_{\mathbb{C}}(n)$. Consequently, by base change, one obtains an operator, also denoted $S_{i,j}$, in the *hyperalgebra* $U_{\mathbb{F}}(n) := U_{\mathbb{Z}}(n) \otimes_{\mathbb{Z}} \mathbb{F}$ of $GL(n)$ over a field $\mathbb{F}$ of arbitrary characteristic p.

The purpose of this article is to give a new description of a remarkable generalisation of Carter's operator $S_{i,j}$, originally discovered by Kleshchev in his work on 'modular branching'. The generalisation, denoted $S_{i,j}(A)$, is parametrised in addition by a subset A of the open interval $(i..j) := \{i+1, \dots, j-1\} \subset \mathbb{N}$. In the special case $A = \varnothing$, $S_{i,j}(A)$ is just the element usually denoted $F_{i,j}$ in $\mathfrak{gl}(n, \mathbb{F})$, that is, the lower triangular matrix with a 1 in the ji-entry and zeros elsewhere. At the other extreme, if $A = (i..j)$, then $S_{i,j}(A)$ is precisely Carter's original operator $S_{i,j}$. These generalised operators are the key new technical tool in the proof of Kleshchev's modular branching rule for the symmetric group $\mathfrak{S}(r)$ [**14, 15, 16**], and hence in Ford and Kleshchev's recent proof [**17, 10**] of the Mullineux conjecture [**20**]. This result describes the irreducible $\mathbb{F}\mathfrak{S}(r)$-module obtained by tensoring an arbitrary irreducible $\mathbb{F}\mathfrak{S}(r)$-module by the sign representation.

Our approach to these operators, based on material in [**2**, Chapters 7–8], results in some significant simplifications to the previously rather technical proofs in the

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work of Kleshchev [**16**], as well as to the earlier work of Carter [**5**]. It also allows us to generalise all definitions to the corresponding *quantum hyperalgebra* $U_{\mathbb{F},v}(n)$.

Once we have described this new formulation of the lowering operators, in both the classsical and quantum cases, we will give two applications. The first application is a construction of an orthogonal basis for Weyl modules for $U_{\mathbb{C}(v),v}(n)$ (ie not at roots of unity), which uses the quantum lowering operators combined with a simplified version of Carter's argument from [**5**] in the classical case.

The second application is to prove the quantum analogue of Kleshchev's modular branching rule, describing the socle of the restriction of an irreducible representation for the Hecke algebra of the symmetric group $\mathfrak{S}(r)$ to the Hecke algebra of $\mathfrak{S}(r-1)$, over $\mathbb{F}$ and at an arbitrary root of unity. As a consequence, this branching rule yields a proof of the 'quantum Mullineux conjecture' for these Hecke algebras. These results are the subject of [**4**], and we only give a brief survey here.

The modular branching rule for the Hecke algebra follows (by a Schur functor argument) from a partial modular branching rule for the restriction of irreducible $U_{\mathbb{F},v}(n)$-modules to $U_{\mathbb{F},v}(n-1)$ – all that is needed is to understand a part of this restriction known as the *first level*. The modular branching rule for levels higher than the first level is not known. At the end of the paper, we discuss this problem of understanding *all levels* in the restriction of an irreducible $U_{\mathbb{F},v}(n)$-module to $U_{\mathbb{F},v}(n-1)$. We explain a computational algorithm to solve this problem, and ask whether there is a purely combinatorial solution.

2. Carter's lowering operator

In this article, we will be working with the *hyperalgebra* $U_{\mathbb{F}}(n)$ corresponding to the algebraic group $GL(n,\mathbb{F})$ over an arbitrary field $\mathbb{F}$. The relationship between the representation theory of this hyperalgebra and rational representations of $GL(n,\mathbb{F})$ is well known (see [**13**, I.7]). The aim in this section is to define this hyperalgebra, and then review the classical theory of the lowering operators from [**5**].

2.1. The hyperalgebra. Let $U_{\mathbb{C}}(n)$ be the universal enveloping algebra of the Lie algebra $\mathfrak{gl}(n,\mathbb{C})$. Let $X_{i,j} \in \mathfrak{gl}(n,\mathbb{C})$ denote the $n \times n$ matrix with a 1 in the ij-entry and zeros elsewhere. Then, $U_{\mathbb{C}}(n)$ can be defined as the $\mathbb{C}$-algebra generated by $\{X_{i,j} \mid 1 \leq i,j \leq n\}$ subject to the relations

$$X_{i,h}X_{k,j} - X_{k,j}X_{i,h} = \delta_{kh}X_{i,j} - \delta_{ij}X_{k,h} \tag{2.1}$$

for all $1 \leq i,h,k,j \leq n$. For $1 \leq i < j \leq n$, we shall adopt the following shorthands in the usual way for elements of $U_{\mathbb{C}}(n)$:

$$\begin{aligned} E_{i,j} &:= X_{i,j}, \qquad E_i := E_{i,i+1};\\ F_{i,j} &:= X_{j,i}, \qquad F_i := F_{i,i+1};\\ H_i &:= X_{ii}, \qquad H_{i,j} := H_i - H_j. \end{aligned}$$

For an arbitrary $X \in U_{\mathbb{C}}$, we write $X^{(r)}$ for $X^r/r!$. Let $U_{\mathbb{Z}}(n)$ be Kostant's $\mathbb{Z}$-form for $U_{\mathbb{C}}(n)$. As is well known (see [**24, 6**]), $U_{\mathbb{Z}}(n)$ is a subring of $U_{\mathbb{C}}(n)$ with $\mathbb{Z}$-basis, known as the *PBW-basis*, as follows:

$$\underbrace{\prod_{1\leq i<j\leq n} F_{i,j}^{(N_{ij})}}_{U_{\mathbb{Z}}^-(n)} \underbrace{\prod_{1\leq i\leq n} \binom{H_i}{N_{ii}}}_{U_{\mathbb{Z}}^0(n)} \underbrace{\prod_{1\leq i<j\leq n} E_{i,j}^{(N_{ji})}}_{U_{\mathbb{Z}}^+(n)}$$

as $N = (N_{ij})_{1\le i,j\le n}$ runs over all $n \times n$ matrices with entries in $\mathbb{Z}_{\ge 0}$. Here, the symbol $\begin{pmatrix} x \\ n \end{pmatrix}$ denotes $\frac{1}{n!}x(x-1)\dots(x-n+1)$. We specify the order of multiplication in the first and last products in this expression as follows. In the $U_{\mathbb{Z}}^-(n)$-product, we choose the order for the tuples (i,j) in the product to be

$$(1,2);(1,3),(2,3);(1,4),\dots,(3,4);\dots;(1,n),\dots,(n-1,n).$$

In the $U_{\mathbb{Z}}^+(n)$-product the order is the opposite of this. Let $U_{\mathbb{Z}}^-(n), U_{\mathbb{Z}}^0(n), U_{\mathbb{Z}}^+(n)$ be the subrings of $U_{\mathbb{Z}}(n)$ generated by the terms of this basis indicated above, so that $U_{\mathbb{Z}}(n) \cong U_{\mathbb{Z}}^-(n) \otimes U_{\mathbb{Z}}^0(n) \otimes U_{\mathbb{Z}}^+(n)$. There is a natural antiautomorphism τ of $U_{\mathbb{C}}(n)$ defined on generators by $\tau(X_{i,j}) = X_{j,i}$. This stabilises $U_{\mathbb{Z}}(n)$.

Now we can define the hyperalgebra $U_{\mathbb{F}}(n)$ (resp. $U_{\mathbb{F}}^-(n), U_{\mathbb{F}}^0(n), U_{\mathbb{F}}^+(n)$) over an arbitrary field $\mathbb{F}$ of characteristic p by tensoring with $\mathbb{F}$, so $U_{\mathbb{F}}(n) := U_{\mathbb{Z}}(n) \otimes_{\mathbb{Z}} \mathbb{F}$ and so on. We write simply $X^{(r)}$ for the image in $U_{\mathbb{F}}(n)$ of the element $X^{(r)} \in U_{\mathbb{Z}}(n)$. We usually work in $U_{\mathbb{F}}(n)$, so no confusion should arise. Note that τ induces an antiautomorphism of $U_{\mathbb{F}}(n)$.

Let $U_{\mathbb{F}}(n-1) < U_{\mathbb{F}}(n)$ be the naturally embedded hyperalgebra corresponding to the subgroup $GL(n-1) < GL(n)$. So, $U_{\mathbb{F}}(n-1)$ is defined by base change starting from $U_{\mathbb{C}}(n-1)$ which is the subalgebra of $U_{\mathbb{C}}(n)$ generated by $\{X_{i,j} \mid 1 \le i,j \le n-1\}$.

Let $\varepsilon_i : U_{\mathbb{C}}^0(n) \to \mathbb{C}$ denote the unique algebra homomorphism such that $\varepsilon_i(H_j) = \delta_{ij}$ for all $1 \le i,j \le n$. We make no distinction in our notation between the homomorphism $\varepsilon_i : U_{\mathbb{C}}^0(n) \to \mathbb{C}$, the homomorphism $\varepsilon_i : U_{\mathbb{Z}}^0(n) \to \mathbb{Z}$ defined by restriction, and the corresponding homomorphism $\varepsilon_i : U_{\mathbb{F}}^0(n) \to \mathbb{F}$ defined by reduction mod p. Let $\mathcal{X}$ denote the free abelian group with generators $\varepsilon_1,\dots,\varepsilon_n$. We shall call $\mathcal{X}$ the *weight lattice*, and elements of $\mathcal{X}$ are *weights*. For a dominant weight $\lambda \in \mathcal{X}$, we shall use the notation $\Delta_n(\lambda), L_n(\lambda)$ or $\nabla_n(\lambda)$ to denote the corresponding Weyl, irreducible or dual Weyl module for $U_{\mathbb{F}}(n)$ respectively.

2.2. Carter's lowering operator. We now define Carter's original operator $S_{i,j} \in U_{\mathbb{Z}}(n)$ for $1 \le i < j \le n$; the corresponding operators in $U_{\mathbb{F}}(n)$ are simply the image of these under the map $U_{\mathbb{Z}}(n) \to U_{\mathbb{F}}(n)$ defined by $X \mapsto X \otimes 1$, and we shall use the same notation for $S_{i,j} \in U_{\mathbb{F}}(n)$, relying on context to determine which we mean. Let $C(i,j) := j - i + H_{i,j}$, an element of $U_{\mathbb{Z}}^0(n)$. For $1 \le i < j \le n$, define $S_{i,j} \in U_{\mathbb{Z}}(n)$ to be the determinant

$$\begin{vmatrix} F_{i,i+1} & F_{i,i+2} & & \dots & F_{i,j-1} & F_{i,j} \\ -C(i,i+1) & F_{i+1,i+2} & & \dots & F_{i+1,j-1} & F_{i+1,j} \\ 0 & -C(i,i+2) & F_{i+2,i+3} & \dots & F_{i+2,j-1} & F_{i+2,j} \\ \vdots & & \ddots & \ddots & \vdots & \vdots \\ & & & -C(i,j-2) & F_{j-2,j-1} & F_{j-2,j} \\ 0 & \dots & & 0 & -C(i,j-1) & F_{j-1,j} \end{vmatrix}$$

Note some care is needed here in interpreting this determinant, since $U_{\mathbb{Z}}(n)$ is a non-commutative ring. We intend every monomial in the expanded determinant to involve terms in order corresponding to the order of the rows in the matrix. That is, we *define* the determinant of an $n \times n$ matrix M over a non-commutative ring to be $\sum_{\pi \in \mathfrak{S}(n)} \varepsilon(\pi) M_{1,1\pi} \dots M_{n,n\pi}$.

The key property from our point of view of these lowering operators is given in the following lemma. We will give a simple inductive proof of this in section 3. Essentially the lemma says that E_k and $S_{i,j}$ 'almost' commute providing $k \neq j-1$.

LEMMA 2.2. *Let $1 \leq i < j \leq n$ and $k \neq j-1$ be given. Then,*

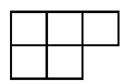
$$E_k S_{i,j} \equiv 0 \quad (\text{modulo } U_{\mathbb{F}}(n).E_k).$$

Thus, $S_{i,n}$ sends $U_{\mathbb{F}}^+(n)$-high weight vectors to $U_{\mathbb{F}}^+(n-1)$-high weight vectors.

One can also define a *raising operator* $R_{i,j}$ to be the image of $S_{i,j}$ under the antiautomorphism τ, as in [**5**]. We will not consider these elements here.

2.3. The classical branching rule. In order to explain the importance of Carter's lowering operator, we first review a result known as the classical branching rule. First, note if $\lambda = \lambda_1\varepsilon_1 + \cdots + \lambda_n\varepsilon_n$ is a dominant weight, then $\Delta_n(\lambda) \cong \Delta_n(\lambda_0) \otimes \det^{\lambda_n}$ where λ_0 is the dominant weight $(\lambda_1 - \lambda_n)\varepsilon_1 + \cdots + (\lambda_{n-1} - \lambda_n)\varepsilon_{n-1}$ and det is the 1-dimensional determinant module. In the questions we are considering, we will always assume that $\lambda_n = 0$ applying this remark if necessary. Then, λ may be identified with the *partition* $(\lambda_1, \ldots, \lambda_{n-1})$.

Recall that a λ-*tableaux* is a function $[\lambda] \to [1..n]$, that is, a way of writing integers in $[1..n] := \{1, \ldots, n\}$ into the boxes of the Young diagram $[\lambda]$ of λ. For example, if $\lambda = (3,2)$, its Young diagram is the following set of boxes in the plane:



The following are examples of λ-tableaux:

2	2	1
2	2	

1	2	2
1	1	

2	3	3
3	4	

A tableaux is *row standard* if the entries increase weakly along the rows (as in the second two tableaux in the above example) and *standard* if the entries increase weakly along rows, strictly down columns (as in the third tableau in the example).

Given a row standard λ-tableau t such that every entry on row i is greater than or equal to i, define

$$F_t := \prod_{1 \leq i < j \leq n} F_{i,j}^{(N_{i,j})}$$

where $N_{i,j}$ is equal to the number of entries equal to j on row i of t, and the order in the product is as in §2.1. Now we state the well-known *standard basis theorem* [**6**]:

THEOREM 2.3. *Let v_λ be a $U_{\mathbb{F}}^+(n)$-high weight vector for $\Delta_n(\lambda)$, over an arbitrary field $\mathbb{F}$. Then,*

$$\{F_t.v_\lambda \mid \textit{for all standard } \lambda\textit{-tableaux } t\}$$

is a basis for $\Delta_n(\lambda)$.

Given two partitions λ, μ, we write $\mu \twoheadleftarrow_i \lambda$ if the Young diagram $[\mu]$ of μ can be obtained from the diagram $[\lambda]$ by removing precisely one node from the bottom of i distinct columns. We write $\mu \twoheadleftarrow \lambda$ if $\mu \twoheadleftarrow_i \lambda$ for some i, in which case we say that μ belongs to the *ith level.* For example, if λ is the partition $(3,2)$, then $\mu \twoheadleftarrow \lambda$ if and only if μ equals $(3,2), (3,1), (3,0), (2,2), (2,1)$ or $(2,0)$.

As a final piece of notation, given $\mu \twoheadleftarrow \lambda$, define $t(\mu)$ to be the *standard* λ-tableau with hk-entry equal to h if (h,k) is the coordinate of a box in the diagram $[\mu]$, or n otherwise. For example, if $\lambda = (3,2)$ and $\mu = (2,1)$, then $t(\mu)$ is the tableau

1	1	n
2	n	

Then, the classical branching rule over an arbitrary field $\mathbb{F}$ can be stated as follows (see [**4**, Theorem 3.19] for a simple proof):

THEOREM 2.4. *Let $\mu_1, \ldots, \mu_N$ be all partitions $\mu \twoheadleftarrow \lambda$ ordered so that $\mu_i < \mu_j$ in the usual dominance order on $\mathcal{X}$ implies that $i > j$. Let v_λ be a $U^+_{\mathbb{F}}(n)$-high weight vector in $V = \Delta_n(\lambda)$, over an arbitrary field $\mathbb{F}$. Then*

(i) *V has a $U_{\mathbb{F}}(n-1)$-stable filtration $0 = V_0 < V_1 < \cdots < V_N = V$ such that $V_i/V_{i-1} \cong \Delta_{n-1}(\mu_i)$ for all i.*

(ii) *The image of $F_{t(\mu_i)}.v_\lambda$ in V_i/V_{i-1} is a $U^+_{\mathbb{F}}(n-1)$-high weight vector.*

In characteristic 0, this result has been known for a long time. In non-zero characteristic, it asserts that $\Delta_n(\lambda)$ has a *Weyl filtration* on restriction to $U_{\mathbb{F}}(n-1)$, which is a special case of Donkin's restriction theorem [**8**]. The more technical part (ii) of the theorem is less well known, but we need this in §5.3.

2.4. Constructing high weight vectors. Now we can describe the original motivation behind the definition of the operators $S_{i,j}$. Suppose that $\mathbb{F} = \mathbb{C}$, when $U_{\mathbb{C}}(n)$ is semisimple. Let $V = \Delta_n(\lambda)$ be an (irreducible) Weyl module, generated by a $U^+_{\mathbb{C}}(n)$-high weight vector v_λ. Since we are working over $\mathbb{C}$, the restriction of V to $U_{\mathbb{C}}(n-1)$ splits as a direct sum of irreducible $U_{\mathbb{C}}(n-1)$-modules, as given by Theorem 2.4. So, the restriction of $\Delta_n(\lambda)$ to $U_{\mathbb{C}}(n-1)$ equals

$$\bigoplus_{\mu \twoheadleftarrow \lambda} \Delta_{n-1}(\mu).$$

Each of these summands is a high weight module for $U_{\mathbb{C}}(n-1)$. Given any one such summand $\Delta_{n-1}(\mu)$ in the restriction, it ought to be possible to find a corresponding element $X_\mu \in U_{\mathbb{C}}(n)$ such that $v_\mu = X_\mu.v_\lambda$ is precisely the $U^+_{\mathbb{C}}(n-1)$-high weight vector in this irreducible.

Because of Lemma 2.2, X_μ can be described as a product of various lowering operators $S_{i,n}$. To explain this, let us generalise the notation F_t from §2.3, for a row standard λ-tableau t such that every entry in row i is greater than or equal to i. Given such a t, define $S_t \in U_{\mathbb{C}}(n)$ by

$$S_t := \prod_{1 \le i < j \le n} S_{i,j}^{(N_{i,j})}$$

where $N_{i,j}$ is equal to the number of entries equal to j on row i of t, and the order in the product is as in §2.1.

We claim that, given $\mu \twoheadleftarrow \lambda$,

$$X_\mu.v_\lambda = S_{t(\mu)}.v_\lambda = S_{1,n}^{(N_{1,n})} \ldots S_{n-1,n}^{(N_{n-1,n})}.v_\lambda$$

is a non-zero $U^+_{\mathbb{C}}(n-1)$-high weight vector in the summand $\Delta_{n-1}(\mu)$, where $N_{i,n}$ is defined to be the number of boxes deleted from row i of $[\lambda]$ to obtain $[\mu]$. To prove the claim, it is clear from Lemma 2.2 that the given vector is a high weight vector. So, it just remains to show that it is non-zero, which follows from the following more general result, proved in [**5**, Theorem 7]:

Lemma 2.5. *Let v_λ be a $U_{\mathbb{C}}^+(n)$-high weight vector for $\Delta_n(\lambda)$, over $\mathbb{C}$. Then,*

$$\{S_t.v_\lambda \mid \text{for all standard } \lambda\text{-tableaux } t\}$$

is a basis for $\Delta_n(\lambda)$.

It is important to note that Lemma 2.5 *only* applies in characteristic 0: essentially, the argument in [**5**] shows that the elements $S_t.v_\lambda$ are related to the elements $F_t.v_\lambda$ in the standard basis by a triangular transition matrix with non-zero entries on the diagonal, using a 'straightening formula'. This is *false* in characteristic p – the diagonal entries in the transition matrix will often be zero in that case.

Finally, some historical remarks are in order. These elements $X_\mu \in U_{\mathbb{C}}(n)$ were first described (in a slightly different form) by Nagel and Moshinsky in 1965 [**22**]. Closely related operators were used by Carter and Lusztig in 1974 [**6**] to study modular representations. Their operators, denoted $T_j^i(t)$ in [**6**], are defined by a very similar non-commutative determinant to the one in §2.2.

Carter and Lusztig used their elements in particular to construct non-zero homomorphisms between Weyl modules for $U_{\mathbb{F}}(n)$ in the modular case. This was done by applying certain products of the lowering operators $T_j^i(t)$ to a high weight vector $v_\lambda \in \Delta_n(\lambda)$ to construct a vector v_μ inside the Weyl module $\Delta_n(\lambda)$, with $\mu < \lambda$. By the above argument, v_μ is easily seen to be a $U_{\mathbb{F}}^+(n-1)$-high weight vector. By exploiting some extra degeneracy dependent on the prime p, it is sometimes even a $U_{\mathbb{F}}^+(n)$-high weight vector, hence giving the required non-zero homomorphism $\Delta_n(\mu) \to \Delta_n(\lambda)$.

The key difficulty in the proof in [**6**] was to show that $v_\mu \neq 0$, hence that a *non-zero* homomorphism had indeed been constructed. Later, in the Carter-Payne theorem [**7**], this step of the proof was improved by a delicate argument involving modifying the lowering operators slightly, to strengthen the results of [**6**]. Nowadays, such homomorphisms between Weyl modules are constructed in a more conceptual way due to Andersen; see [**13**, II.6.25].

2.5. Orthogonal bases for Weyl modules. To conclude this section, we describe the rather stronger fact about the basis from Lemma 2.5 noticed by Carter in [**5**].

Recall that given a $U_{\mathbb{F}}(n)$-module V, a contravariant form $(.,.)$ on V is a symmetric bilinear form such that $(X.u, v) = (u, \tau(X).v)$ for all $u, v \in V$ and all $X \in U_{\mathbb{F}}(n)$, where τ is the antiautomorphism defined in §2.1. High weight modules for $U_{\mathbb{F}}(n)$ possess non-zero contravariant forms, unique up to scalars. If V is a Weyl module, then the radical of V with respect to the contravariant form coincides with the unique maximal submodule of V. In particular, in characteristic 0, the contravariant form on a Weyl module is non-degenerate.

In [**5**], Carter proved the following result. We include a detailed proof, slightly simpler than Carter's original argument, since we wish to generalise this to quantum $GL(n)$ in section 4.

Theorem 2.6. *Let v_λ be a $U_{\mathbb{C}}^+(n)$-high weight vector for $V = \Delta_n(\lambda)$, over $\mathbb{C}$. Then,*

$$\{S_t.v_\lambda \mid \text{for all standard } \lambda\text{-tableaux } t\}$$

is an orthogonal basis for $\Delta_n(\lambda)$ with respect to the usual contravariant form $(.,.)$.

PROOF. We prove this by induction on n, the result being trivial in the case $n = 1$ (when any basis is an orthogonal basis!). The restriction of V to $U_{\mathbb{C}}(n-1)$ splits as a direct sum of (irreducible) $U_{\mathbb{C}}(n-1)$-Weyl modules

$$\bigoplus_{\mu \twoheadleftarrow \lambda} \Delta_{n-1}(\mu).$$

By §2.4, $v_\mu := S_{t(\mu)}.v_\lambda$ is a non-zero $U_{\mathbb{C}}^+(n-1)$-high weight vector in V, so the summand $V_\mu = \Delta_{n-1}(\mu)$ in this decomposition is precisely the module $U_{\mathbb{C}}(n-1).v_\mu$.

We now show that V_μ, V_ν, for $\mu, \nu \twoheadleftarrow \lambda$, $\mu \neq \nu$, are orthogonal relative to $(.,.)$. Arbitrary elements of V_μ, V_ν can be written as $Y_\mu.v_\mu$, $Y_\nu.v_\nu$ for $Y_\mu, Y_\nu \in U_{\mathbb{C}}^-(n-1)$. Now, $(Y_\mu.v_\mu, Y_\nu.v_\nu) = (v_\mu, \tau(Y_\mu)Y_\nu.v_\nu)$. We may write $\tau(Y_\mu)Y_\nu.v_\nu$ as $Y.v_\nu$ for some $Y \in U_{\mathbb{C}}^-(n-1)$. Hence, this equals $(v_\mu, Y.v_\nu) = (\tau(Y).v_\mu, v_\nu)$. Now, $\tau(Y).v_\mu$ is a (possibly zero) scalar multiple of v_μ. So, it suffices to show that $(v_\mu, v_\nu) = 0$. But this is clear because they lie in different weight spaces.

In particular, since the contravariant form on $U_{\mathbb{C}}(n)$ is non-degenerate over $\mathbb{C}$, this argument implies that for all $\mu \twoheadleftarrow \lambda$, $(v_\mu, v_\mu) \neq 0$. Hence, the restriction of the contravariant form on V to V_μ is a non-zero multiple of the $U_{\mathbb{C}}(n-1)$-contravariant form on V_μ. Now the result follows by induction and the definition of S_t (this is why we chose the ordering for the bases in §2.1 with some care). □

It is interesting to note that there is a quite different approach to constructing essentially the same (up to scalars) orthogonal basis for Weyl modules over $\mathbb{C}$. This argument is due to James and Mathas; in [**12**], they give the argument in full for the quantum analogue of Weyl modules, working with the q-Schur algebra. Their argument involves certain operators known as *Murphy operators* in the q-Schur algebra. In particular, James-Mathas use the orthogonal basis to understand the *Jantzen filtration* of the corresponding Weyl module in the non-semisimple case, from which they are able to give a purely algebraic (ie non-geometric) proof of the Jantzen-Schaper theorem.

3. Generalised lowering operators: the classical case

We begin this section by discussing the problems with Carter's operators in non-zero characteristic, and the possibility of obtaining *modular branching rules.* It is to overcome these problems that we need the generalised lowering operators. We give the new definition of these operators, explain the relationship between the original operators of Carter and Kleshchev, and use them to give a very simple proof of Lemma 2.2.

3.1. Modular branching rules. The branching rule in Theorem 2.4 shows that, in arbitrary characteristic, the restriction of $\Delta_n(\lambda)$ to $U_{\mathbb{F}}(n-1)$ has a Weyl filtration with factors $\Delta_{n-1}(\mu)$ occuring precisely once for each $\mu \twoheadleftarrow \lambda$. Unlike in characteristic 0, there need not be a $U_{\mathbb{F}}^+(n-1)$-high weight vector in $\Delta_n(\lambda)$ corresponding to each of these factors. However, the following remains true in non-zero characteristic [**3**, Theorem A]:

THEOREM 3.1. *Let $\mu = \mu_1\varepsilon_1 + \cdots + \mu_{n-1}\varepsilon_{n-1}$ be a dominant weight. Then,*

$$\dim \operatorname{Hom}_{U_{\mathbb{F}}(n-1)}(\Delta_{n-1}(\mu), \nabla_n(\lambda)) = \begin{cases} 1 & \text{if } \mu \twoheadleftarrow \lambda \\ 0 & \text{otherwise.} \end{cases}$$

(Here, $\mathrm{Hom}_{U_{\mathbb{F}}(n-1)}$ *denotes homomorphisms of* $U_{\mathbb{F}}(n-1)$*-modules.) Hence, each of the spaces*

$$\mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)) \cong \mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(L_n(\lambda), \nabla_{n-1}(\mu)),$$
$$\mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(L_{n-1}(\mu), \nabla_n(\lambda)) \cong \mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(\Delta_n(\lambda), L_{n-1}(\mu)),$$
$$\mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(L_{n-1}(\mu), L_n(\lambda)) \cong \mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(L_n(\lambda), L_{n-1}(\mu))$$

are at most 1*-dimensional, and they are non-zero only if* $\mu \twoheadleftarrow \lambda$*.*

Proof. The first statement is immediate from Theorem 2.4 and standard properties of good filtrations [**13**, II.4.16]. The second follows immediately from the first applying the universal property of Weyl modules [**13**, II.2.13]. □

In particular, the theorem implies that the socle of the restriction of $L_n(\lambda)$ to $U_{\mathbb{F}}(n-1)$ is multiplicity-free, a result which was first noticed by Kleshchev [**14**, Theorem A]. The proof given here is taken from [**3**] (where this multiplicity-free phenomenon is explained conceptually in terms of density of certain double cosets in reductive algebraic groups).

Because of the theorem, we make the following definitions. Let $\mu \twoheadleftarrow \lambda$.

(i) Say μ is *normal* (for λ) if $\dim \mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)) = 1$.

(ii) Say μ is *good* (for λ) if $\dim \mathrm{Hom}_{U_{\mathbb{F}}(n-1)}(L_{n-1}(\mu), L_n(\lambda)) = 1$.

It is reasonable to ask for a combinatorial description of normal and good partitions, and we refer to such results as *modular branching rules.* We will return to these matters again in section 5, when we discuss Kleshchev's modular branching rules for the first level.

Observe that if μ is normal for λ, then a non-zero $U_{\mathbb{F}}^+(n-1)$-high weight vector of weight μ does indeed exist in $L_n(\lambda)$. One might hope to use Carter's lowering operators to give an explicit construction of such high weight vectors, as in 2.4. However, this does not work in general – the problem is that in non-zero characteristic, $S_{i,n}$ often act as zero on high weight vectors in $L_n(\lambda)$. This is original motivation behind Kleshchev's generalisation of Carter's operators.

3.2. The definition. We wish to define operators $S_{i,j}(A)$ for all $1 \le i < j \le n$ and all subsets A of the open interval $(i..j) := \{i+1, \dots, j-1\}$. First, we define the operators that we have been calling *Murphy operators* because they play a similar role to operators defined by Murphy [**21**] in the representation theory of symmetric groups. For $1 < j \le n$, define

$$L_j := \sum_{1 \le i < j} F_{i,j} E_{i,j}.$$

These operators commute: for all j, k, $L_j L_k = L_k L_j$ (this is an easy exercise using the relations 2.1).

Now define $\tilde{C}(i,j) := C(i,j) + 1$, where $C(i,j)$ is as in §2.2. Note that $\tilde{C}(i,k)F_{i,j} = F_{i,j}C(i,k)$ and $C(k,j)F_{i,k} = F_{i,j}\tilde{C}(k,j)$ for $i < k < j$, and that $\tilde{C}(i,k)$ commutes with L_t for all t (because any element of $U_{\mathbb{F}}^0(n)$ does). For $A \subset (i..j)$, define the operator $\tilde{S}_{i,j}(A) \in U_{\mathbb{F}}(n)$ by

$$\tilde{S}_{i,j}(A) := \prod_{t \in A} (\tilde{C}(i,t) - L_{i+1} - \cdots - L_t).F_{i,j}.$$

Because the Murphy operators commute, we do not need to specify the order in the product.

Finally, we define the required operator $S_{i,j}(A)$. Well, we can expand $\tilde{S}_{i,j}(A)$ in terms of the PBW-basis from §2.1. Define $S_{i,j}(A)$ to be the sum of those terms in this expansion which lie in $U_{\mathbb{F}}^-(n)U_{\mathbb{F}}^0(n)$. The point of this is that any PBW-basis element of the form FHE for $E \neq 1$ acts as zero on $U_{\mathbb{F}}^+(n)$-high weight vectors, so is irrelevant for the questions we are considering.

To compute the lowering operators $S_{i,j}(A)$ in practise, the following lemma is useful:

LEMMA 3.2. *Let $A \subset (i..j)$. For any $k \in [1..n]$, let I_k be the left ideal of $U_{\mathbb{F}}(n)$ generated by $\{E_{h,l}^{(r)} \mid 1 \leq h < l \leq k, r \geq 1\}$. Then,*
(i) $L_t F_{i,j} \equiv -F_{i,t}F_{t,j}$ *(modulo I_{j-1});*
(ii) *For $i < t < j$, $(\tilde{C}(i,t) - L_{i+1} - \cdots - L_t)L_t F_{i,j} \equiv 0$ (modulo I_{j-1}).*

PROOF. (i) is clear from the defining relations. Now, for (ii), working modulo I_{j-1} always,

$$(\tilde{C}(i,t) - L_{i+1} - \cdots - L_{t-1})L_t F_{i,j} \equiv -\left(\tilde{C}(i,t)F_{i,t}F_{t,j} + \sum_{i<s<t} F_{i,s}F_{s,t}F_{t,j}\right).$$

We show that $L_t^2 F_{i,j} \equiv -L_t F_{i,t}F_{t,j}$ is congruent to this expression modulo I_{j-1} to prove (ii). For this, one first checks using the defining relations that

$$F_{s,t}E_{s,t}F_{i,t}F_{t,j} \equiv \begin{cases} F_{i,s}F_{s,t}F_{t,j} + F_{i,t}F_{t,j} & \text{if } i < s < t \\ (H_{i,t}+2)F_{i,t}F_{t,j} & \text{if } s = i \\ 0 & \text{if } s < i. \end{cases}$$

Hence,

$$\begin{aligned} L_t F_{i,t}F_{t,j} &\equiv \left(\sum_{s<i} F_{s,t}E_{s,t} + F_{i,t}E_{i,t} + \sum_{i<s<t} F_{s,t}E_{s,t}\right) F_{i,t}F_{t,j} \\ &\equiv (H_{i,t} + t - i + 1)F_{i,t}F_{t,j} + \sum_{i<s<t} F_{i,s}F_{s,t}F_{t,j}, \end{aligned}$$

precisely as required. □

Let us now illustrate the definition by computing $S_{1,4}(\{3\})$ and $S_{1,4}(\{2,3\})$. By definition, working modulo I_3 as in the lemma,

$$S_{1,4}(\{3\}) \equiv (\tilde{C}(1,3) - L_2 - L_3)F_{1,4} \equiv F_{1,4}C(1,3) + F_{1,2}F_{2,4} + F_{1,3}F_{3,4}$$

using Lemma 3.2(i). Similarly, this time using Lemma 3.2(ii) as well,

$$\begin{aligned} S_{1,4}(\{2,3\}) &\equiv (\tilde{C}(1,2) - L_2)(\tilde{C}(1,3) - L_2 - L_3)F_{1,4} \\ &\equiv (\tilde{C}(1,2) - L_2)(\tilde{C}(1,3) - L_3)F_{1,4} \\ &\equiv (\tilde{C}(1,2) - L_2)(F_{1,4}C(1,3) + F_{1,3}F_{3,4}) \\ &\equiv F_{1,4}C(1,2)C(1,3) + F_{1,3}F_{3,4}C(1,2) + F_{1,2}F_{2,4}C(1,3) + F_{1,2}F_{2,3}F_{3,4}. \end{aligned}$$

This last expression is easily checked to be the same as $S_{1,4}$ from §2.2.

More generally, suppose that $A = (i..j)$. Then, by Lemma 3.2,

$$S_{i,j}(A) \equiv \prod_{i<t<j} (\tilde{C}(i,t) - L_t)F_{i,j} \equiv \sum_{B \subset A} \left(F_{i,j}^B \prod_{t \in A \setminus B} C(i,t)\right) \tag{3.3}$$

where if $B = \{b_1 < \cdots < b_s\}$, then $F^B_{i,j} = F_{i,b_1}F_{b_1,b_2}\dots F_{b_s,j}$. The next lemma now follows immediately from this expression, expanding the determinant in the definition of $S_{i,j}$:

LEMMA 3.4. *If $A = (i..j)$, then $S_{i,j}(A) = S_{i,j}$.*

Thus the operators here are indeed generalisations of Carter's lowering operators. The equation 3.3 also gives the link between our approach to constructing orthogonal bases for Weyl modules and the James-Mathas approach in [**12**].

3.3. The recurrence relation. We now prove a fundamental property of the operators $S_{i,j}(A)$, namely that they satisfy a certain recurrence relation. This recurrence relation is the key to giving elementary proofs to most facts about the lowering operators.

Two pieces of notation are convenient here. First, given a property $\mathcal{P}$, we let $\delta_{\mathcal{P}}$ be 1 if $\mathcal{P}$ is true, or 0 if it is false. Second, given any subset $A \subset \mathbb{N}$, and $i \le j$, we let $A_{i..j}$ denote $\{a \in A \,|\, i < a < j\}$. The key result is:

PROPOSITION 3.5. *Let $A \subset (i..j)$. If $A = \varnothing$, then $S_{i,j}(A) = F_{i,j}$. Otherwise, take any $k \in A$ and let $h = \max([i..k-1] \setminus A)$. Then,*

$$S_{i,j}(A) = S_{i,j}(A \setminus \{k\})C(h,k) + \delta_{h \ne i} S_{i,j}(\{h\} \cup A \setminus \{k\}) + S_{i,k}(A_{i..k})S_{k,j}(A_{k..j}).$$

PROOF. We work modulo the left ideal of $U_{\mathbb{F}}(n)$ generated by the non-constant elements of $U^+_{\mathbb{F}}(n)$. We consider the case $h \ne i$ here, leaving the (easier) case $h = i$ to the reader. Rewrite the definition of $S_{i,j}(A)$ using the identity $(\tilde{C}(i,k) - \cdots - L_k) = C(h,k) + (\tilde{C}(i,h) - \cdots - L_h) - (L_{h+1} + \cdots + L_{k-1}) - L_k$ to deduce that $S_{i,j}(A) \equiv P_1 + P_2 + P_3 + P_4$, where

$$P_1 = \prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t) C(h,k) \prod_{t \in A_{k..j}} (\tilde{C}(i,t) - L_{i+1} \cdots - L_t) F_{i,j},$$

$$P_2 = \prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t)(\tilde{C}(i,h) - \cdots - L_h) \prod_{t \in A_{k..j}} (\tilde{C}(i,t) - \cdots - L_t) F_{i,j},$$

$$P_3 = -\prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t)(L_{h+1} + \cdots + L_{k-1}) \prod_{t \in A_{k..j}} (\tilde{C}(i,t) - \cdots - L_t) F_{i,j}$$

which is zero by Lemma 3.2(ii), and

$$\begin{aligned} P_4 &= -\prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t) \prod_{t \in A_{k..j}} (\tilde{C}(i,t) - \cdots - L_t) L_k F_{i,j} \\ &\equiv \prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t) \prod_{t \in A_{k..j}} (C(k,t) - L_{k+1} - \cdots - L_t) F_{i,k} F_{k,j} \\ &\equiv \prod_{t \in A_{i..k}} (\tilde{C}(i,t) - \cdots - L_t) F_{i,k} \prod_{t \in A_{k..j}} (\tilde{C}(k,t) - L_{k+1} - \cdots - L_t) F_{k,j}, \end{aligned}$$

using Lemma 3.2(ii) twice in this last step. Now, $P_1 + \cdots + P_4$ is easily seen to equal the required right hand side. □

In particular, this recurrence relation allows us to explain the connection between our lowering operators and the operators originally defined by Kleshchev in [**16**], working over $\mathbb{Z}$. Fix a partition $\lambda \in \mathcal{X}$. Define a map

$$e_\lambda : U^-_{\mathbb{Z}}(n) U^0_{\mathbb{Z}}(n) U^+_{\mathbb{Z}}(n) \to U^-_{\mathbb{Z}}(n) U^+_{\mathbb{Z}}(n)$$

by "evaluation at λ"; on a basis element FHE, $e_\lambda(FHE) := F\lambda(H)E$. For any $X \in U_{\mathbb{Z}}(n)$, let $X^\lambda := e_\lambda(X)$. In particular, this defines operators $S^\lambda_{i,j}(A) = e_\lambda(S_{i,j}(A))$. In [**16**], Kleshchev defines certain operators $T_{i,j-1}(M) \in U^-_{\mathbb{Z}}(n)$ for $M \subset (i..j)$, and shows that they satisfy a recurrence relation given in [**16**, Lemma 2.4]. On rearranging Proposition 3.5, together with Lemma 3.4, one can easily see that

$$T_{i,j-1}(M) = S^\lambda_{i,j}(A)$$

where $A = (i..j)\backslash M$. So, our operator $S_{i,j}(A)$ specialises to the operator $T_{i,j-1}(M)$ defined by Kleshchev in [**16**].

To conclude this section, we illustrate the use of the recurrence relation by giving the simple proof of Lemma 2.2. In fact, we prove something slightly more general; Lemma 2.2 follows from this using Lemma 3.4:

Lemma 3.6. *Let $A \subset (i..j)$ and $1 \le l < n$. Suppose one of the following holds:*
(a) $l + 1 \in A$;
(b) $l \notin \{i\} \cup A$ *and* $l + 1 \notin A \cup \{j\}$.
Then, $E_l S_{i,j}(A) \equiv 0 \quad$ *(modulo $U_{\mathbb{F}}(n).E_l$).*

Proof. Use induction on height, where $\mathrm{ht}(A) = \sum_{a\in A} a$. If $\mathrm{ht}(A) = 0$, $S_{i,j}(A) = F_{i,j}$ and the result is immediate from 2.1. If $\mathrm{ht}(A) = i + 1$, $A = \{i + 1\}$ and

$$S_{i,j}(A) = F_{i,j}C(i, i + 1) + F_{i,i+1}F_{i+1,j}.$$

The conclusion is immediate from this and 2.1 if (b) holds. So, suppose (a) holds, so that $l = i$. Then, by 2.1

$$E_l S_{i,j}(A) \equiv -F_{i+1,j}(1 + H_{i,i+1}) + H_{i,i+1}F_{i+1,j} = 0.$$

So now suppose that $\mathrm{ht}(A) > i + 1$ and that the result has been proved for all A of smaller height. Suppose first that $A = \{l + 1\}$ where $i < l < j - 1$. Applying Proposition 3.5 twice, $S_{i,j}(A)$ equals

$$F_{i,l+1}F_{l+1,j} + F_{i,l}F_{l,j} + F_{i,j}C(l - 1, l + 1) + \delta_{l-1\neq i}S_{i,j}(\{l - 1\}).$$

By induction, $E_l S_{i,j}(\{l - 1\}) \equiv 0 \equiv E_l, F_{i,j}$. Now the conclusion follows since by 2.1 again, E_l commutes with $F_{i,l+1}F_{l+1,j} + F_{i,l}F_{l,j}$. So, we may assume that we can choose some $k \in A$ with $k \neq l + 1$. Let $h = \max([i..k - 1] \setminus A)$ and apply Proposition 3.5. The conclusion follows in either case (a) or case (b) using the induction hypothesis. □

4. Generalised lowering operators: the quantum case

In this section, we explain briefly how to generalise the lowering operators $S_{i,j}(A)$ to the quantum analogue $U_{\mathbb{F},v}(n)$ of $U_{\mathbb{F}}(n)$. In particular, this will enable us to give the correct definition of the quantum analogue of Carter's operators $S_{i,j}$, hence to construct orthogonal bases for quantum Weyl modules. Further details here can be found in [**2**, Chapter 8] and [**4**].

4.1. The quantum hyperalgebra. We begin by briefly sketching the definition from [**18, 9**] of the quantum hyperalgebra $U_{\mathbb{F},v}(n)$ over our arbitrary field $\mathbb{F}$, where v is a fixed element of $\mathbb{F}^\times$. The definition is by base change, much as in the classical case in §2.1. First, we let $U_{\mathbb{C}(v),v}(n)$ be the Drinfeld-Jimbo generic quantised enveloping algebra corresponding to $U_{\mathbb{C}}(n)$, where here v is an indeterminate over $\mathbb{C}$. This is defined by generators and relations; we use notation as in [**9, 4**], so denote the generators by E_i, F_i, K_j, K_j^{-1} $(1 \leq i < n, 1 \leq j \leq n)$. The full relations can be found in [**9**] or [**4**], and we omit the details here.

For $t, u \in \mathbb{N}$, we define the *quantum factorial* and the *quantum binomial coefficient* by

$$[t]! := \prod_{s=1}^{t} \frac{v^s - v^{-s}}{v - v^{-1}}, \qquad \left[\begin{array}{c} t \\ u \end{array}\right] := \prod_{s=1}^{u} \frac{v^{t-s+1} - v^{-t+s-1}}{v^s - v^{-s}}.$$

For $X \in U_{\mathbb{C}(v),v}(n)$, $X^{(s)}$ now denotes the *divided power* $X^s/[s]!$ and

$$\left[\begin{array}{c} K_j \\ u \end{array}\right] := \prod_{s=1}^{u} \frac{K_j v^{-s+1} - K_j^{-1} v^{s-1}}{v^s - v^{-s}}.$$

Lusztig [**18**] has constructed an integral form for $U_{\mathbb{C}(v),v}(n)$ over the ring of Laurent polynomials $\mathbb{Z}[v, v^{-1}]$. To describe a PBW-basis for this integral form, we need to define elements $E_{i,j}, F_{i,j}$ for arbitrary $1 \leq i < j \leq n$, as in the classical case. Unfortunately, there are many ways of doing this as shown in [**18**]. We want to be definite here, so fix our choice to be the one described in [**18**, Example 4.4] (see also [**4**]). This is quite arbitrary, and with appropriate modifications, any of Lusztig's definitions of $E_{i,j}, F_{i,j}$ for general i, j could be made to work.

Having fixed this choice, we can describe the integral form. Du [**9**, Section 2] and Lusztig [**18**, 4.5] have shown that there is a free $\mathbb{Z}[v, v^{-1}]$-subalgebra $U_{\mathbb{Z}[v,v^{-1}],v}(n)$ of $U_{\mathbb{C}(v),v}(n)$ with $\mathbb{Z}[v, v^{-1}]$-basis

$$\underbrace{\prod_{1 \leq i < j \leq n} F_{i,j}^{(N_{ij})}}_{U^-} \underbrace{\prod_{1 \leq i \leq n} \left(K_i^{\delta_i} \left[\begin{array}{c} K_i \\ N_{ii} \end{array}\right] \right)}_{U^0} \underbrace{\prod_{1 \leq i < j \leq n} E_{i,j}^{(N_{ji})}}_{U^+}$$

as $N = (N_{ij})_{1 \leq i,j \leq n}$ runs over all $n \times n$ matrices with entries in $\mathbb{Z}_{\geq 0}$ and $\delta = (\delta_i)_{1 \leq i \leq n}$ runs over all vectors with entries in $\{0, 1\}$. The order of multiplication in the first and last products is fixed as in §2.1.

We now obtain the quantum hyperalgebra $U_{\mathbb{F},v}(n)$ by base change. So now $\mathbb{F}$ is an arbitrary field of characteristic p and v is an arbitrary unit in $\mathbb{F}$. Regard $\mathbb{F}$ as a $\mathbb{Z}[v, v^{-1}]$-algebra by letting the indeterminate $v \in \mathbb{Z}[v, v^{-1}]$ act as multiplication by $v \in \mathbb{F}$. Then,

$$U_{\mathbb{F},v}(n) := U_{\mathbb{Z}[v,v^{-1}],v}(n) \otimes_{\mathbb{Z}[v,v^{-1}]} \mathbb{F}.$$

The construction is precisely analogous to the classical case in §2.1. In fact, if $v^2 = 1$, then $U_{\mathbb{F},v}(n)$ is just (a covering of) the classical hyperalgebra $U_{\mathbb{F}}(n)$, so our results about $U_{\mathbb{F},v}(n)$ contain as a special case results about the classical case.

One then defines $U^-_{\mathbb{F},v}(n), U^0_{\mathbb{F},v}(n), U^+_{\mathbb{F},v}(n)$ to be span of the image of those parts of the PBW-basis indicated above, as in the classical case. A weight $\lambda \in \mathcal{X}$ is now regarded as a homomorphism $U^0_{\mathbb{F},v}(n) \to \mathbb{F}$ by base change from the generic case, when $\varepsilon_i : U^0_{\mathbb{C}(v),v}(n) \to \mathbb{C}(v)$ is defined by $K_j \mapsto v^{\delta_{i,j}}$. Again, $U_{\mathbb{F},v}(n)$ admits an antiautomorphism τ as in the classical case defined by base change from the generic case where τ acts as $\tau(E_i) = F_i$, $\tau(F_i) = E_i$, $\tau(K_i) = K_i$ for all i.

Finally, we note that all the classical notions of Weyl, irreducible and dual Weyl modules have analogues in the quantum case. Denote these by $\Delta_n(\lambda), L_n(\lambda), \nabla_n(\lambda)$ as before. Also, the standard basis theorem and the analogue of the classical branching rule (Theorem 2.3 and Theorem 2.4) generalise easily to the quantum case [**4**, Section 3].

4.2. Quantum lowering operators. To define the quantum analogue of $S_{i,j}(A)$, we renormalise $F_{i,j}$. For $1 \le i < j \le n$, define

$$\hat{F}_{i,j} := v^{-j} K_j F_{i,j} K_i v^{-i}.$$

The following relations hold for the renormalised $\hat{F}_{i,j}$: for all $1 \le i < j \le n$ and all $1 \le l < n$,

$$E_l \hat{F}_{i,j} = \begin{cases} \hat{F}_{i,j} E_l + \frac{v^{-2i}}{v - v^{-1}}(K_i^2 - K_{i+1}^2) & \text{if } l = i, l+1 = j \\ v^{-1}\hat{F}_{i,j} E_i - v\hat{F}_{i+1,j} & \text{if } l = i, l+1 \neq j \\ v\hat{F}_{i,j} E_i + \hat{F}_{i,j-1} & \text{if } l \neq i, l+1 = j \\ \hat{F}_{i,j} E_l & \text{if } l \notin \{i,j\}, l+1 \notin \{i,j\} \\ v\hat{F}_{i,j} E_l & \text{if } l+1 = i \\ v^{-1}\hat{F}_{i,j} E_l & \text{if } l = j. \end{cases} \tag{4.1}$$

Also define the quantum $C(i,j)$ for $1 \le i < j \le n$ by

$$C(i,j) := \frac{v^{-2i-1}K_i^2 - v^{-2j-1}K_j^2}{v - v^{-1}}.$$

Now we can define $S_{i,j}(A)$ in the quantum case. We want to do this simply by giving the analogue of the recurrence relation in Proposition 3.5. So, if $A = \varnothing$, we let $S_{i,j}(A)$ be $\hat{F}_{i,j}$. Otherwise, if A is non-empty and $k \in A$ is any element, we let $h = \max([i..k-1] \setminus A)$ and set

$$S_{i,j}(A) := S_{i,j}(A \setminus \{k\})C(h,k) + \delta_{h \neq i} S_{i,j}(\{h\} \cup A \setminus \{k\}) + S_{i,k}(A_{i..k})S_{k,j}(A_{k..j}).$$

Unfortunately, it is not immediately clear that this recurrence relation is well-defined (since there is freedom to choose $k \in A$). So, we have to proceed with slightly more care: certainly, there are well-defined operators $S_{i,j}(A)$ defined by this recurrence relation where we prescribe at all times that k is chosen to be $\max(A)$ in the inductive definition of $S_{i,j}(A)$. One then needs to prove by induction that the operator thus defined satisfies the given recurrence relation *for all* $k \in A$. This is not hard to do, and we leave the details to the reader. A rather different approach to defining this quantum $S_{i,j}(A)$ is given in [**4**, Section 4].

The key fact about the quantum operators $S_{i,j}(A)$ is that the analogue of Lemma 3.6 holds in precisely the same way. The proof is identical to the classical case, using 4.1 in place of the classical relations 2.1 (see [**4**, Lemma 4.11]). In particular, we can now give the definition of the quantum analogue $S_{i,j}$ of Carter's original lowering operator $S_{i,j}$. By definition, this should be $S_{i,j}(A)$ in the case $A = (i..j)$. It is easy to see (using the recurrence relation) that this is just the element of $U_{\mathbb{F},v}(n)$ obtained by expanding the non-commutative determinant of §2.2 with every entry $C(i,j)$ replaced with the quantum $C(i,j)$ and every entry $F_{i,j}$ replaced with $\hat{F}_{i,j}$. We remark that, as $\hat{F}_{i,j}$ and $C(i,j)$ both specialise to the classical objects $F_{i,j}$ and $C(i,j)$, the operators we have defined do indeed specialise to the classical objects from section 3.

4.3. Orthogonal bases revisited. We now give a first application of these quantum lowering operators to construct orthogonal bases for quantum Weyl modules using the quantum $S_{i,j}$. We work now in the semisimple case $\mathbb{F} = \mathbb{C}(v)$, where v is an indeterminate.

Given a standard λ tableau t, define S_t as in §2.4, now using quantum divided powers and the quantum version of $S_{i,j}$. The Weyl module $\Delta_n(\lambda)$ again possesses a (unique up to scalars) non-degenerate contravariant form over $\mathbb{C}(v)$, and we have the analogous orthogonal basis given by our lowering operators:

THEOREM 4.2. *Let v_λ be a $U^+_{\mathbb{C}(v),v}(n)$-high weight vector for $\Delta_n(\lambda)$ over $\mathbb{C}(v)$. Then,*

$$\{S_t.v_\lambda \mid \text{for all standard } \lambda\text{-tableaux } t\}$$

is an orthogonal basis for $\Delta_n(\lambda)$ with respect to the usual contravariant form $(.,.)$.

PROOF. Precisely the argument described in the proof of Theorem 2.6 carries over to the quantum case, since all that we used is that $S_{i,n}$ sends $U_{\mathbb{F}}(n)^+$-high weight vectors to $U_{\mathbb{F}}(n-1)^+$-high weight vectors, which follows because the analogue of Lemma 3.6 holds for our quantum lowering operators. The only problem in doing this is to show that if $\mu \twoheadleftarrow \lambda$, then $S_{t(\mu)}.v_\lambda$ is actually non-zero. But this follows because this element specialises under the map $v \mapsto 1$ to the classical element $S_{t(\mu)}.v_\lambda$, which is known to be non-zero by Lemma 2.5. $\square$

We stress again that this orthogonal basis over $\mathbb{C}(v)$ does not give a basis at roots of unity. The basis in Theorem 4.2 is the same (up to scalars) as the basis constructed by James and Mathas [**12**] working within the q-Schur algebra.

5. Modular branching rules

Now we survey the original application of the lowering operators $S_{i,j}(A)$ to prove a modular branching rule for the first level. In the classical case, this is a result of Kleshchev [**16**], which has recently been generalised [**4**] to the quantum case, using the operators in section 4. We refer the reader to [**4**] for detailed proofs. The material in §5.3 is new.

5.1. Normal and good nodes. We first give the combinatorial definitions of normal and good nodes from [**16**]. These definitions are the key to understanding Kleshchev's modular branching rule. Our formulation of the definitions is somewhat different from Kleshchev's original formulation, though equivalent by [**4**, Remark 2.5].

Fix an integer $e \in \mathbb{N}$, and let $\lambda = (\lambda_1, \dots, \lambda_{n-1})$ be our fixed partition. Given the coordinate (i,j) of a box in the diagram $[\lambda]$, define the corresponding *e-residue* $\mathrm{res}_e(i,j)$ to be $i - j$ regarded as an element of the ring $\mathbb{Z}/e\mathbb{Z}$. In the example $\lambda = (3,2)$, the 3-residues are:

0	2	1
1	0	

Say (i,j) is a *removable node* if the diagram obtained from $[\lambda]$ by removing the box in position (i,j) is the diagram of a proper partition. We may parametrise removable nodes by the set

$$\mathscr{R} := \{i \mid 1 \leq i < n, \lambda_i \neq \lambda_{i+1}\},$$

so $i \in \mathscr{R}$ corresponds to the removable node with coordinate (i, λ_i). If $i \in \mathscr{R}$, let $\lambda(i)$ be the partition obtained from the diagram of λ by removing (i, λ_i).

Define a partial order on subsets of $[1..n]$, which we call the *lattice order*, denoted by $\downarrow$. Let $A, B \subset [1..n]$. Then, $A \downarrow B$ if there exists an injection $\theta : A \hookrightarrow B$ such that $\theta(a) \leq a$ for all $a \in A$.

For $1 \leq i \leq j \leq n$, let

$$\mathscr{B}(i) := \{j \mid i \leq j < n, \operatorname{res}_e(i, \lambda_i) = \operatorname{res}_e(j+1, \lambda_{j+1}+1)\},$$
$$\mathscr{C}(i) := \{j \mid i < j < n, \operatorname{res}_e(i, \lambda_i) = \operatorname{res}_e(j, \lambda_j)\}.$$

Fix $i \in \mathscr{R}$ and $r \in \mathbb{Z}/e\mathbb{Z}$. Say i is *r-normal* if $\operatorname{res}_e(i, \lambda_i) = r$ and $\mathscr{B}(i) \downarrow \mathscr{C}(i)$. Let $\mathscr{R}_{\text{normal}}$ be the set of all $i \in \mathscr{R}$ such that i is r-normal for some r. Finally, say $i \in \mathscr{R}$ is *r-good* if i is r-normal and there is no r-normal node $j \in \mathscr{R}$ with $j < i$. Let $\mathscr{R}_{\text{good}}$ be the set of all $i \in \mathscr{R}$ such that i is r-good for some r.

It is interesting to note that the concept of r-good node has also proved important to understanding the combinatorics behind computing crystal bases for the 'Fock space'. This can be found in the work of Lascoux, Leclerc and Thibon [**19**], where they conjecture an algorithm for computing decomposition numbers for Hecke algebras of type **A** (at roots of unity over $\mathbb{C}$) using these crystal bases. This algorithm has now been proved (independently) in [**1, 11**].

5.2. A modular branching rule. We work in this subsection with the quantum hyperalgebra $U_{\mathbb{F},v}(n)$. Given any $U_{\mathbb{F},v}(n)$-module W and $i \in \mathbb{Z}$, define the *ith level of W* to be

$$W^i := \left\{ w \in W \,\middle|\, K_n.w = v^i w, \begin{bmatrix} K_n \\ r \end{bmatrix}.w = \begin{bmatrix} i \\ r \end{bmatrix} w \text{ for all } r \in \mathbb{N} \right\}.$$

Since K_n and $\begin{bmatrix} K_n \\ r \end{bmatrix}$ centralise $U_{\mathbb{F},v}(n-1)$, this is a $U_{\mathbb{F},v}(n-1)$-submodule of W.

Now we note that all the results and definitions in §3.1 generalise without complication to the quantum hyperalgebra (details can be found in [**4**, Section 3]). In particular, one has analogous definitions for normal and good partitions to those in §3.1: a partition $\mu \twoheadleftarrow \lambda$ is *normal* if $\dim \operatorname{Hom}_{U_{\mathbb{F}}(n-1)}(\Delta_{n-1}(\mu), L_n(\lambda)) = 1$, and *good* if $\dim \operatorname{Hom}_{U_{\mathbb{F}}(n-1)}(L_{n-1}(\mu), L_n(\lambda)) = 1$.

Suppose that $\mu \twoheadleftarrow_i \lambda$, so that μ belongs to the ith level. It is clear then the image of $\Delta_{n-1}(\mu)$ or $L_{n-1}(\mu)$ in $L_n(\lambda)$ (under these homomorphisms) must lie in the ith level $L_n(\lambda)^i$. Thus, when considering normal and good partitions, it is natural to look at each level separately.

A purely combinatorial description for normal and good partitions belonging to the first level is known [**16, 4**]. To state these results, we need to define the integer e used in the definition of r-normal and r-good nodes in §5.1. Let e be the smallest positive integer such that

$$v^{1-e} + v^{3-e} + \cdots + v^{e-3} + v^{e-1} = 0$$

in $\mathbb{F}$, or 0 if no such number exists. Observe that if $v^2 = 1$, then e is precisely the characterstic p of the field $\mathbb{F}$, but if $v^2 \neq 1$, then (somewhat surprisingly) e is independent of p.

THEOREM 5.1. *Let $\mu \twoheadleftarrow_1 \lambda$ belong to the first level. Then*
(i) *μ is normal if and only if $\mu = \lambda(i)$ for some $i \in \mathscr{R}_{\text{normal}}$;*
(ii) *μ is good if and only if $\mu = \lambda(i)$ for some $i \in \mathscr{R}_{\text{good}}$.*
In particular, the socle of the restriction of the first level $L_n(\lambda)^1$ to $U_{\mathbb{F},v}(n-1)$ is precisely

$$\bigoplus_{i \in \mathscr{R}_{\text{good}}} L_{n-1}(\lambda(i)).$$

In the classical case $v^2 = 1$, these facts are proved in [**16**], and in the quantum case, in [**4**]. In fact, in the proof of Theorem 5.1(i), slightly more is proved – an explicit construction for a $U^+_{\mathbb{F},v}(n-1)$-high weight vector is given, using the generalised lowering operators. This explicit construction is the key to proving the criterion for good partitions in Theorem 5.1(ii). The construction is as follows. Suppose that $\mu = \lambda(i)$ is normal, so that $\mathscr{B}(i) \downarrow \mathscr{C}(i)$. Take any injection $\theta : \mathscr{B}(i) \hookrightarrow \mathscr{C}(i)$ such that $\theta(b) \leq b$ for all $b \in \mathscr{B}(i)$, and let $A = (i..n) \setminus \operatorname{im} \theta$. Then, the proof shows

$$S_{i,n}(A).v_\lambda$$

is a non-zero $U^+_{\mathbb{F},v}(n-1)$-high weight vector in $L_n(\lambda)$. That is, the generalised lowering operators $S_{i,n}(A)$ suffice to construct all $U^+_{\mathbb{F},v}(n-1)$-high weight vectors in the first level of $L_n(\lambda)$ – precisely as Carter's original lowering operator does in characteristic 0.

This modular branching rule has an important application to proving a modular branching rule for symmetric groups and (in the quantum case) Hecke algebras of type **A**, by a Schur functor argument due to Kleshchev [**15**]. As a consequence, one obtains a combinatorial description of the corresponding Mullineux map. For full details of these matters, we refer the reader to [**17, 4**]. We also remark that the algorithm for computing decomposition matrices for Hecke algebras of type **A** from [**19**] now gives an alternative way to construct the Mullineux map.

5.3. Higher levels. The results of Theorem 5.1 give a purely combinatorial description of normal and good partitions belonging to the first level of λ. There is at present no elementary combinatorial description of normal and good partitions for higher levels.

The main problem here is likely to be the normal partitions: it is reasonable to expect the good partitions to be minimal amongst all normal partitions in the same block, as in the first level. To prove this, one would first need some sort of explicit construction of the high weight vectors corresponding to normal partitions, hopefully by iterating the operators $S_{i,n}(A)$. We now wish to discuss briefly the problem of computing normal partitions for arbitrary levels. We consider only the classical case $v = 1$ for simplicity, but the same argument applies to the quantum case as well.

So fix some $\mu \twoheadleftarrow_i \lambda$. We wish to determine whether or not μ is normal, that is whether the (unique up to scalars) $U_{\mathbb{F}}(n-1)$-homomorphism $\theta_\mu : \Delta_n(\lambda) \to \nabla_{n-1}(\mu)$ given in Theorem 3.1 factors to give a homomorphism $\bar{\theta}_\mu : L_n(\lambda) \to \nabla_{n-1}(\mu)$. We have the following criterion for normal partitions:

THEOREM 5.2. *The map θ_μ factors through $L_n(\lambda)$ if and only if there is no vector w lying in the radical of $\Delta_n(\lambda)$ such that the $(F_{t(\mu)}.v_\lambda)$-coefficient of w is non-zero when w is expanded in terms of the standard basis.*

PROOF. Recall that $\nabla_{n-1}(\mu)$ is just an induced module, induced from the 1-dimensional $U_{\mathbb{F}}^-(n-1)U_{\mathbb{F}}^0(n-1)$-module $\mathbb{F}_\mu$ of weight μ. Thus, by Frobenius reciprocity, the condition that θ_μ factors through $L_n(\lambda)$ is equivalent to determining when the $U_{\mathbb{F}}^-(n-1)U_{\mathbb{F}}^0(n-1)$-homomorphism $\phi_\mu : \Delta_n(\lambda) \to \mathbb{F}_\mu$ factors to give a homomorphism $\bar{\phi}_\mu : L_n(\lambda) \to \mathbb{F}_\mu$. Here, ϕ_μ is the image on θ_μ under the isomorphism in Frobenius reciprocity.

Take $w \in \Delta_n(\lambda)$. Observe from Theorem 2.4(ii) that all vectors of weight μ in the standard basis for $\Delta_n(\lambda)$, except for $F_{t(\mu)}.v_\lambda$ itself, lie in strictly lower factors than $\nabla_{n-1}(\mu)$ in the Weyl filtration of Theorem 2.4. Consequently, ϕ_μ simply picks out the $(F_{t(\mu)}.v_\lambda)$-coefficient of w when written in terms of the standard basis.

Now suppose $\bar{\phi}_\mu$ exists; then, for all w in the radical of $\Delta_n(\lambda)$, $\phi_\mu(w) = 0$, or equivalently, the $(F_{t(\mu)}.v_\lambda)$-coefficient is zero, as required. Conversely, if some w exists in the radical of $\Delta_n(\lambda)$ with non-zero $(F_{t(\mu)}.v_\lambda)$-coefficient, then $\phi_\mu(w) \neq 0$, so ϕ_μ cannot factor through $L_n(\lambda)$. □

The condition in Theorem 5.2 can be tested computationally. Let G_μ be the Gram matrix of the contravariant form on the μ-weight space of $\Delta_n(\lambda)$, written with respect to the standard basis $b_1, \dots, b_m$ of Theorem 2.3 for this weight space ordered so that $b_1 = F_{t(\mu)}.v_\lambda$. Then, by the theorem, we see that μ is normal if and only if the first row of G_μ cannot be written as a linear combination of the remaining rows of G_μ.

We have implemented this algorithm in the GAP language [**23**]. The algorithm is computationally very intensive, since it involves computing the Gram matrix for certain weight spaces of $\Delta_n(\lambda)$, and thus is only useful in small cases. We conclude by presenting some data to illustrate the complexity of the problem of giving a purely combinatorial criterion for normal partitions. In Tables 1 and 2, we list the normal partitions $\mu \twoheadleftarrow_i \lambda$ for all i, and all $n \leq 8$, when $p = 3$. In fact, we only list those normal partitions μ for which $\mu_1 \neq \lambda_1$, since it is easy to see that if $\mu_1 = \lambda_1$, then μ is normal for λ if and only if $(\mu_2, \dots, \mu_{n-1})$ is normal for $(\lambda_2, \dots, \lambda_{n-1})$ which can be found earlier in the table. We also omit λ if $\lambda = (h)$ has just one non-zero part, since here it is obvious that $\mu = (k)$ is normal for λ if and only if $0 \leq k \leq h$ and $\binom{h}{k} \not\equiv 0 (\text{mod } p)$. The GAP program used to compute this output is available on request from the author.

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TABLE 1. Normal partitions for $p = 3, n \leq 7$

n	λ	Level	μ
3	$(2,1)$	2	(1)
4	$(2,1^2)$	1	(1^3)
		2	(1^2)
	$(3,1)$	1	$(2,1)$
		2	$(2),(1^2)$
		3	(1)
5	$(2,1^3)$	1	(1^4)
		2	(1^3)
	$(3,1^2)$	1	$(2,1^2)$
		2	$(2,1),(1^3)$
		3	(1^2)
	$(3,2)$	1	(2^2)
		2	$(2,1)$
		3	(2)
	$(4,1)$	3	(1^2)
		4	(1)
6	$(2,1^4)$	2	(1^4)
	$(3,1^3)$	2	$(2,1^2)$
		3	(1^3)
	$(3,2,1)$	3	$(2,1)$
	$(4,1^2)$	3	(1^3)
		4	(1^2)
	$(4,2)$	1	$(3,2)$
		2	$(3,1),(2^2)$
		3	$(3),(2,1)$
		4	(2)
	$(5,1)$	2	(4)
		3	$(2,1)$
		4	(2)
		5	(1)

n	λ	Level	μ
7	$(2,1^5)$	1	(1^6)
		2	(1^5)
	$(3,1^4)$	1	$(2,1^4)$
		2	$(2,1^3),(1^5)$
		3	(1^4)
	$(3,2,1^2)$	2	$(2,1^3)$
		3	$(2,1^2)$
	$(3,2^2)$	1	(2^3)
		2	$(2^2,1)$
		3	(2^2)
	$(4,1^3)$	3	(1^4)
		4	(1^3)
	$(4,2,1)$	1	$(3,2,1)$
		2	$(3,2)$
		3	$(3,1),(2^2)$
		4	$(2,1)$
	$(4,3)$	1	(3^2)
		4	(3)
	$(5,1^2)$	1	$(4,1^2)$
		2	$(4,1)$
		3	$(2,1^2)$
		4	$(2,1),(1^3)$
		5	(1^2)
	$(5,2)$	3	(2^2)
		4	$(2,1)$
		5	(2)
	$(6,1)$	1	$(5,1)$
		2	$(5),(4,1)$
		3	$(4),(3,1)$
		4	$(3),(2,1)$
		5	$(2),(1^2)$
		6	(1)

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TABLE 2. Normal partitions for $p = 3, n = 8$

n	λ	Level	μ
8	$(2,1^6)$	1	(1^7)
		2	(1^6)
	$(3,1^5)$	1	$(2,1^5)$
		2	$(2,1^4),(1^6)$
		3	(1^5)
	$(3,2,1^3)$	2	$(2,1^4)$
		3	$(2,1^3)$
	$(3,2^2,1)$	1	$(2^3,1)$
		2	(2^3)
		3	$(2^2,1)$
	$(4,1^4)$	3	(1^5)
		4	(1^4)
	$(4,2,1^2)$	1	$(3,2,1^2)$
		2	$(3,2,1),(3,1^3),(2^2,1^2)$
		3	$(3,1^2),(2^2,1),(2,1^3)$
		4	$(2,1^2)$
	$(4,2^2)$	3	$(3,2)$
		4	(2^2)
	$(4,3,1)$	2	$(3^2),(3,2,1)$
		3	$(3,2)$
		4	$(3,1)$
	$(5,2,1)$	3	$(2^2,1)$
		4	(2^2)
		5	$(2,1)$
8	$(5,1^3)$	1	$(4,1^3)$
		2	$(4,1^2)$
		3	$(2,1^3)$
		4	$(2,1^2),(1^4)$
		5	(1^3)
	$(5,3)$	1	$(4,3)$
		2	(3^2)
		4	(4)
		5	(3)
	$(6,1^2)$	1	$(5,1^2)$
		2	$(5,1),(4,1^2)$
		3	$(4,1),(3,1^2)$
		4	$(3,1),(2,1^2)$
		5	$(2,1),(1^3)$
		6	(1^2)
	$(6,2)$	1	$(5,2)$
		2	$(5,1)$
		3	$(5),(3,2)$
		4	$(3,1),(2^2)$
		5	$(3),(2,1)$
		6	(2)
	$(7,1)$	3	$(4,1)$
		4	(4)
		6	(1^2)
		7	(1)

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