ODD GRASSMANNIAN BIMODULES AND DERIVED EQUIVALENCES FOR SPIN SYMMETRIC GROUPS

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Abstract. We prove odd analogs of results of Chuang and Rouquier on $\mathfrak{sl}_2$-categorification. Combined also with recent work of the second author with Livesey, this allows us to complete the proof of Broué's Abelian Defect Conjecture for the double covers of symmetric groups. The article also develops the theory of odd symmetric functions initiated a decade ago by Ellis, Khovanov and Lauda. A key role in our approach is played by a 2-category consisting of *odd Grassmannian bimodules* over superalgebras which are odd analogs of equivariant cohomology algebras of Grassmannians. This is the odd analog of the category of Grassmannian bimodules which was at the heart of Lauda's independent approach to categorification of $\mathfrak{sl}_2$. We also construct an action of the odd Kac-Moody 2-category $\mathfrak{U}(\mathfrak{sl}_2)$ on the 2-category of odd Grassmannian bimodules, and use this to give a new proof of its non-degeneracy.

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1. Introduction

This paper establishes "odd" analogs of results of Chuang and Rouquer [CR]. The motivating problem is to prove Broué's Abelian Defect Group Conjecture for the double covers of symmetric groups. In the ordinary even theory, Broué's conjecture for symmetric groups was proved in two steps. First came the work of Chuang and Kessar [CK], which established a Morita equivalence reducing the proof of Broué's conjecture to proving that all blocks of symmetric groups in characteristic $p > 0$ with the same defect are derived equivalent. Then the second part of the proof came in Chuang and Rouquier's work which deduced this assertion from a special case of a remarkable general theory of $\mathfrak{sl}_2$-categorification.

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Their theory has had many other significant applications and generalizations, especially following the works of Rouquier [R1, R2] and Khovanov and Lauda [KL], which upgraded from $\mathfrak{sl}_2$ to an arbitrary symmetrizable Kac-Moody algebra $\mathfrak{g}$.

The analogous story for the double covers of symmetric groups has an equally long history, being initiated of course by Schur soon after the ordinary representation theory of symmetric groups was worked out. In [BK], we uncovered a connection in the same spirit as Grojnowski's work [G]—an important predecessor of [CR]—between modular representations of spin symmetric groups in odd characteristic $p = 2l + 1$ and the Kac-Moody group of type $A_{2l}^{(2)}$. A few years later, the odd theory was given new life by work of Ellis, Khovanov and Lauda [EK, EKL, EL, E], whose motivation came from the completely different direction of the categorification program related to odd Khovanov homology. They developed a substantial theory of *odd symmetric functions* which plays a key role in this article. Soon after the work of Ellis, Khovanov and Lauda, a major breakthrough was made in work of Kang, Kashiwara, Oh and Tsuchioka [KKT, KKO1, KKO2]. They introduced so-called *quiver Hecke superalgebras*, which are the odd analogs of the Khovanov-Lauda-Rouquier algebras that underpin all current approaches to categorification of Kac-Moody algebras. In fact, quiver Hecke superalgebras categorify the positive part of the so-called *covering quantum group* $U_{q,\pi}(\mathfrak{g})$ associated to a super Kac-Moody datum with underlying symmetrizable Kac-Moody algebra $\mathfrak{g}$. These covering quantum groups were defined independently and studied in great detail by Clark, Wang and Hill [CW, CHW, CHW2, C]. Then there was a lull in activity, until work of the first author with Ellis [BE2] which simplified the definition of the odd analog of the 2-category associated to $\mathfrak{sl}_2$ made originally by Ellis and Lauda [EL] and extended it to an arbitrary super Kac-Moody datum. Recently, Dupont, Ebert and Lauda [DEL] have used "rewriting theory" to prove that the odd $\mathfrak{sl}_2$ 2-category from [BE2] is non-degenerate, but this is still an open problem for other odd types.

It has in fact been expected for long time that there should exist a comprehensive odd analog of the Chuang and Rouquier theory, and that this should play a role in constructing the derived equivalences required to prove Broué Conjecture for spin symmetric groups. However, due in part to the lack of an appropriate analog of the first part of the proof for symmetric groups—the part provided by the work of Chuang and Kessar—it was not investigated seriously until now. This analog has recently been established, in work of the second author with Livesey [KLi], and is in fact highly non-trivial. The arguments in [KLi] depend essentially on the Morita equivalence between cyclotomic quiver Hecke superalgebras and group algebras of spin symmetric groups constructed in [KKT], and also rely on the new approach to the study of RoCK blocks developed by the second author and Evseev in [EvK].

This article completes the second step of this program for spin symmetric groups. In order to do this, one needs to be able to compute explicitly with some realization of the categorification of the analog $V(-\ell)$ of the $\mathfrak{sl}_2$-module of lowest weight $-\ell$ for the covering quantum group $U_{q,\pi}(\mathfrak{sl}_2)$. We do this in this article by developing a non-trivial theory of *odd Grassmannian bimodules*. These are bimodules over pairs of algebras which we refer to as the *equivariant odd Grassmannian cohomology algebras* since they are analogous to the $GL_\ell(\mathbb{C})$-equivariant cohomology algebras of the usual Grassmannian of n-dimensional subspaces of $\mathbb{C}^\ell$. The specialized versions of these algebras with the word "equivariant" removed were worked out already by Ellis, Khovanov and Lauda [EKL], but the generalization to the equivariant setting is not obvious due to the non-commutativity of the algebra *OSym* of odd symmetric functions. The definition of equivariant odd Grassmannian cohomology algebras—which are purely algebraic in nature rather than coming from any known cohomology theory—is given in Definition 8.1, and then the all-important 2-category $\mathcal{OGBim}_\ell$ of bimodules over these algebras is introduced in Definition 10.6. The key property of this, its *rigidity*, is established in Theorems 11.3 and 11.5.

With this theory in place, in Definition 12.2, we are able to write down the odd analog of the *singular Rouquier complex* in the category $\mathcal{OGBim}_\ell$, proving the necessary homological properties of this needed to be able to obtain derived equivalences between the module categories over odd Grassmannian cohomology algebras. After that, we digress to explain the relationship between the odd $\mathfrak{sl}_2$ 2-category $\mathfrak{U}(\mathfrak{sl}_2)$ from [EL, BE2] (Definition 13.1) and the category $\mathcal{OGBim}_\ell$, namely, there is a 2-functor from the former to the latter (Theorem 13.2). This is the odd analog of the main result about the ordinary $\mathfrak{sl}_2$ 2-category obtained by Lauda in [L1, L2]. We use this 2-functor to give another proof of the non-degeneracy of the odd $\mathfrak{sl}_2$ 2-category established originally in [DEL]; see Theorem 13.5. This implies that the Grothendieck ring of the super Karoubi envelope of $\mathfrak{U}(\mathfrak{sl}_2)$ is isomorphic to the appropriate integral form of the covering quantum group $U_{q,\pi}(\mathfrak{sl}_2)$. Then, in Section 14, we develop some of the basic theory of 2-representations of the odd $\mathfrak{sl}_2$ 2-category, following [R1] quite closely. This is applied in the next section to prove Theorem 15.5, which may be paraphrased as follows:

Theorem. *The bounded homotopy category $K^b(\mathcal{V})$ of any integrable Karoubian 2-representation $\mathcal{V}$ of the odd $\mathfrak{sl}_2$ 2-category $\mathfrak{U}(\mathfrak{sl}_2)$ admits an auto-equivalence categorifying the action of the simple reflection in the associated Weyl group.*

In the final Section 16, we apply this, together with its even analog from [CR], to establish the key derived equivalences between blocks of double covers of symmetric and alternating groups predicted by Kessar and Schaps [KS]. In fact, we establish derived equivalences between the corresponding cyclotomic quiver Hecke superalgebras of type $A_{2l}^{(2)}$, which were shown to be Morita equivalent to spin blocks of symmetric groups up to Clifford twists in [KKT]. Our arguments here also rest crucially on the results of [KKO1, KKO2] in order to check that representations of cyclotomic quiver Hecke superalgebras do admit the structure of a super Kac-Moody 2-representation. Combined with the results in [KLi], this is sufficient to complete the proof of the Broué Conjecture for double covers of symmetric and alternating groups.

We would finally like to discuss some significant overlaps between the results of this article and the independent work of Ebert, Lauda and Vera [ELV]. Their work also introduces the equivariant odd Grassmannian cohomology algebras studied here, relating them to deformed odd cyclotomic nilHecke algebras in exactly the same way as in Theorem 9.2 below, and they also establish the derived equivalences necessary to complete the proof of Broué's Conjecture for spin symmetric groups. We view their general approach as complementary to ours, and it is reassuring to have an independent proof of this difficult place in the theory. The strategy adopted by Ebert, Lauda and Vera is modelled on Vera's new treatment of derived equivalences in the ordinary even case developed in [V]. In particular, it uses a version of the results of Kang, Kashiwara and Oh [KKO1, KKO2] to construct the universal categorification of the $U_{q,\pi}(\mathfrak{sl}_2)$-module $V(-\ell)$ in terms of representations of deformed odd cyclotomic nil-Hecke algebras. This is the place where we use instead the theory of odd Grassmannian bimodules developed in this article, making our article more self-contained.

We expect the results here will have further applications, notably, to the representation theory of the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$. This article also initiates the study of 2-representations of super Kac-Moody 2-categories in the spirit of Rouquier's 2-representation theory for ordinary Kac-Moody 2-categories.

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General conventions. With the exception of Section 3, we work over an algebraically closed field $\mathbb{F}$ of characteristic different from 2, and all categories, functors, etc. are assumed to be $\mathbb{F}$-linear without further comment. The symbol $\otimes$ with no additional subscript denotes tensor product over $\mathbb{F}$. We use the shorthand $X \in \mathcal{C}$ to indicate that X is an element of the object set of a category $\mathcal{C}$.

Let Λ^+ be the set of all *partitions* $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots)$. We adopt standard notations such as $\lambda^{\mathtt{t}} = (\lambda_1^{\mathtt{t}}, \lambda_2^{\mathtt{t}}, \dots)$ for the transpose partition and $\mathrm{ht}(\lambda)$ for the number $\lambda_1^{\mathtt{t}}$ of non-zero parts. The usual dominance ordering is denoted $\leq$. The lexicographic ordering $\leq_{\mathrm{lex}}$ is a total order refining $\leq$. We use the English convention for Young diagrams and tableaux, so rows and columns are indexed like for matrices. Let $\Lambda_n^+ := \{\lambda \in \Lambda^+ \mid \mathrm{ht}(\lambda) \leq n\}$ be the set of partitions of height at most n and $\Lambda_{m\times n}^+$ be the set of partitions λ whose Young diagram fits into an $m \times n$ rectangle, i.e. $\mathrm{ht}(\lambda) \leq m$ and $\lambda_1 \leq n$. Note that

$$\left|\Lambda^+_{m\times n}\right| = \binom{m+n}{n}.$$

For $\lambda \in \Lambda^+$, the following will be needed in various formulae for signs, following [E, Sec. 2.2]:

- $N(\lambda)$ is the number of pairs (A, B) such that B is strictly north of A (strictly above in any column);
- $NE(\lambda)$ is the number of pairs of boxes (A, B) such that B is strictly northeast of A (strictly above and strictly to the right);
- $\overline{NE}(\lambda)$ is the number of pairs of boxes (A, B) such that B is weakly northeast of A ([above or in the same row] and [to the right or in the same column]);
- $dN(\lambda)$ is the number of pairs (A, B) of boxes in the Young diagram of λ such that B is due north of A (strictly above and in the same column);
- $dE(\lambda)$ is the number of pairs of boxes (A, B) in the Young diagram such that B is due east of A (strictly to the right and in the same row).

Some equivalent definitions: $N(\lambda) = \sum_{1\leq i<j} \lambda_i\lambda_j$; $dN(\lambda) = \sum_{i\geq 1}(i-1)\lambda_i$; $dE(\lambda) = \sum_{i\geq 1}\binom{\lambda_i}{2} = dN(\lambda^{\mathrm{t}})$; $\overline{NE}(\lambda) = |\lambda| + dN(\lambda) + dE(\lambda) + NE(\lambda)$.

Let $\Lambda(k, n)$ be the set of all *compositions* of n with k parts, that is, k-tuples $\alpha = (\alpha_1, \dots, \alpha_k)$ of non-negative integers such that $|\alpha| := \alpha_1 + \cdots + \alpha_k = n$. Let $N(\alpha) := \sum_{1\leq i<j\leq k} \alpha_i\alpha_j$ (like for partitions). The reversed composition is $\alpha^{\mathrm{r}} := (\alpha_k, \dots, \alpha_1)$. Also $\alpha \sqcup \beta$ denotes concatenation of compositions α and β.

We denote the symmetric group by S_n acting on the left on $\{1, \dots, n\}$. The ith basic transposition is $s_i = (i\ \ i{+}1)$ and $\ell : S_n \to \mathbb{N}$ is the associated length function. We use the notation w_n to denote the longest element of S_n of length $\ell(w_n) = \binom{n}{2}$. We will often use the identities

$$\binom{r+s}{2} = \binom{r}{2} + \binom{s}{2} + rs, \qquad \binom{-r}{2} = \binom{r+1}{2}$$

for $r, s \in \mathbb{Z}$. For $\alpha \in \Lambda(k, n)$, there is a corresponding parabolic subgroup S_α of S_n; it is the subgroup generated by all s_i for $i \in \{1, \dots, n\} - \{\alpha_1, \alpha_1 + \alpha_2, \dots, \alpha_1 + \cdots + \alpha_k\}$. We use $[S_n/S_\alpha]_{\min}$ and $[S_\alpha\backslash S_n]_{\min}$ to denote the sets of minimal length left and right coset representatives, respectively. Also let w_α be the longest element of S_α and w^α be the longest element of $[S_n/S_\alpha]_{\min}$, so that $w_n = w^\alpha w_\alpha$. For example, $S_{(n-1,1)}$ is S_{n-1} embedded into S_n as the permutations that fix n. These natural embeddings define a tower of subgroups $S_0 < S_1 < S_2 < \cdots$. There is also the *shifted* embedding $\mathrm{sh}_n : S_{n'} \hookrightarrow S_{n+n'}$, $s_j \mapsto s_{n+j}$ for $n, n' \geq 0$.

We may implicitly identify $\lambda \in \Lambda^+$ with the "dominant" composition $(\lambda_1, \dots, \lambda_k) \in \Lambda(k, n)$ where $n := |\lambda|$ and $k := \mathrm{ht}(\lambda)$. Note then that $NE(\lambda)$ defined combinatorially above is the length of the unique minimal length $S_{\lambda^{\mathrm{t}}}\backslash S_n/S_\lambda$-double coset representative w such that $\left|S_{\lambda^{\mathrm{t}}} \cap wS_\lambda w^{-1}\right| = 1$. For $n \in \mathbb{Z}, r \geq 0$, we let

$$n\#r := n + (n+1) + \cdots + (n+r-1) = nr + \binom{r}{2}.$$

2. Graded superalgebra

In this section, we review some basic language, referring the reader to the exposition in the introduction of [BE1] for more details; see also [BE1, Sec. 6] which discusses gradings. For a commutative ring R, we write R^π for the ring $R[\pi]/(\pi^2 - 1)$. Assuming that 2 is invertible in R, the Chinese Remainder Theorem gives a ring isomorphism $R^\pi \xrightarrow{\sim} R \times R, a \mapsto (a_+, a_-)$ for $a_\pm \in R$ defined by evaluating π at ± 1. Then we have that $a \in (R^\pi)^\times$ if and only if both $a_+ \in R^\times$ and $a_- \in R^\times$. For example, $\pi q - q^{-1} \in \mathbb{Q}(q)^\pi$ is invertible because both $q - q^{-1}$ and $-q - q^{-1}$ are invertible in $\mathbb{Q}(q)$.

A *graded vector superspace* is a $\mathbb{Z} \times \mathbb{Z}/2$-graded vector space $V = \bigoplus_{d\in\mathbb{Z}, p\in\mathbb{Z}/2} V_{d,p}$. We may also write V_p for $\bigoplus_{d\in\mathbb{Z}} V_{d,p}$, so $V_{\bar 0}$ is the *even part* and $V_{\bar 1}$ is the *odd part* of A. For a homogeneous vector

$v \in V_{d,p}$, we write $\deg(v)$ for its *degree* d (the $\mathbb{Z}$-grading) and $\operatorname{par}(v)$ for its *parity* p (the $\mathbb{Z}/2$-grading). We write $\underline{\mathcal{gsVec}}$ for the closed symmetric monoidal category of graded vector superspaces with morphisms that preserve both degree and parity of vectors. Its symmetric braiding is defined on graded vector superspaces V and W by

$$B_{V,W} : V \otimes W \to W \otimes V, \qquad v \otimes w \mapsto (-1)^{\operatorname{par}(v)\operatorname{par}(w)} w \otimes v. \tag{2.1}$$

This only makes sense if v and w are homogeneous, but we adopt the usual abuse of notation by suppressing this assumption. We use the notation Π for the *parity switch functor* and Q for the upward *grading shift functor*, using π and q for the induced maps at the level of Grothendieck groups.

A *graded superalgebra* is an associative, unital algebra in $\underline{\mathcal{gsVec}}$. Any graded superalgebra A has the *parity involution*

$$\mathrm{p} : A \to A, \qquad a \mapsto (-1)^{\operatorname{par}(a)} a. \tag{2.2}$$

For graded superalgebras A and B, their tensor product $A \otimes B$ is the tensor product of the underlying graded vector superspaces viewed as a graded superalgebra so that

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\operatorname{par}(b_1)\operatorname{par}(a_2)} a_1 a_2 \otimes b_1 b_2$$

for $a_1, a_2 \in A, b_1, b_2 \in B$. We also write A^{sop} for the opposite superalgebra, whose multiplication is defined from $a \cdot b := (-1)^{\operatorname{par}(a)\operatorname{par}(b)} ba$.

Very important in this article is the graded superalgebra $OPol_n$ of *odd polynomials*. By definition, this is the tensor product

$$OPol_n := \underbrace{OPol_1 \otimes \cdots \otimes OPol_1}_{n \text{ times}}, \tag{2.3}$$

where $OPol_1 := \mathbb{F}[x]$ is the usual commutative polynomial algebra in an indeterminate x, viewed as a graded superalgebra so that x is odd of degree 2. We let $x_i := 1 \otimes \cdots \otimes x \otimes \cdots \otimes 1$ where x is in the ith tensor position from the left. Here are some further observations.

- For $\alpha \in \Lambda(k, n)$, the tensor product $OPol_{\alpha_1} \otimes \cdots \otimes OPol_{\alpha_k}$ is canonically identified with $OPol_n$ so that $1^{\otimes(j-1)} \otimes x_i \otimes 1^{\otimes(k-j)} \equiv x_{\alpha_1+\cdots+\alpha_{j-1}+i}$.
- The elements $x_1, \ldots, x_n$ generate $OPol_n$ subject to the relations $x_j x_i = -x_i x_j$ for $1 \leq i < j \leq n$.
- There are no (non-zero) zero divisors in $OPol_n$, although it does not have unique factorization, e.g., $(x_1 - x_2)^2 = (x_1 + x_2)^2$.
- The monomials $x^\kappa := x_1^{\kappa_1} \cdots x_n^{\kappa_n} \in OPol_n$ for $\kappa = (\kappa_1, \ldots, \kappa_n) \in \mathbb{N}^n$ give a linear basis for $OPol_n$.

From the last point, it follows that

$$\dim_{q,\pi} OPol_n = \frac{1}{(1 - \pi q^2)^n} \in \mathbb{Z}[\![q]\!]^\pi, \tag{2.4}$$

meaning that the coefficient of $q^d \pi^p$ in this generating function is the dimension of the homogeneous component in degree $d \in \mathbb{Z}$ and parity $p \in \mathbb{Z}/2$.

A *graded supercategory* is a category enriched in $\underline{\mathcal{gsVec}}$, and a *graded superfunctor* is an enriched functor. In particular, graded superfunctors preserve degrees and parities of morphisms. Given graded supercategories $\mathcal{A}$ and $\mathcal{B}$, we use the notation $\mathcal{gsHom}(\mathcal{A}, \mathcal{B})$ for the graded supercategory of graded superfunctors and graded *super*natural transformations in the sense of [BE1, Ex. 1.1]. In particular, $\mathcal{gsEnd}(\mathcal{A}) := \mathcal{gsHom}(\mathcal{A}, \mathcal{A})$ is a strict *graded monoidal supercategory*; see [BE1, Ex. 1.5(ii)]. More generally, there is a strict *graded 2-supercategory* $\mathcal{gsCat}$ consisting of (small) graded supercategories, graded superfunctors and graded supernatural transformations; see [BE1, Sec. 6]. For graded superfunctors $F, G : \mathcal{A} \to \mathcal{B}$, we denote the morphism space $\operatorname{Hom}_{\mathcal{gsHom}(\mathcal{A},\mathcal{B})}(F, G)$, that is, the graded superspace consisting of all graded supernatural transformations from F to G, simply by $\operatorname{gsHom}(F, G)$. If $F = G$ we denote it by $\operatorname{gsEnd}(F)$, this being a graded superalgebra.

For any graded supercategory $\mathcal{A}$, we denote the *underlying ordinary category* consisting of the same objects and the morphisms that are even of degree 0 by $\underline{\mathcal{A}}$. We will systematically write $\cong$ to denote the existence of an isomorphism that is not necessarily homogeneous, and $\simeq$ to denote the existence of an isomorphism that preserves parities and degrees. For example, the category $\underline{\mathcal{gsVec}}$ is the underlying ordinary category of a graded monoidal supercategory $\mathcal{gsVec}$. In $\mathcal{gsVec}$, a linear map $f : V \to W$ is homogeneous of degree $d \in \mathbb{Z}$ and parity $p \in \mathbb{Z}/2$ if $f(V_{d',p'}) \subseteq W_{d+d',p+p'}$ for all $d' \in \mathbb{Z}$, $p' \in \mathbb{Z}/2$. Then an arbitrary morphism in $\mathcal{gsVec}$ is a *graded linear map*, that is, a linear map with the property that it can be written as a finite sum of homogeneous linear maps of different degrees.

In fact, $\mathcal{gsVec}$ is a *graded monoidal* (Q,Π)*-supercategory*; see [BE1, Def. 1.12, Def. 6.5] for the formal definition. The unit object is the field $\mathbb{F}$ viewed as a graded superspace so that it is even in degree 0, the parity shift functor is $\Pi\mathbb{F} \otimes -$ where $\Pi\mathbb{F}$ is $\mathbb{F}$ in degree 0 and odd parity, and grading shift functor is $Q\mathbb{F} \otimes -$ where $Q\mathbb{F}$ is $\mathbb{F}$ in degree one and even parity. For any graded (Q,Π)-supercategory $\mathcal{V}$, its underlying ordinary category is a (Q,Π)*-category* in the sense of [BE1, Def. 6.12]. In fact, $\underline{\mathcal{gsVec}}$ is a *monoidal* (Q,Π)*-category* in the sense of [BE1, Def. 1.14, Def. 6.14].

Given two graded superalgebras A and B, we write A-gsMod-B for the graded supercategory of graded (A,B)-superbimodules and graded (A,B)-superbimodule homomorphisms; such a homomorphism is a finite sum of homogeneous (A,B)-superbimodule homomorphisms of different degrees and parities. We adopt the usual sign convention for morphisms as in [BE1, Ex. 1.8]. So a morphism $f : V \to W$ in A-gsMod-B satisfies $f(avb) = (-1)^{\operatorname{par}(f)\operatorname{par}(a)} af(v)b$ for $a \in A, b \in B, v \in M$. The category A-gsMod-B is a graded (Q,Π)-supercategory in the sense of [BE1, Def. 1.7, Def. 6.4]. We use the notation $\operatorname{Hom}_{A\text{-}B}(V,W)$ to denote a morphism space in this category, which is a graded vector superspace. The parity switching functor Π takes a graded (A,B)-superbimodule V to the same underlying graded vector space viewed as a superspace with the opposite parities and actions of $a \in A$ and $b \in B$ on $v \in \Pi V$ defined in terms of the original action so that

$$a \cdot v \cdot b := (-1)^{\operatorname{par}(a)} avb. \tag{2.5}$$

On a morphism $f : V \to W$, $\Pi f : \Pi V \to \Pi W$ is defined so that $(\Pi f)(v) = (-1)^{\operatorname{par}(f)} f(v)$. The grading shift functor Q takes V to the same underlying superbimodule with the new grading $(QV)_d := V_{d-1}$. This is less delicate since it does not introduce any additional signs. The definition of graded (Q,Π)-supercategory also involves some additional data of supernatural isomorphisms $\zeta : \Pi \overset{\sim}{\Rightarrow} \mathrm{Id}$, $\sigma : Q \overset{\sim}{\Rightarrow} \mathrm{Id}$ and $\bar\sigma : Q^{-1} \overset{\sim}{\Rightarrow} \mathrm{Id}$, which in this case all come from the identity function on the underlying vector space. We will not need these in any significant way, so refer the reader to [BE1] for the details.

The graded (Q,Π)-supercategories A-gsMod and gsMod-B of graded left A-supermodules and right B-supermodules can now be defined to be A-gsMod-$\mathbb{F}$ and $\mathbb{F}$-gsMod-B, respectively. We use the notation $\operatorname{Hom}_{A\text{-}}(V,W)$ and $\operatorname{Hom}_{\text{-}B}(V,W)$ to denote the morphism spaces in these categories. We use A-gsmod, A-pgsMod and A-pgsmod to denote the full subcategories of A-gsMod consisting of the finite-dimensional, projective and finitely generated projective left A-supermodules, respectively. The underlying ordinary categories are A-$\underline{\text{gsMod}}$, A-$\underline{\text{gsmod}}$, A-$\underline{\text{pgsMod}}$ and A-$\underline{\text{pgsmod}}$.

For graded supercategories $\mathcal{A}, \mathcal{B}$, an *adjoint pair* (E,F) of graded superfunctors $E : \mathcal{A} \to \mathcal{B}$ and $F : \mathcal{B} \to \mathcal{A}$ means an adjoint pair of $\mathbb{F}$-linear functors in the usual sense, such that in addition the unit and the counit of the adjunction are both even supernatural transformations of degree 0. It follows that the restrictions of E and F to the underlying ordinary categories also form an adjoint pair.

Now suppose that A and B are graded superalgebras such that A is a (unital) subalgebra of B. Then there is an adjoint triple of graded superfunctors $(\operatorname{Ind}_A^B, \operatorname{Res}_A^B, \operatorname{Coind}_A^B)$:

$$\operatorname{Ind}_A^B := B \otimes_A - : A\text{-gsMod} \to B\text{-gsMod}, \tag{2.6}$$

$$\operatorname{Res}_A^B := \operatorname{Hom}_{B\text{-}}(B,-) \simeq B \otimes_B - : B\text{-gsMod} \to A\text{-gsMod}, \tag{2.7}$$

$$\operatorname{Coind}_A^B := \operatorname{Hom}_{A\text{-}}(B,-) : A\text{-gsMod} \to B\text{-gsMod}. \tag{2.8}$$

Following [PS] (which explicitly treats graded superalgebras), we say that B is a *Frobenius extension* of A of degree $d \in \mathbb{Z}$ and parity $p \in \mathbb{Z}/2$ if there exists a *trace map* $\mathrm{tr} : B \to A$ that is a homogeneous graded (A, A)-superbimodule homomorphism of degree $-d$ and parity p, together with homogeneous elements $b_1, \dots, b_m, b_1^\vee, \dots, b_m^\vee$ of B such that

$$\deg(b_i) + \deg(b_i^\vee) = d, \qquad \mathrm{par}(b_i) + \mathrm{par}(b_i^\vee) = p, \tag{2.9}$$

$$\sum_{i=1}^m b_i \,\mathrm{tr}(b_i^\vee b) = b, \qquad \sum_{i=1}^m (-1)^{p\,\mathrm{par}(b) + p\,\mathrm{par}(b_i^\vee)} \mathrm{tr}(b b_i) b_i^\vee = b \tag{2.10}$$

for any $b \in B$. The associated *comultiplication* is the homogeneous graded (B, B)-superbimodule homomorphism

$$\Delta : B \to B \otimes_A B, \qquad 1 \mapsto \sum_{j=1}^m (-1)^{p\,\mathrm{par}(b_j^\vee)} b_j \otimes b_j^\vee, \tag{2.11}$$

which is of degree d and parity p. In the adjoint pair $(\mathrm{Ind}_A^B, \mathrm{Res}_A^B)$, the unit and counit of the canonical adjunction making $(\mathrm{Ind}_A^B, \mathrm{Res}_A^B)$ into an adjoint pair are induced by the superbimodule homomorphisms $\eta : A \to B$ and $\mu : B \otimes_A B \to B$ given by the canonical inclusion and multiplication, respectively. Assuming that we have a Frobenius extension, there is also an adjunction making $(\mathrm{Res}_A^B, Q^{-d}\Pi^p \,\mathrm{Ind}_A^B)$ into an adjoint pair, with unit and counit of adjunction induced by the superbimodule homomorphisms tr and Δ viewed now as homogeneous graded supermodule homomorphisms $\mathrm{tr} : Q^{-d}\Pi^p B \to A$ and $\Delta : B \to Q^d \Pi^p B \otimes_A B$ that are even of degree 0. This adjunction induces a canonical even degree 0 isomorphism $\mathrm{Ind}_A^B \simeq Q^d \Pi^p \,\mathrm{Coind}_A^B$ Conversely, if there exists such an isomorphism of graded superfunctors then B is a Frobenius extension of A of degree $d \in \mathbb{Z}$ and parity $p \in \mathbb{Z}/2$; see [PS, Th. 6.2].

We will only use the definitions from the previous paragraph in situations in which B is positively graded and connected (i.e., $B_0 = \mathbb{F}$). In that case, the trace map is unique up to multiplication by a non-zero scalar; see [PS, Prop. 4.7] for a more general uniqueness statement. Moreover, the elements $b_1, \dots, b_m$ (resp., $b_1^\vee, \dots, b_m^\vee$) can be chosen so that they give a basis for B as a free right (resp., left) A-supermodule, in which case (2.10) can be replaced simply by the condition

$$\mathrm{tr}(b_i^\vee b_j) = \delta_{i,j} \tag{2.12}$$

for all i, j, i.e., the two bases are *dual* to each other.

Now suppose that A is a *supercommutative* graded superalgebra, i.e., its multiplication satisfies $ab = (-1)^{\mathrm{par}(a)\,\mathrm{par}(b)} ba$ for all $a, b \in A$. A *graded A-superalgebra* B is a graded superalgebra together with a structure map $\eta : A \to Z(B)$ which is a (unital) graded superalgebra homomorphism from A to the *supercenter*

$$Z(B) = \{c \in B \,\big|\, bc = (-1)^{\mathrm{par}(b)\,\mathrm{par}(c)} cb \text{ for all } b \in B\}. \tag{2.13}$$

In particular, such a superalgebra B is a *graded A-supermodule*, by which we mean a graded (A, A)-superbimodule such that the left and right actions of $a \in A$ on a vector v are related by

$$av = (-1)^{\mathrm{par}(a)\,\mathrm{par}(v)} va. \tag{2.14}$$

We say that a graded A-superalgebra B is a *graded Frobenius superalgebra* over A of degree d and parity p if the structure map $\eta : A \to Z(B)$ is injective, and B is a Frobenius extension of $\eta(A)$ of degree d and parity p in the sense from the previous paragraph.

By the *graded super Karoubi envelope* $\mathrm{gsKar}(\mathcal{A})$ of a graded supercategory $\mathcal{A}$ we mean the graded (Q, Π)-supercategory obtained by first passing to its (Q, Π)-envelope $\mathcal{A}_{q,\pi}$ (see the next paragraph), then to the additive envelope of $\mathcal{A}_{q,\pi}$, then finally to the completion of that at all homogeneous idempotents. The underlying ordinary category $\underline{\mathrm{gsKar}}(\mathcal{A})$, which is an additive and idempotent complete (Q, Π)-category, is what is called the graded super Karoubi envelope in the final paragraph of [BE1, Sec. 6]. Any graded superfunctor $F : \mathcal{A} \to \mathcal{B}$ to an additive graded (Q, Π)-supercategory $\mathcal{B}$ whose underlying ordinary category is idempotent complete induces a canonical graded superfunctor $F_{q,\pi} : \text{gsKar}(\mathcal{A}) \to \mathcal{B}$; this follows from [BE1, Th. 6.3] combined with the usual universal properties of additive envelopes and idempotent completions. The split Grothendieck group $K_0(\underline{\text{gsKar}}(\mathcal{A}))$ is naturally a $\mathbb{Z}[q,q^{-1}]^\pi$-module. For example, if $\mathcal{A}$ is the graded supercategory with one object whose endomorphisms are given by some graded superalgebra A then, by Yoneda Lemma, $\text{gsKar}(\mathcal{A})$ is contravariantly graded superequivalent to A-pgsmod. In this case, we denote $K_0(\underline{\text{gsKar}}(\mathcal{A})) \cong K_0(A\text{-pg}\underline{\text{smod}})$ simply by $K_0(A)$.

The only part of the construction of $\text{gsKar}(\mathcal{A})$ just outlined which is not standard is the notion of the (Q,Π)-*envelope* of $\mathcal{A}$. According to [BE1, Def. 6.8], this is the graded supercategory $\mathcal{A}_{q,\pi}$ with objects given by the symbols $Q^d\Pi^p X$ for $d \in \mathbb{Z}$, $p \in \mathbb{Z}/2$ and $X \in \mathcal{A}$, with

$$\text{Hom}_{\mathcal{A}_{q,\pi}}(Q^d\Pi^p X, Q^e\Pi^q Y) := Q^{e-d}\Pi^{p+q}\,\text{Hom}_{\mathcal{A}}(X,Y),$$

where Q and Π on the right hand side are the grading and parity shift functors on $\mathcal{gsVec}$. Denoting $f \in \text{Hom}_{\mathcal{A}}(X,Y)$ viewed as a morphism in $\text{Hom}_{\mathcal{A}_{q,\pi}}(Q^d\Pi^p X, Q^e\Pi^q Y)$ by $f_{d,p}^{e,q}$ as in [BE1], the composition law in $\mathcal{A}_{q,\pi}$ is induced by the composition law in $\mathcal{A}$ in the obvious way:

$$g_{d,q}^{e,r} \circ f_{c,p}^{d,q} := (g \circ f)_{c,p}^{e,r}. \tag{2.15}$$

The grading and parity shift functors making $\mathcal{A}_{q,\pi}$ into a graded (Q,Π) are defined on objects so that $Q(Q^d\Pi^p X) = Q^{d+1}\Pi^p X$ and $\Pi(Q^d\Pi^p X) = Q^d\Pi^{p+\bar{1}}X$, and on morphisms so that $Q(f_{d,p}^{e,q}) = f_{d+1,p}^{e+1,q}$ and $\Pi f_{d,p}^{e,q} = (-1)^{\text{par}(f)+p+q} f_{d,p+\bar{1}}^{e,q+\bar{1}}$. For a more complete account (including also the definitions of the required supernatural isomorphisms ζ, σ and $\bar{\sigma}$) we refer to [BE1, Def. 6.8]. We will always *identify* $\mathcal{A}$ with the full subcategory of $\mathcal{A}_{q,\pi}$ via the obvious graded superfunctor $\mathcal{A} \to \mathcal{A}_{q,\pi}, X \mapsto Q^0\Pi^{\bar{0}}X, f \mapsto f_{0,\bar{0}}^{0,\bar{0}}$. The graded supercategory $\mathcal{A}$ is called (Q,Π)-*complete* if this functor is a graded superequivalence; equivalently, for every object $X \in \mathcal{A}$, $d \in \mathbb{Z}$ and $p \in \mathbb{Z}/2$ there is another object Y and an isomorphism $f : Y \xrightarrow{\sim} X$ that is homogeneous of degree d and parity p; cf. [BE1, Lem. 4.1].

It also makes sense to take the graded super Karoubi envelope $\text{gsKar}(\mathfrak{A})$ of a graded 2-supercategory $\mathfrak{A}$, which is a graded (Q,Π)-2-supercategory. The split Grothendieck ring $K_0(\underline{\text{gsKar}}(\mathfrak{A}))$ is naturally a locally unital $\mathbb{Z}[q,q^{-1}]^\pi$-algebra equipped with a distinguished collection of mutually orthogonal idempotents indexed by the objects of $\mathfrak{A}$. We just explain the non-trivial step here, which is the construction of the (Q,Π)-envelope $\mathfrak{A}_{q,\pi}$ of a graded 2-supercategory, referring to [BE1, Def. 6.10] and the subsequent discussion for a fuller account. In the case that $\mathfrak{A}$ is a strict graded 2-supercategory, $\mathfrak{A}_{q,\pi}$ is a strict graded (Q,Π)-2-supercategory with the same objects as $\mathfrak{A}$ and morphism supercategories that are the (Q,Π)-envelopes of the morphism supercategories in $\mathfrak{A}$. For 1-morphisms $X, Y : \lambda \to \mu$ in $\mathfrak{A}$, we represent the 2-morphism $Q^d\Pi^p X \Rightarrow Q^e\Pi^q Y$ in $\mathfrak{A}_{q,\pi}$ associated to the 2-morphism $\alpha : X \Rightarrow Y$ by $\alpha_{d,p}^{e,q}$, which is of parity $\text{par}(\alpha)+p+q$ and degree $\deg(\alpha)+e-d$. Vertical composition is defined in the obvious way as in (2.15). Horizontal composition is defined in terms of horizontal composition in $\mathfrak{A}$ but there are some surprising signs:

$$\alpha_{d,p}^{e,q}(\alpha')_{d',p'}^{e',q'} := (-1)^{p(\text{par}(\alpha')+p'+q')+\text{par}(\alpha)q'}(\alpha\alpha')_{d+d',p+p'}^{e+e',q+q'} \tag{2.16}$$

These signs play an important role in the following lemma, which when $p = \bar{1}$ is the idea of "odd adjunction".

Lemma 2.1. *Let $\mathfrak{A}$ be a strict graded 2-supercategory. Suppose that $E : \lambda \to \mu$ and $F : \mu \to \lambda$ are 1-morphisms and $\varepsilon : E \circ F \Rightarrow 1_\mu$ and $\eta : 1_\lambda \Rightarrow F \circ E$ are 2-morphisms of parity p and degrees d and $-d$, respectively, such that $(\varepsilon 1_E) \circ (1_E \eta) = 1_E$ and $(1_F \varepsilon) \circ (\eta 1_F) = (-1)^p 1_F$.*

(1) The degree 0 even 2-morphisms $\varepsilon_{d,p}^{0,\bar{0}} : E \circ (Q^d\Pi^p F) \Rightarrow \text{Id}$ and $\eta_{0,\bar{0}}^{d,p} : \text{Id} \Rightarrow (Q^d\Pi^p F) \circ E$ give the unit and counit of an adjunction making $Q^d\Pi^p F$ into a right dual of E in $\mathfrak{A}_{q,\pi}$.

(2) *For a supernatural transformation $\alpha : E \Rightarrow E$ in $\mathfrak{A}$, its mate $\left(1_{Q^d\Pi^pF}\varepsilon^{0,\bar{0}}_{d,p}\right)\circ\left(1_{Q^d\Pi^pF}\alpha 1_{Q^d\Pi^pF}\right)\circ\left(\eta^{d,p}_{0,\bar{0}}1_{Q^d\Pi^pF}\right) : Q^d\Pi^pF \Rightarrow Q^d\Pi^pF$ with respect to the adjunction constructed in (1) is equal to $(-1)^p((1_F\varepsilon)\circ(1_F\alpha 1_F)\circ(\eta 1_F))^{d,p}_{d,p}$.*

Proof. (1) We need to show that $\left(\varepsilon^{0,\bar{0}}_{d,p}1_E\right)\circ\left(1_E\eta^{d,p}_{0,\bar{0}}\right) = 1_E$ and $\left(1_{Q^d\Pi^pF}\varepsilon^{0,\bar{0}}_{d,p}\right)\circ\left(\eta^{d,p}_{0,\bar{0}}1_{Q^d\Pi^pF}\right) = 1_{Q^d\Pi^pF}$. We just check the second of these (the signs are more interesting!). We have that $1_{Q^d\Pi^pF} = Q^d\Pi^p1_F = (1_F)^{d,p}_{d,p}$. So

$$\begin{aligned}\left(1_{Q^d\Pi^pF}\varepsilon^{0,\bar{0}}_{d,p}\right)\circ\left(\eta^{d,p}_{0,\bar{0}}1_{Q^d\Pi^pF}\right) &= \left((1_F)^{d,p}_{d,p}\varepsilon^{0,\bar{0}}_{d,p}\right)\circ\left(\eta^{d,p}_{0,\bar{0}}(1_F)^{d,p}_{d,p}\right) = (1_F\varepsilon)^{d,p}_{2d,\bar{0}}\circ\left(\eta^{d,p}_{0,\bar{0}}(1_F)^{d,p}_{d,p}\right)\\ &= (-1)^p(1_F\varepsilon)^{d,p}_{2d,\bar{0}}\circ(\eta 1_F)^{2d,\bar{0}}_{d,p} = (-1)^p((1_F\varepsilon)\circ(\eta 1_F))^{d,p}_{d,p}\\ &= (1_F)^{d,p}_{d,p} = 1_{Q^d\Pi^pF}.\end{aligned}$$

(2) This is another such explicit calculation. We just note that $1_{Q^d\Pi^pF}\alpha 1_{Q^d\Pi^pF} = (1_F\alpha 1_F)^{2d,\bar{0}}_{2d,\bar{0}}$ regardless of the parity of α. ◻

3. Combinatorics of (q,π)-binomial coefficients and odd quantum $\mathfrak{sl}_2$

In this section, we recall briefly the definition of the enveloping algebra of "odd quantum $\mathfrak{sl}_2$" discovered by Clark and Wang [CW] and developed in much greater generality in [CHW]. We work initially over $\mathbb{Q}(q)^\pi$; cf. the opening paragraph of Section 2. The most significant difference compared to [CHW, C] is that our q is q^{-1} in [CHW] and v^{-1} in [C]. We define the (q,π)-*integers*

$$[n]_{q,\pi} := \frac{(\pi q)^n - q^{-n}}{\pi q - q^{-1}} = \begin{cases} q^{1-n} + \pi q^{3-n} + \cdots + \pi^{n-1}q^{n-1} & \text{if } n \geq 0,\\ -\pi^n(q^{n+1} + \pi q^{n+3} + \cdots + \pi^{-n-1}q^{-n-1}) & \text{if } n \leq 0\end{cases} \tag{3.1}$$

for any $n \in \mathbb{Z}$. This is exactly the same as the definition given in [CHW, Sec. 1.6]—but because our q is the q^{-1} in [CHW] it is actually a different convention! For $n \neq 0$, the (q,π)-integer $[n]_{q,\pi}$ is invertible in the ring $\mathbb{Q}(q)^\pi$; this follows because the elements of $\mathbb{Q}(q)$ obtained from $[n]_{q,\pi}$ by setting $\pi = \pm 1$ are both non-zero. Note also that

$$[-n]_{q,\pi} = -\pi^n[n]_{q,\pi}. \tag{3.2}$$

There are corresponding (q,π)-factorials $[n]^!_{q,\pi}$ for $n \geq 0$:

$$[n]^!_{q,\pi} := [n]_{q,\pi}[n-1]_{q,\pi}\cdots[1]_{q,\pi} = q^{-\binom{n}{2}}\sum_{w\in S_n}(\pi q^2)^{\ell(w)}, \tag{3.3}$$

where the last equality is a consequence of the well-known factorization of the Poincaré polynomial for the symmetric group. Then we have the (q,π)-binomial coefficients $\begin{bmatrix} n\\ r\end{bmatrix}_{q,\pi}$, which make sense as written for any $n \in \mathbb{Z}$ and $r \geq 0$:

$$\begin{bmatrix} n\\ r\end{bmatrix}_{q,\pi} := \frac{[n]_{q,\pi}[n-1]_{q,\pi}\cdots[n-r+1]_{q,\pi}}{[r]^!_{q,\pi}}. \tag{3.4}$$

We also adopt the convention that $\begin{bmatrix} n\\ r\end{bmatrix}_{q,\pi} = 0$ for any $n \in \mathbb{Z}$ and $r < 0$. Note by (3.2) that

$$\begin{bmatrix} -n\\ r\end{bmatrix}_{q,\pi} = (-1)^r\pi^{nr+\binom{r}{2}}\begin{bmatrix} n+r-1\\ r\end{bmatrix}_{q,\pi}. \tag{3.5}$$

We also need quantum *tri*nonomial coefficients for $n \in \mathbb{Z}$ and $r, s \geq 0$:

$$\begin{bmatrix} n\\ r,s\end{bmatrix}_{q,\pi} := \frac{[n]_{q,\pi}[n-1]_{q,\pi}\cdots[n-r-s+1]_{q,\pi}}{[r]^!_{q,\pi}[s]^!_{q,\pi}} = \begin{bmatrix} n\\ r\end{bmatrix}_{q,\pi}\begin{bmatrix} n-r\\ s\end{bmatrix}_{q,\pi}. \tag{3.6}$$

Again we interpret $\begin{bmatrix} n \\ r,s \end{bmatrix}_{q,\pi}$ as zero if $r<0$ or $s<0$. More generally, for $\alpha \in \Lambda(k,n)$, let

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,\pi} := \frac{[n]^!_{q,\pi}}{[\alpha_1]^!_{q,\pi}\cdots[\alpha_k]^!_{q,\pi}} \tag{3.7}$$

be the (q,π)-multinomial coefficient. The identity (3.3) implies that

$$\begin{bmatrix} n \\ \alpha \end{bmatrix}_{q,\pi} = q^{-N(\alpha)} \sum_{w\in[S_n/S_\alpha]_{\min}} (\pi q^2)^{\ell(w)}. \tag{3.8}$$

We let $- : \mathbb{Q}(q)^\pi \to \mathbb{Q}(q)^\pi$ be the $\mathbb{Q}^\pi$-algebra involution with $\overline{q} = q^{-1}$. We use the word *anti-linear* for a $\mathbb{Z}$-module homomorphism $f : V \to W$ between $\mathbb{Q}(q)^\pi$- or $\mathbb{Z}[q,q^{-1}]^\pi$-modules such that $f(cv) = \overline{c}f(v)$. We have that

$$\overline{[n]_{q,\pi}} = \pi^{n-1}[n]_{q,\pi}, \qquad \overline{[n]^!_{q,\pi}} = \pi^{\binom{n}{2}}[n]^!_{q,\pi}, \tag{3.9}$$

$$\overline{\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi}} = \pi^{(n-r)r}\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi}, \qquad \overline{\begin{bmatrix} n \\ r,s \end{bmatrix}_{q,\pi}} = \pi^{(n-r)(r+s)+s}\begin{bmatrix} n \\ r,s \end{bmatrix}_{q,\pi}. \tag{3.10}$$

We stress that our bar involution is *not* the same as the bar involution introduced in [CHW, C]; the latter takes q to πq^{-1} and *fixes* the (q,π)-integers, (q,π)-factorials and (q,π)-binomial. Some further properties of (q,π)-binomial and trinomial coefficients are proved in the next two lemmas.

Lemma 3.1. *The (q,π)-binomial and trinomial coefficients have the following properties.*

(1) For $n \in \mathbb{Z}$ and $r \geq 0$, we have that

$$\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi} = q^{-r}\begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi} + (\pi q)^{n-r}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi} = (\pi q)^r\begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi} + q^{r-n}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi}.$$

(2) For $n \in \mathbb{Z}$ and $r,s \geq 0$, we have that

$$\begin{bmatrix} n \\ r,s \end{bmatrix}_{q,\pi} = \pi^s q^{s-r}\begin{bmatrix} n-1 \\ r,s \end{bmatrix}_{q,\pi} + (\pi q)^{n-r}\begin{bmatrix} n-1 \\ r-1,s \end{bmatrix}_{q,\pi} + q^{s-n}\begin{bmatrix} n-1 \\ r,s-1 \end{bmatrix}_{q,\pi}.$$

(3) For $n \in \mathbb{Z}$ and $r \geq 0$, we have that

$$\sum_{s+t=r} \pi^{\binom{t}{2}}(-q)^{-t}\begin{bmatrix} n+s \\ s,t \end{bmatrix}_{q,\pi} = (\pi q)^{nr}.$$

Proof. (1) The first equality follows from the definition of (q,π)-binomial coefficient by replacing the $[n]_{q,\pi}$ in the numerator with $q^{-r}[n-r]_{q,\pi} + (\pi q)^{n-r}[r]_{q,\pi}$ and then splitting the result into two fractions. The second follows by replacing it instead with $\pi^r q^r[n-r]_{q,\pi} + q^{r-n}[r]_{q,\pi}$.

(2) Using (1) twice, we have that

$$\begin{aligned}
\pi^s q^{s-r}\begin{bmatrix} n-1 \\ r,s \end{bmatrix}_{q,\pi} + q^{s-n}\begin{bmatrix} n-1 \\ r,s-1 \end{bmatrix}_{q,\pi} &= q^{-r}\begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi}\left((\pi q)^s\begin{bmatrix} n-r-1 \\ s \end{bmatrix}_{q,\pi} + q^{r+s-n}\begin{bmatrix} n-r-1 \\ s-1 \end{bmatrix}_{q,\pi}\right)\\
&= q^{-r}\begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi}\begin{bmatrix} n-r \\ s \end{bmatrix}_{q,\pi} = \left(\begin{bmatrix} n \\ r \end{bmatrix} - (\pi q)^{n-r}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi}\right)\begin{bmatrix} n-r \\ s \end{bmatrix}_{q,\pi}\\
&= \begin{bmatrix} n \\ r,s \end{bmatrix}_{q,\pi} - (\pi q)^{n-r}\begin{bmatrix} n-1 \\ r-1,s \end{bmatrix}_{q,\pi}.
\end{aligned}$$

(3) Let $a_n(r) := \sum_{s+t=r} \pi^{\binom{t}{2}}(-q)^{-t}\begin{bmatrix} n+s \\ s,t \end{bmatrix}_{q,\pi}$. The goal is to show that $a_n(r) = (\pi q)^{nr}$. It is easy to check that this is true when $nr = 0$. This gives the base of an induction. For the induction step, we also need the

identity

$$a_n(r) = \pi^{nr}(\pi q)^{-r}\,\overline{a_{n-1}(r)} + (\pi q)^n a_n(r-1) - \pi^{nr}(\pi q)^{1-n-r}\,\overline{a_{n-1}(r-1)},$$

which will be verified in the next paragraph. Using this, it is easy to complete the proof for all $n \geq 0$ and $r \geq 0$ by induction on $n + r$. The proof for $n \leq 0$ and $r \geq 0$ goes instead by induction on $r - n$ using the following:

$$a_n(r) = \pi^{nr}q^{-r}\,\overline{a_{n+1}(r)} - \pi^{nr}(\pi q)^{-n-1}q^{-r}\,\overline{a_{n+1}(r-1)} + (\pi q)^n a_n(r-1)$$

This follows by applying $-$ to the previous identity, then replacing n by $n + 1$ and rearranging.

It remains to prove the first identity. Using (2), we have that

$$a_n(r) = \sum_{s+t=r} \pi^{\binom{t}{r}}(-q)^{-t}\left(\pi^t q^{t-s}\begin{bmatrix} n+s-1 \\ s,t \end{bmatrix}_{q,\pi} + (\pi q)^n\begin{bmatrix} n+s-1 \\ s-1,t \end{bmatrix}_{q,\pi} + q^{t-s-n}\begin{bmatrix} n+s-1 \\ s,t-1 \end{bmatrix}_{q,\pi}\right).$$

Moving the sum inside the parentheses produces three terms which we simplify separately, reindexing the second sum by replacing s by $s + 1$ and the third sum by replacing t by $t + 1$. We also use

$$\overline{a_n(r)} = \pi^{nr}\sum_{s+t=r}\pi^{\binom{t+1}{2}}(-q)^t\begin{bmatrix} n+s \\ s,t \end{bmatrix}_{q,\pi},$$

which follows by (3.10). In this way, the three terms become:

$$q^{-r}\sum_{s+t=r}\pi^{\binom{t+1}{2}}(-q)^t\begin{bmatrix} n+s-1 \\ s,t \end{bmatrix}_{q,\pi} = \pi^{(n-1)r}q^{-r}\,\overline{a_{n-1}(r)},$$

$$(\pi q)^n\sum_{s+t=r-1}\pi^{\binom{t}{2}}(-q)^{-t}\begin{bmatrix} n+s \\ s,t \end{bmatrix}_{q,\pi} = (\pi q)^n a_n(r-1),$$

$$-q^{1-n-r}\sum_{s+t=r-1}\pi^{\binom{t+1}{2}}(-q)^t\begin{bmatrix} n+s-1 \\ s,t \end{bmatrix}_{q,\pi} = -q^{1-n-r}\pi^{(n-1)(r-1)}\overline{a_{n-1}(r-1)}.$$

The sum of these three produces the right hand side of the identity we are proving. □

Corollary 3.2. *For $0 \leq r \leq n$, we have that*

$$q^{(n-r)r}\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi} = \sum_{\lambda\in\Lambda^+_{r\times(n-r)}}(\pi q^2)^{|\lambda|}.$$

Proof. This is an induction exercise using Lemma 3.1(1). □

Recall from the *General conventions* that $n\#r$ denotes $n + (n + 1) + \cdots + (n + r - 1)$.

Lemma 3.3. *For $m, n \in \mathbb{Z}$ and $r \geq 0$, we let*

$$b_{m,n}(r) := (\pi q^{-2})^{(n-r)\#r}\sum_{s=0}^{r-1}(\pi q^2)^{n-r+m(r-s-1)}q^{(m-n+r-1)(n-r+s+1)+(n-r)s}\begin{bmatrix} m+s \\ n-r+s+1 \end{bmatrix}_{q,\pi}\begin{bmatrix} n-r+s \\ s \end{bmatrix}_{q,\pi},$$

$$c_{m,n}(r) := (\pi q^{-2})^{(n-r)\#r}q^{(m-n+r)n+(n-r)r}\begin{bmatrix} m+r \\ n \end{bmatrix}_{q,\pi}\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi}.$$

Then we have that $c_{m,n}(r) = b_{m,n}(r) + b_{m,n}(r + 1)$ for any $m, n \in \mathbb{Z}$ and $r \geq 0$.

Proof. Proceed by induction on r. The base case $r = 0$ is easily checked. For the induction step, take $r > 0$. We have that

$$b_{m,n}(r) = (\pi q^{-2})^{n-m-1}b_{m,n-1}(r-1) + (\pi q^{-2})^{(n-r)\#r-n+r}q^{(m-n+r-1)n+(n-r)(r-1)}\begin{bmatrix} m+r-1 \\ n \end{bmatrix}_{q,\pi}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi},$$

$$b_{m,n}(r+1) = (\pi q^{-2})^{n-m-1} b_{m,n-1}(r) + (\pi q^{-2})^{(n-r-1)\#(r+1)-n+r+1} q^{(m-n+r)n+(n-r-1)r} \begin{bmatrix} m+r \\ n \end{bmatrix}_{q,\pi} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi}.$$

Note that $(n-r)\#(r-1) = (n-r)\#r - n + 1$ and $(n-r-1)\#(r+1) - n + r + 1 = (n-r)\#r$. Adding the above equations and using the induction hypothesis gives that

$$\begin{aligned} b_{m,n}(r) + b_{m,n}(r+1) = (\pi q^{-2})^{(n-r)\#r-m} q^{(m-n+r)(n-1)+(n-r)(r-1)} \begin{bmatrix} m+r-1 \\ n-1 \end{bmatrix}_{q,\pi} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi} \\ + (\pi q^{-2})^{(n-r)\#r-n+r} q^{(m-n+r-1)n+(n-r)(r-1)} \begin{bmatrix} m+r-1 \\ n \end{bmatrix}_{q,\pi} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi} \\ + (\pi q^{-2})^{(n-r)\#r} q^{(m-n+r)n+(n-r-1)r} \begin{bmatrix} m+r \\ n \end{bmatrix} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi}. \end{aligned}$$

Using the identity $(\pi q)^{m-n+r}\begin{bmatrix} m+r-1 \\ n-1 \end{bmatrix}_{q,\pi} + q^{-n}\begin{bmatrix} m+r-1 \\ n \end{bmatrix}_{q,\pi} = \begin{bmatrix} m+r \\ n \end{bmatrix}_{q,\pi}$ from Lemma 3.1(1), the first two terms combine into one leaving us with

$$(\pi q^{-2})^{(n-r)\#r-n+r} q^{(m-n+r)n+(n-r)(r-1)} \begin{bmatrix} m+r \\ n \end{bmatrix}_{q,\pi} \begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi} + (\pi q^{-2})^{(n-r)\#r} q^{(m-n+r)n+(n-r-1)r} \begin{bmatrix} m+r \\ n \end{bmatrix}_{q,\pi} \begin{bmatrix} n-1 \\ r \end{bmatrix}_{q,\pi}.$$

Then we use the identity $(\pi q)^{n-r}\begin{bmatrix} n-1 \\ r-1 \end{bmatrix}_{q,\pi} + q^{-r}\begin{bmatrix} n-1 \\ r \end{bmatrix} = \begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi}$ to see finally that this is equal to $c_{m,n}((r)$. □

Corollary 3.4. $q^{(n-r)r}\begin{bmatrix} n \\ r \end{bmatrix}_{q,\pi} = \sum_{s=0}^{r} (\pi q^2)^{(n-r)(r-s)} q^{(n-r-1)s} \begin{bmatrix} n-r+s-1 \\ s \end{bmatrix}_{q,\pi}$ *for* $0 \leq r \leq n$.

Proof. Take $m = n - r$ in Lemma 3.3. □

Let $U_{q,\pi}(\mathfrak{sl}_2)$ denote the locally unital $\mathbb{Q}(q)^\pi$-algebra with distinguished idempotents $\{1_k \mid k \in \mathbb{Z}\}$ and generators $E1_k = 1_{k+2}E, F1_k = 1_{k-2}F$ subject to the relations

$$EF1_k - \pi FE1_k = \overline{[k]}_{q,\pi} 1_k \tag{3.11}$$

for all $k \in \mathbb{Z}$. Note there is some flexibility in writing the idempotents 1_k—in any given monomial one just needs to include one idempotent somewhere in the word for the notation to be unambiguous. Let

$$E^{(d)}1_k = 1_{k+2d}E^{(d)} := \frac{E^d 1_k}{[d]^!_{q,\pi}}, \qquad 1_k F^{(d)} = F^{(d)} 1_{k+2d} := \frac{F^d 1_k}{[d]^!_{q,\pi}}, \tag{3.12}$$

$$\overline{E}^{(d)}1_k = 1_{k+2d}\overline{E}^{(d)} := \frac{E^d 1_k}{\overline{[d]}^!_{q,\pi}}, \qquad 1_k \overline{F}^{(d)} = \overline{F}^{(d)} 1_{k+2d} := \frac{F^d 1_k}{\overline{[d]}^!_{q,\pi}}. \tag{3.13}$$

By (3.10), we have that

$$\overline{E}^{(d)}1_k = \pi^{\binom{d}{2}} E^{(d)} 1_k, \qquad 1_k\overline{F}^{(d)} = \pi^{\binom{d}{2}} 1_k F^{(d)}. \tag{3.14}$$

There is an anti-linear involution

$$\varpi : U_{q,\pi}(\mathfrak{sl}_2) \to U_{q,\pi}(\mathfrak{sl}_2), \qquad 1_k \mapsto 1_{-k},\ E1_k \mapsto F1_{-k},\ F1_k \mapsto E1_{-k}. \tag{3.15}$$

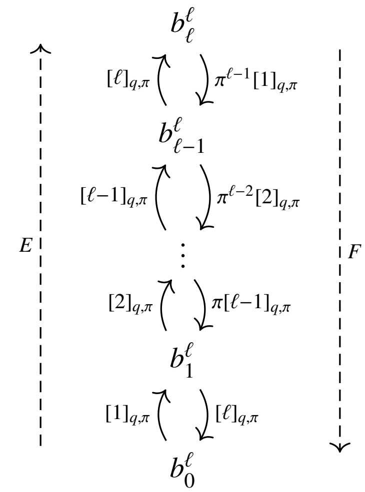
This sends $E^{(d)}1_k \mapsto \overline{F}^{(d)}1_{-k}$ and $F^{(d)}1_k \mapsto \overline{E}^{(d)}1_{-k}$. We warn the reader that this is different from the involution ω in [CHW].

By a $U_{q,\pi}(\mathfrak{sl}_2)$-module we mean a locally unital left module $V = \bigoplus_{k\in\mathbb{Z}} 1_k V$. We call $1_k V$ the *k-weight space* of V. We say that V is *integrable* if any weight vector $v \in 1_k V$ for $k \in \mathbb{Z}$ is annihilated by $E^n 1_k$ and $F^n 1_k$ for $n \gg 0$ (depending on v). For $\ell \in \mathbb{N}$—a dominant weight for $\mathfrak{sl}_2$—there is a $U_{q,\pi}(\mathfrak{sl}_2)$-module $V(-\ell)$ which is free as a $\mathbb{Q}(q)^\pi$-module with basis $\{b_n^\ell \mid 0 \leq n \leq \ell\}$ such that

- b_n^ℓ is of weight $2n - \ell$, i.e., $1_{2n-\ell} b_n^\ell = b_n^\ell$;

- b^ℓ_ℓ is a highest weight vector and b^ℓ_0 is a lowest weight vector, i.e., $Eb^\ell_\ell = Fb^\ell_0 = 0$;
- for $0 \le n < \ell$, we have that $Eb^\ell_n = [n+1]_{q,\pi} b^\ell_{n+1}$ and $Fb^\ell_{n+1} = \pi^n [\ell - n]_{q,\pi} b^\ell_n$.

We visualize the action with the familiar $\mathfrak{sl}_2$-type picture showing how the operators E and F raise and lower basis vectors to multiples of basis vectors:



(3.16)

For $0 \le n \le \ell - d$, we have that

$$E^{(d)} b^\ell_n = \begin{bmatrix} n+d \\ d \end{bmatrix}_{q,\pi} b^\ell_{n+d}, \qquad F^{(d)} b^\ell_{n+d} = \pi^{\binom{d}{2}+nd} \begin{bmatrix} \ell - n \\ d \end{bmatrix}_{q,\pi} b^\ell_n. \tag{3.17}$$

There is an anti-linear involution

$$\varpi : V(-\ell) \to V(-\ell), \qquad b^\ell_n \mapsto \pi^{n(\ell-n)} b^\ell_{\ell-n}. \tag{3.18}$$

This has the key property that

$$\varpi(uv) = \varpi(u)\varpi(v) \tag{3.19}$$

for all $u \in U_{q,\pi}(\mathfrak{sl}_2)$, $v \in V(-\ell)$.

Let $V_\pm(-\ell) := \frac{1}{2}(1 \pm \pi)V(-\ell)$. These are irreducible $U_{q,\pi}(\mathfrak{sl}_2)$-modules generated by the highest weight vectors $\frac{1}{2}(1 \pm \pi)b^\ell_\ell$ of weight ℓ on which π acts by the scalar ± 1. In particular, these modules are not isomorphic for different ℓ or different choices of sign.

Theorem 3.5 ([CHW, Cor. 3.3.3])**.** *Any integrable $U_{q,\pi}(\mathfrak{sl}_2)$-module decomposes as a direct sum of the modules $V_\pm(-\ell)$ for $\ell \in \mathbb{N}$.*

Now we can prove the main result of the section.

Theorem 3.6. *Let V be an integrable $U_{q,\pi}(\mathfrak{sl}_2)$-module. There is a linear automorphism $T : V \xrightarrow{\sim} V$ sending $1_{-k}V$ to 1_kV for each $k \in \mathbb{Z}$ such that*

$$T(v) = \sum_{d \ge \max(0,-k)} (-q)^d E^{(k+d)} F^{(d)} v$$

on a vector $v \in 1_{-k}V$. The inverse is given explicitly by the formula

$$T^{-1}(v_k) = \sum_{d \ge \max(0,-k)} (-q)^{-d} \overline{F}^{(k+d)} \overline{E}^{(d)} v$$

on a vector $v \in 1_kV$.

Proof. In view of Theorem 3.5, it suffices to check this when $V = V(-\ell)$ for $\ell \in \mathbb{N}$. Take $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod 2$ and set $n := \frac{\ell+k}{2}$ and $n' := \frac{\ell-k}{2}$, so $n' + n = \ell$ and $n' - n = k$. The space $1_{-k}V(-\ell)$ is spanned by b^ℓ_n and $1_kV(-\ell)$ is spanned by $b^\ell_{n'}$. Since $F^{(d)}b^\ell_n = 0$ for $d > n$, we have by the definition in the statement of the theorem that $T(b^\ell_n) = ub^\ell_n$ where

$$u := \sum_{d=\max(0,-k)}^{n} (-q)^d E^{(k+d)} F^{(d)} \in U_{q,\pi}(\mathfrak{sl}_2).$$

In the next paragraph, we show that

$$ub^\ell_n = (-1)^n \pi^{\binom{n}{2}+nn'} q^{n+nn'} b^\ell_{n'}. \tag{3.20}$$

Assuming this, the proof can be completed as follows. Applying ϖ to (3.20) using (3.18) and (3.19), we also have that

$$\varpi(u) b^\ell_{n'} = (-1)^n \pi^{\binom{n}{2}+nn'} q^{-n-nn'} b^\ell_n. \tag{3.21}$$

From (3.20) and (3.21), it follows that $\varpi(u)ub^\ell_n = b^\ell_n$ and $u\varpi(u)b^\ell_{n'} = b^\ell_{n'}$. Hence, $T : 1_{-k}V(-\ell) \to 1_kV(-\ell)$ is an isomorphism with inverse T^{-1} defined by multiplication by $\varpi(u)$. Finally we observe that $\overline{E}^{(d)} b^\ell_{n'} = 0$ for $d > n$ so

$$\varpi(u) b^\ell_{n'} = \sum_{d \geq \max(0,-k)} (-q)^{-d} \overline{F}^{(k+d)} \overline{E}^{(d)} b^\ell_{n'},$$

which agrees with the formula for $T^{-1}(b^\ell_{n'})$ in the statement of the theorem.

It remains to prove (3.20). First we make some elementary computations using (3.17):

$$\begin{aligned}
ub^\ell_n &= \sum_{d=\max(0,n-n')}^{n} (-q)^d E^{(n'-n+d)} F^{(d)} b^\ell_n \\
&= \sum_{d=\max(0,n-n')}^{n} \pi^{\binom{d}{2}+(n-d)d} (-q)^d \begin{bmatrix} n'+d \\ n' \end{bmatrix}_{q,\pi} E^{(n'-n+d)} b^\ell_{n-d} \\
&= \sum_{d=\max(0,n-n')}^{n} \pi^{\binom{d}{2}+(n-d)d} (-q)^d \begin{bmatrix} n'+d \\ n' \end{bmatrix}_{q,\pi} \begin{bmatrix} n' \\ n-d \end{bmatrix}_{q,\pi} b^\ell_{n'} \\
&= \sum_{d=0}^{n} \pi^{\binom{d}{2}+(n-d)d} (-q)^d \begin{bmatrix} n'+d \\ n' \end{bmatrix}_{q,\pi} \begin{bmatrix} n' \\ n-d \end{bmatrix}_{q,\pi} b^\ell_{n'},
\end{aligned}$$

noting in the last step that $\begin{bmatrix} n' \\ n-d \end{bmatrix}_{q,\pi} = 0$ if $d < n - n'$ so that we can remove the restriction on the summation. Then we switch to another variable $s := n - d$ and sum instead over $d, s \geq 0$ with $d + s = n$ to get that

$$ub^\ell_n = \sum_{d+s=n} \pi^{\binom{n-s}{2}+s(n-s)} (-q)^{n-s} \begin{bmatrix} n'+d \\ n' \end{bmatrix}_{q,\pi} \begin{bmatrix} n' \\ s \end{bmatrix}_{q,\pi} b^\ell_{n'} = \pi^{\binom{n}{2}} (-q)^n \sum_{d+s=n} \pi^{\binom{s}{2}} (-q)^{-s} \begin{bmatrix} n'+d \\ d,\ s \end{bmatrix}_{q,\pi} v_{n',n}.$$

Now an application of Lemma 3.1(3) completes the proof of (3.20). □

Remark 3.7. The specialization of $U_{q,\pi}(\mathfrak{sl}_2)$ at $\pi = 1$, that is, the algebra $U_q(\mathfrak{sl}_2) := U_{q,\pi}(\mathfrak{sl}_2) \otimes_{\mathbb{Q}(q)^\pi} \mathbb{Q}(q)$ where $\mathbb{Q}(q)$ is viewed here as a $\mathbb{Q}(q)^\pi$-module so that π acts as 1, is the usual quantized enveloping algebra of SL_2. Theorem 3.6 is well known in this case. The specialization at $\pi = -1$ is the quantized enveloping algebra $U_q(\mathfrak{osp}_{1|2})$ of Clark and Wang [CW].

The algebra $U_{q,\pi}(\mathfrak{sl}_2)$ has a $\mathbb{Z}[q, q^{-1}]^\pi$-form we denote by $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$, namely, the $\mathbb{Z}[q, q^{-1}]^\pi$-algebra generated by the divided powers $E^{(r)}1_k, F^{(r)}1_k$ for $r \geq 1, k \in \mathbb{Z}$. The module $V(-\ell)$ is also defined over

$\mathbb{Z}[q,q^{-1}]^\pi$, with its integral form $\mathbf{V}(-\ell)$ being the $\mathbb{Z}[q,q^{-1}]^\pi$-submodule of $V(-\ell)$ generated by the basis vectors chosen above.

Theorem 3.8 ([C, Lem. 3.5])**.** *The algebra $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ is free as a $\mathbb{Z}[q,q^{-1}]^\pi$-module with basis given by the monomials $\left\{F^{(r)}E^{(s)}1_k \,\middle|\, r,s\geq 0, k\in\mathbb{Z}\right\}$.*

We say that a $\mathbb{Z}[q,q^{-1}]^\pi$-free $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$-module $\mathbf{V}$ is *integrable* if any weight vector $v\in 1_k\mathbf{V}$ is annihilated by $E^{(n)}1_k$ and $F^{(n)}1_k$ for $n\gg 0$ (depending on v). Equivalently, the $\mathbb{Q}(q)^\pi$-free $U_{q,\pi}(\mathfrak{sl}_2)$-module $\mathbb{Q}(q)^\pi\otimes_{\mathbb{Z}[q,q^{-1}]^\pi}\mathbf{V}$ is integrable in the earlier sense. It is clear that the automorphism T from Theorem 3.6 descends to an automorphism of any integrable $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$-module that is free as a $\mathbb{Z}[q,q^{-1}]^\pi$-module.

4. Odd symmetric functions

This section is largely an exposition of results from [EK, EKL], and assumes the reader is already familiar with the classical theory of symmetric functions as in [Mac]. However, we have made one substantial modification to the setup: instead of the elementary odd symmetric functions denoted ε_r in [EKL], we usually prefer to work with the renormalized odd elementary symmetric functions $e_r := (-1)^{\binom{r}{2}}\varepsilon_r$. We will explain the implications of this more thoroughly as we proceed. We also warn the reader that in [EK] the notation e_r is used for the same thing as the element denoted ε_r in [EKL], so the e_r of [EK] is *not* the one here.

The algebra *OSym* of *odd symmetric functions* is the graded superalgebra over the ground field $\mathbb{F}$ generated by elements h_r ($r\geq 1$) of degree $2r$ and parity r (mod 2) subject to the relations of [EK, Cor. 2.13]:

$$h_rh_s = h_sh_r \qquad \text{if } r\equiv s \pmod 2 \tag{4.1}$$

$$h_rh_s + (-1)^r h_sh_r = (-1)^r h_{r+1}h_{s-1} + h_{s-1}h_{r+1} \qquad \text{if } r\not\equiv s \pmod 2 \tag{4.2}$$

for $r\geq 0, s\geq 1$, interpreting h_0 as 1. We also define elements $e_r (r\geq 0)$ so that the *infinite Grassmannian relation*

$$\sum_{s=0}^{r}(-1)^s e_sh_{r-s} = \delta_{r,0} \tag{4.3}$$

holds for all $r\geq 0$. The element h_r is exactly the rth *complete odd symmetric function* from [EK]. We call e_r the rth *elementary odd symmetric function*.

In [EK, Cor. 2.13, Prop. 2.10], it is shown that their elements $\{\varepsilon_r \mid r\geq 1\}$ generate *OSym* subject to exactly the same relations as the h_r. Noting that $\binom{r}{2}+\binom{s}{2}\equiv\binom{r+1}{2}+\binom{s-1}{2}$ (mod 2) when $r\not\equiv s$ (mod 2), this means that our elements $\{e_r \mid r\geq 1\}$ also generate *OSym* subject to the same relations

$$e_re_s = e_se_r \qquad \text{if } r\equiv s \pmod 2 \tag{4.4}$$

$$e_re_s + (-1)^r e_se_r = (-1)^r e_{r+1}e_{s-1} + e_{s-1}e_{r+1} \qquad \text{if } r\not\equiv s \pmod 2 \tag{4.5}$$

for $r\geq 0, s\geq 1$, again interpreting e_0 as 1. There are also mixed relations, which are derived in [EK, Prop. 2.11]. These look slightly different with our modified odd elementary symmetric functions:

$$e_rh_s = h_se_r \qquad \text{if } r\equiv s \pmod 2 \tag{4.6}$$

$$e_rh_s + (-1)^r h_se_r = e_{r+1}h_{s-1} + (-1)^r h_{s-1}e_{r+1} \qquad \text{if } r\not\equiv s \pmod 2 \tag{4.7}$$

for $r\geq 0, s\geq 1$. The following is equivalent to [EK, (2.6)]:

$$e_r = \det\left(h_{i-j+1}\right)_{i,j=1,\dots,r} \tag{4.8}$$

where det should be interpreted as the usual Laplace expansion of determinant ordering monomials in the same way as the elements appear in the rows of the matrix. For example:

$$e_0 = 1, \qquad e_1 = h_1, \qquad e_2 = h_1^2 - h_2, \qquad e_3 = h_1^3 - h_1h_2 - h_2h_1 + h_3 = h_1^3 - h_3.$$

In fact, (4.8) is a formal consequence of the infinite Grassmannian relation which does not require any commutativity. The same thing holds for ordinary symmetric functions, indeed, (4.3) is the same relation as for the algebra *Sym* of symmetric functions from [Mac, (I.2.6')], and (4.8) is [Mac, Ex. I.2.8].

It is often useful to work with the generating functions

$$e(t) = \sum_{r\geq 0}(-1)^r e_r t^{-r}, \qquad h(t) = \sum_{r\geq 0} h_r t^{-r}, \tag{4.9}$$

which are elements of $OSym[\![t^{-1}]\!]$ for a formal even variable t. Now the infinite Grassmannian relation becomes the first of the following:

$$e(t)h(t) = 1, \qquad h(t)e(t) = 1. \tag{4.10}$$

Since $h(t)$ is invertible in the formal power series ring, its left inverse $e(t)$ is also its right inverse, proving the second equality. In other words, we have that

$$\sum_{s=0}^{r}(-1)^s h_s e_{r-s} = \delta_{r,0} \tag{4.11}$$

for all $r \geq 0$. Consequently,

$$h_r = \det\left(e_{i-j+1}\right)_{i,j=1,\dots,r}. \tag{4.12}$$

The evident symmetry between complete and elementary odd symmetric functions is best expressed in terms of the algebra automorphism

$$\psi : OSym \to OSym, \qquad h_r \mapsto (-1)^r e_r. \tag{4.13}$$

Extending ψ trivially to $OSym[\![t^{-1}]\!]$, we have that $\psi(h(t)) = e(t)$. As $e(t)$ is the two-sided inverse of $h(t)$, it follows that $\psi(e(t)) = \psi^2(h(t))$ is the two-sided inverse of $\psi(h(t)) = e(t)$. Hence, $\psi^2(h(t)) = h(t)$, and we have shown that ψ is an involution. So we also have that

$$\psi(e_r) = (-1)^r h_r. \tag{4.14}$$

For $\lambda \in \Lambda^+$, we let

$$h_\lambda := h_{\lambda_1} h_{\lambda_2} \cdots, \qquad e_\lambda := e_{\lambda_1} e_{\lambda_2} \cdots. \tag{4.15}$$

Similarly, we define h_α and e_α for a composition $\alpha \in \Lambda(k,n)$. As in [EKL, (2.25)], the relations (4.1) and (4.2) imply for $r < s$ that

$$h_r h_s = \begin{cases} h_s h_r & \text{if } r \text{ and } s \text{ have the same parity} \\ h_s h_r + 2\sum_{t=1}^{r}(-1)^{\binom{t}{2}} h_{s+t} h_{r-t} & \text{if } r \text{ is even and } s \text{ is odd} \\ -h_s h_r - 2\sum_{t=1}^{r}(-1)^{\binom{t+1}{2}} h_{s+t} h_{r-t} & \text{if } r \text{ is odd and } s \text{ is even.} \end{cases} \tag{4.16}$$

Similarly, by (4.4) and (4.5), we have for $r < s$ that

$$e_r e_s = \begin{cases} e_s e_r & \text{if } r \text{ and } s \text{ have the same parity} \\ e_s e_r + 2\sum_{t=1}^{r}(-1)^{\binom{t}{2}} e_{s+t} e_{r-t} & \text{if } r \text{ is even and } s \text{ is odd} \\ -e_s e_r - 2\sum_{t=1}^{r}(-1)^{\binom{t+1}{2}} e_{s+t} e_{r-t} & \text{if } r \text{ is odd and } s \text{ is even.} \end{cases} \tag{4.17}$$

Consequently, any monomial h_α or e_α for $\alpha \in \Lambda(k,n)$ can be rearranged into decreasing order modulo a linear combination of lexicographically greater monomials of the same degree. This proves the easy spanning part of the next theorem.

Theorem 4.1 ([EK, Cor. 2.12])**.** *The set $\{h_\lambda \mid \lambda \in \Lambda^+\}$ is a linear basis for OSym. Equivalently, applying ψ, the set $\{e_\lambda \mid \lambda \in \Lambda^+\}$ is a basis.*

There is a comultiplication $\Delta^- : OSym \to OSym \otimes OSym$ making $OSym$ into a graded Hopf superalgebra such that

$$\Delta^-(h_r) = \sum_{s=0}^{r} h_s \otimes h_{r-s} \tag{4.18}$$

for all $r \geq 0$. This can be written more concisely in terms of generating functions as

$$\Delta^-(h(t)) = h(t) \otimes h(t). \tag{4.19}$$

By [EK, Prop. 2.17], the antipode $S^- : OSym \to OSym$, which we remind the reader is both a superalgebra anti-automorphism and a cosuperalgebra anti-automorphism, satisfies $S^-(h_r) = (-1)^r e_r$ or, equivalently, $S^-(h(t)) = e(t)$.

So far, apart from a lack of commutativity, there have been many similarities between $OSym$ and the ordinary theory for Sym, but now some more significant differences come into view. Unlike in the ordinary theory, S^- is *not* an involution; indeed, we have that $S^-(h_1) = -h_1$ and $S^-(h_2) = h_1^2 - h_2$, hence, $S^{-n}(h_2) = (-1)^n(h_2 - nh_1^2)$ for any $n \geq 0$. Another important point is that $OSym$ is *not* a cocommutative cosuperalgebra, e.g., $\Delta^-(h_2) = h_2 \otimes 1 + h_1 \otimes h_1 + 1 \otimes h_2$ is not invariant under the braiding $B_{OSym,OSym}$. So the opposite comultiplication

$$\Delta^+ := B_{OSym,OSym} \circ \Delta^- : OSym \to OSym \otimes OSym \tag{4.20}$$

gives a second graded Hopf superalgebra structure on $OSym$ (the multiplication is the same as before). Remembering that our e_r is $(-1)^{\binom{r}{2}}\varepsilon_r$, [EK, Prop. 2.5] implies that

$$\Delta^+(e_r) = \sum_{s=0}^{r} e_s \otimes e_{r-s} \tag{4.21}$$

or, equivalently,

$$\Delta^+(e(t)) = e(t) \otimes e(t). \tag{4.22}$$

It follows that

$$\Delta^- \circ \psi = (\psi \otimes \psi) \circ \Delta^+, \qquad \Delta^+ \circ \psi = (\psi \otimes \psi) \circ \Delta^-, \tag{4.23}$$

because both sides of the left hand equation agree on $e(t)$ and both sides of the right hand equation agree on $h(t)$. This shows that ψ is a cosuperalgebra *anti*-involution. The antipode S^+ for the second Hopf superalgebra structure is the inverse of S^-, so it takes e_r to $(-1)^r h_r$.

We will use two more useful symmetries

$$\gamma : OSym \to OSym, \qquad e_r \mapsto (-1)^{\binom{r}{2}} e_r, \tag{4.24}$$

$$* : OSym \to OSym, \qquad e_r \mapsto e_r, \tag{4.25}$$

the first of which is an algebra involution, and the second is a superalgebra *anti*-involution. It is a routine check using (4.4) and (4.5) to see that these make sense. Note in particular that γ takes our e_r to the ε_r of [EK, EKL]. The symmetries γ and $*$ commute. Neither γ nor $*$ commutes with ψ, but it is still true that $* \circ \gamma$ commutes with ψ; see Lemma 4.10 below for the proof of this. We have that

$$\Delta^- \circ \gamma = (\gamma \otimes \gamma) \circ \Delta^+, \qquad \Delta^+ \circ \gamma = (\gamma \otimes \gamma) \circ \Delta^-, \tag{4.26}$$

$$\Delta^+ \circ * = (* \otimes *) \circ \Delta^+, \qquad \Delta^- \circ * = (* \otimes *) \circ \Delta^-. \tag{4.27}$$

To justify these, it suffices to check the left hand equations, then the right hand ones follow because $B_{OSym,OSym} \circ (\gamma \otimes \gamma) = (\gamma \otimes \gamma) \circ B_{OSym,OSym}$ and $B_{OSym,OSym} \circ (* \otimes *) = (* \otimes *) \circ B_{OSym,OSym}$. To check the left hand equation from (4.26), one instead shows that $B_{OSym,OSym} \circ \Delta^+ \circ \gamma = (\gamma \otimes \gamma) \circ \Delta^+$ by checking that both sides do the same thing on e_r. The left hand equation form (4.27) holds because both sides take $e(t)$ to $e(t) \otimes e(t)$.

Lemma 4.2. *For $\lambda \in \Lambda^+$, $\gamma(e_\lambda)^* = (-1)^{dN(\lambda)+dE(\lambda)} e_\lambda$+(a $\mathbb{Z}$-linear combination of e_μ for $\mu >_{\mathrm{lex}} \lambda$).*

Proof. Let $k := \mathrm{ht}(\lambda)$. By the definitions, we have that $\gamma(e_\lambda)^* = (-1)^{\binom{|\lambda|}{2}} e_{\lambda_k} \cdots e_{\lambda_1}$. Then we use (4.17) to rewrite $e_{\lambda_k} \cdots e_{\lambda_1}$ as $\pm e_\lambda$ plus a sum of lexicographically higher e_μ. It remains to compute the sign. We get a sign change each time we commute e_{λ_j} with e_{λ_i} for $1 \le i < j \le k$ such that λ_j is odd and λ_i is even. So the overall sign is $(-1)^{\binom{|\lambda|}{2} + \sum_{1 \le i < j \le k} (\lambda_i - 1)\lambda_j}$. This simplifies to $(-1)^{dN(\lambda)+dE(\lambda)}$. □

Remark 4.3. In [EK, Sec. 2.3], symmetries denoted ψ_1, ψ_2 and ψ_3 are introduced. These are related to our ψ, γ and $*$ by $\psi_1 = \gamma \circ \mathrm{p} \circ \psi$ (because the latter takes h_r to $\varepsilon_r = (-1)^{\binom{r}{2}} e_r$), $\psi_2 = \psi \circ \mathrm{p} \circ \gamma \circ \psi$ (because the latter sends h_r to $(-1)^{\binom{r+1}{2}} h_r$), and $\psi_3 = \psi \circ * \circ \psi$ (because the latter is a superalgebra anti-involution taking h_r to h_r). We emphasize that our $\psi = \psi_1 \circ \psi_2$ is an involution, whereas ψ_1 is not.

In [EK], the definition of *OSym* is motivated by the definition of a non-degenerate symmetric[1] bilinear form $(\cdot,\cdot)^- : OSym \otimes OSym \to \mathbb{F}$. Extending the bilinear form $(\cdot,\cdot)^-$ on *OSym* to $OSym \otimes OSym$ so that $(a_1 \otimes a_2, b_1 \otimes b_2)^- = (a_1, b_1)^-(a_2, b_2)^-$, the form is characterized uniquely by the following properties:

$$(h_r, h_s)^- = \delta_{r,s}, \qquad (ab, c)^- = (a \otimes b, \Delta^-(c))^- \tag{4.28}$$

for $r, s \ge 0, a, b, c \in OSym$. For symmetry's sake, one can also consider a form $(\cdot,\cdot)^+$ which is defined in a similar way so that

$$(e_r, e_s)^+ = \delta_{r,s}, \qquad (ab, c)^+ = (a \otimes b, \Delta^+(c))^+ \tag{4.29}$$

for $r, s \ge 0, a, b, c \in OSym$. The forms $(\cdot,\cdot)_\pm$ are related by the first of the following properties:

$$(\psi(a), \psi(b))^\pm = (a, b)^\mp, \qquad (\gamma(a^*), \gamma(b^*))^\pm = (a, b)^\pm \tag{4.30}$$

for any $a, b \in OSym$. This and the second property are both checked by induction on degree, using (4.13) and (4.23) to (4.27). The first of the next two properties follows from [EK, (2.10)], then the second follows by applying ψ:

$$(e_r, e_s)^- = (-1)^{\binom{r}{2}} \delta_{r,s}, \qquad (h_r, h_s)^+ = (-1)^{\binom{r}{2}} \delta_{r,s}. \tag{4.31}$$

The following allows $(h_\lambda, e_\mu)^\pm$ to be computed:

$$(h_\lambda, e_r)^- = (-1)^{\binom{r}{2}} \delta_{\lambda,(1^r)}, \qquad (h_r, e_\lambda)^+ = (-1)^{\binom{r}{2}} \delta_{\lambda,(1^r)}. \tag{4.32}$$

The first equality here is established in [EK, Prop. 2.5], again remembering that our normalization of the elements e_r is different; then the second follows on applying ψ. In particular, (4.32) can be used to show that

$$(h_\lambda, e_\mu)^- = \begin{cases} (-1)^{NE(\lambda)+dN(\lambda)} & \text{if } \lambda = \mu^t \\ 0 & \text{if } \lambda \not\le_{\mathrm{lex}} \mu^t \end{cases} \qquad (h_\lambda, e_\mu)^+ = \begin{cases} (-1)^{NE(\mu)+dN(\mu)} & \text{if } \lambda = \mu^t \\ 0 & \text{if } \lambda \not\le_{\mathrm{lex}} \mu^t \end{cases} \tag{4.33}$$

for $\lambda, \mu \in \Lambda^+$; see [EK, Prop. 2.14] for the first equality. This "semi-orthogonality" is used to complete the proof of Theorem 4.1 in [EK].

Recall that $OPol_n$ is the algebra of odd polynomials from (2.3). Define a superalgebra involution γ_n and a superalgebra anti-involution $*$ of $OPol_n$ by

$$\gamma_n : OPol_n \to OPol_n, \qquad x_i \mapsto x_{n+1-i}, \tag{4.34}$$

$$* : OPol_n \to OPol_n, \qquad x_i \mapsto x_i. \tag{4.35}$$

The following theorem gives another way to motivate the definition of *OSym*, as was explained originally in [EKL].

[1] We really do mean symmetric rather than supersymmetric here!

Theorem 4.4. *There is a graded superalgebra homomorphism $\pi_n : OSym \to OPol_n$ taking e_r and h_r to the polynomials*

$$e_r(x_1,\dots,x_n) := \sum_{1\le i_1<\dots<i_r\le n} x_{i_1}\cdots x_{i_r} \tag{4.36}$$

$$h_r(x_n,\dots,x_1) := \sum_{n\ge i_r\ge\dots\ge i_1\ge 1} x_{i_r}\cdots x_{i_1}, \tag{4.37}$$

respectively. Moreover, π_n intertwines the involution γ of $OSym$ with the algebra involution γ_n of $OPol_n$ from (4.34), and it intertwines the anti-involution $*$ of $OSym$ with the superalgebra anti-involution $*$ of $OPol_n$ from (4.35). Finally, if $n = a + b$, the diagram

$$\begin{array}{ccc} OSym & \xrightarrow{\Delta^+} & OSym\otimes OSym \\ {\scriptstyle\pi_n}\downarrow & & \downarrow{\scriptstyle\pi_a\otimes\pi_b} \\ OPol_n & = & OPol_a\otimes OPol_b \end{array} \tag{4.38}$$

commutes, where the identification at the bottom is as explained after (2.3).

Proof. Note our x_i is the variable $\tilde{x}_i = (-1)^{i-1}x_i$ in the notation of [EKL], hence, our $e_r(x_1,\dots,x_n)$ is the polynomial denoted $\varepsilon_r(x_1,\dots,x_n)$ in [EKL]. With this in mind, [EKL, Lem. 2.3] checks that the polynomials $e_r(x_1,\dots,x_n) \in OPol_n$ satisfy the defining relations of $OSym$ from (4.4) and (4.5). Hence, there is a unique homomorphism $\pi_n : OSym \to OPol_n$ such that $\pi_n(e_r) = e_r(x_1,\dots,x_n)$ for all $r \ge 0$.

The involution γ of $OSym$ takes e_r to $(-1)^{\binom{r}{2}}e_r$. The involution γ_n of $OPol_n$ takes $e_r(x_1,\dots,x_n)$ to

$$e_r(x_n,\dots,x_1) = \sum_{n\ge i_r>\dots>i_1\ge 1} x_{i_n}\cdots x_{i_2}x_{i_1}.$$

Rearranging these monomials into increasing order of x_i produces a sign of $(-1)^{\binom{r}{2}}$. Hence, γ_n takes $e_r(x_1,\dots,x_n)$ to $(-1)^{\binom{r}{2}}e_r(x_1,\dots,x_n)$. This checks that $\pi_n\circ\gamma = \gamma_n\circ\pi_n$. Similarly, we see that $\pi_n\circ * = *\circ\pi_n$ because $*$ on $OSym$ fixes e_r and $*$ on $OPol_n$ fixes $e_r(x_1,\dots,x_n)$.

In [EKL, Lem. 2.8], again using that our x_i is $\tilde{x}_i$ in [EKL], it is checked that the polynomials

$$h_r(x_1,\dots,x_n) = \sum_{1\le i_1\le\dots\le i_r\le n} x_{i_1}x_{i_2}\cdots x_{i_n}$$

satisfy $\sum_{s=0}^r (-1)^{\binom{s+1}{2}} e_s(x_1,\dots,x_n)h_{r-s}(x_1,\dots,x_n) = \delta_{r,s}$ for all $r \ge 0$. Applying γ_n, it follows that $\sum_{s=0}^r (-1)^{\binom{s+1}{2}} e_s(x_n,\dots,x_1)h_{r-s}(x_n,\dots,x_1) = \delta_{r,s}$. We already know that $e_s(x_n,\dots,x_1) = (-1)^{\binom{s}{2}}\pi_n(e_s)$, so this shows that $\sum_{s=0}^r (-1)^s\pi_n(e_s)h_{r-s}(x_n,\dots,x_1) = \delta_{r,s}$. Comparing with (4.3), this proves that $\pi_n(h_r) = h_r(x_n,\dots,x_1)$.

Finally, to see that (4.38) commutes, use (4.21) and the definition of $e_r(x_1,\dots,x_n)$. □

Now we *define $OSym_n$*, the algebra of *odd symmetric polynomials*, to be the subalgebra of $OPol_n$ that is the image of the homomorphism π_n from Theorem 4.4. For any $a \in OSym$, we use the notation $a^{(n)}$ to denote its canonical image in $OSym_n$. Note from (4.36) that $e_r^{(n)} = 0$ for $r > n$. For $\lambda \in \Lambda^+$, we have that

$$e_{\lambda^t}^{(n)} = \begin{cases} (-1)^{NE(\lambda)}x^\lambda + (\text{a }\mathbb{Z}\text{-linear combination of } x^\kappa \text{ for } \kappa\in\mathbb{N}^n \text{ with } \kappa<\lambda) & \text{if } \mathrm{ht}(\lambda)\le n \\ 0 & \text{if } \mathrm{ht}(\lambda)>n, \end{cases} \tag{4.39}$$

where $x^\kappa = x_1^{\kappa_1}\cdots x_n^{\kappa_n}$ and x^λ is defined similarly, identifying $\lambda\in\Lambda^+$ with $\mathrm{ht}(\lambda)\le n$ with $(\lambda_1,\dots,\lambda_n)\in\mathbb{N}^n$. This is easily checked from the definition and gives the clearest explanation for the sign $NE(\lambda)$.

Theorem 4.5. *The set $\left\{e_{\lambda^t}^{(n)} \;\middle|\; \lambda\in\Lambda_n^+\right\}$ is a basis for $OSym_n$.*

Proof. The set $\left\{e^{(n)}_{\lambda^t} \;\middle|\; \lambda \in \Lambda^+_n\right\}$ spans $OSym_n$ since the images of all other e_μ in the basis for $OSym$ from Theorem 4.1 are zero. Linear independence is clear from (4.39). ◻

Corollary 4.6. *The quotient maps $\pi_n : OSym \twoheadrightarrow OSym_n$ induce an isomorphism*

$$OSym \xrightarrow{\sim} \varprojlim OSym_n,$$

where on the right we have the inverse limit of the inverse system $\cdots \twoheadrightarrow OSym_1 \twoheadrightarrow OSym_0$ taken in the category of graded superalgebras, with the map $OSym_{n+1} \twoheadrightarrow OSym_n$ taking $e^{(n+1)}_r$ to $e^{(n)}_r$. Moreover, $OSym_n$ may be identified with the quotient $OSym / \langle e_r \mid r > n\rangle$.

Corollary 4.7 ([EKL, (2.20)])**.** $\dim_{q,\pi} OSym_n = \displaystyle\prod_{r=1}^{n} \frac{1}{1-(\pi q^2)^r}$.

Proof. Theorem 4.5 shows that $OSym_n$ has the same graded superdimension as a commutative polynomial algebra with generators $x_1, \dots, x_n$ such that x_r is of degree $2r$ and parity $r \pmod 2$. ◻

Corollary 4.8. $\dim_{q,\pi} OPol_n = \dim_{q,\pi} OSym_n \times q^{\binom{n}{2}}[n]^!_{q,\pi} = \dim_{q,\pi} OSym_n \times \displaystyle\sum_{w\in S_n} (\pi q^2)^{\ell(w)}$.

Proof. The second equality follows from the first by (3.3). To obtain the first equality, we use Corollary 4.7 to see that

$$\begin{aligned}\dim_{q,\pi} OSym_n \times q^{\binom{n}{2}}[n]^!_{q,\pi} &= q^{\binom{n}{2}}[n]^!_{q,\pi} \prod_{r=1}^{n} \frac{1}{1-(\pi q^2)^r} = \frac{q^{\binom{n}{2}}[n]^!_{q,\pi}}{(1-\pi q^2)^n} \prod_{r=1}^{n} \frac{1-\pi q^2}{1-(\pi q^2)^r}\\ &= \frac{[n]^!_{q,\pi}}{(1-\pi q^2)^n} \prod_{r=1}^{n} \frac{\pi q - q^{-1}}{(\pi q)^r - q^{-r}} = \frac{1}{(1-\pi q^2)^n} \overset{(2.4)}{=} \dim_{q,\pi} OPol_n.\end{aligned}$$

◻

The next technical lemma about relations in $OSym_{n+1}$ will be needed at a key place later on; see Lemma 11.2 which is used to prove Theorem 11.3.

Lemma 4.9. *For any $0 \le p, q, k \le n$ with $p + q \le n$, we have that*

$$\sum_{m=1}^{n-p-q} (-1)^{(n+k)(m+p+k)} e^{(n+1)}_{n-k+m} e^{(n+1)}_{n-p-q-m} = \sum_{m=1}^{n-p-q} (-1)^{(n+k)(m+q)} e^{(n+1)}_{n-p-q-m} e^{(n+1)}_{n-k+m}$$

in $OSym_{n+1}$.

Proof. If $p + q + k$ is even then $(-1)^{(n+k)(m+p+k)} = (-1)^{(n+k)(m+q)}$, and $e^{(n+1)}_{n-k+m}$ commutes with $e^{(n+1)}_{n-p-q-m}$ by the relation (4.4). The result obviously follows in this situation since corresponding terms on each side are equal.

Next, assume that $p + q + k$ is odd and $n \equiv p + q \pmod 2$, in which case $n + k$ is odd. There are an even number of terms in the summations in the identity we are trying to prove. It suffices to show that sums of consecutive pairs of terms on each side are equal, i.e.,

$$\begin{aligned}&(-1)^{(n+k)(m+p+k)} e^{(n+1)}_{n-k+m} e^{(n+1)}_{n-p-q-m} + (-1)^{(n+k)(m+1+p+k)} e^{(n+1)}_{n-k+m+1} e^{(n+1)}_{n-p-q-m-1} =\\ &\qquad (-1)^{(n+k)(m+q)} e^{(n+1)}_{n-p-q-m} e^{(n+1)}_{n-k+m} + (-1)^{(n+k)(m+1+q)} e^{(n+1)}_{n-p-q-m-1} e^{(n+1)}_{n-k+m+1}\end{aligned}$$

for every odd m with $1 \le m < n - p - q$. Multiplying both sides by $(-1)^{(n+k)(m+p+k)} = (-1)^{(n+k)(m+q+1)}$, this simplifies to

$$e^{(n+1)}_{n-k+m} e^{(n+1)}_{n-p-q-m} - e^{(n+1)}_{n-k+m+1} e^{(n+1)}_{n-p-q-m-1} = -e^{(n+1)}_{n-p-q-m} e^{(n+1)}_{n-k+m} + e^{(n+1)}_{n-p-q-m-1} e^{(n+1)}_{n-k+m+1}.$$

This follows because when m is odd, $n-k+m$ is even, so we have that

$$e_{n-k+m}e_{n-p-q-m} + e_{n-p-q-m}e_{n-k+m} = e_{n-k+m+1}e_{n-p-q-m-1} + e_{n-p-q-m-1}e_{n-k+m+1}$$

by (4.5).

Finally we treat the case that $p+q+k$ is odd and $n \not\equiv p+q \pmod 2$, when $n+k$ is even. Now the signs on both sides of the identity we are trying to prove are always + so can be omitted. There are an odd number of terms in the summations. The $m = n-p-q$ terms in both summations are equal, indeed, they both equal $e^{(n+1)}_{2n-p-q-k}$ as $e^{(n+1)}_0 = 1$. To see that the remaining terms in the summations are equal, we show that sums of consecutive pairs of terms are equal like in the previous paragraph, i.e.,

$$e^{(n+1)}_{n-k+m}e^{(n+1)}_{n-p-q-m} + e^{(n+1)}_{n-k+m+1}e^{(n+1)}_{n-p-q-m-1} = e^{(n+1)}_{n-p-q-m}e^{(n+1)}_{n-k+m} + e^{(n+1)}_{n-p-q-m-1}e^{(n+1)}_{n-k+m+1}$$

for each odd m with $1 \leq m < n-p-q-1$. This follows because when $n-k+m$ is odd we have that

$$e_{n-k+m}e_{n-p-q-m} - e_{n-p-q-m}e_{n-k+m} = -e_{n-k+m+1}e_{n-p-q-m-1} + e_{n-p-q-m-1}e_{n-k+m+1}$$

by (4.5) again. □

It is time to say a little more about variants of the odd complete and elementary symmetric functions. The following lemma, which is another application of Theorem 4.4, is helpful to understand the possibilities.

Lemma 4.10. *We have that* $\gamma(h_r) = (-1)^{\binom{r}{2}}h_r^*$ *and* $\gamma(e_r) = (-1)^{\binom{r}{2}}e_r^*$ *for any* $r \geq 0$. *Hence,* $* \circ \gamma \circ \psi = \psi \circ * \circ \gamma$.

Proof. It is immediate from the definitions that $\gamma(e_r) = (-1)^{\binom{r}{2}}e_r^*$. To see the analogous thing for h_r, it suffices to show that $\gamma_n(h_r^{(n)})^* = (-1)^{\binom{r}{2}}h_r^{(n)}$ for all $r \geq 0$. This follows from the explicit descriptions of these polynomials and maps given in Theorem 4.4. To deduce finally that $* \circ \gamma$ and ψ commute, it suffices to check that $(* \circ \gamma \circ \psi)(e_r) = (\psi \circ * \circ \gamma)(e_r)$ for all $r \geq 0$, which is clear at this point. □

As we have said before, our odd complete symmetric function h_r is the same as the h_r in [EK, EKL], but our odd elementary symmetric function e_r is different from the one there, which is

$$\varepsilon_r := \gamma(e_r) = (-1)^{\binom{r}{2}}e_r^* = (-1)^{\binom{r}{2}}e_r, \tag{4.40}$$

where the non-trivial equality follows by Lemma 4.10. There is also a natural variant on the odd complete symmetric function h_r, namely,

$$\eta_r := \gamma(h_r) = (-1)^{\binom{r}{2}}h_r^*. \tag{4.41}$$

Since $h_r^* \neq h_r$ for $r > 1$, it is *not* the case that $\eta_r = (-1)^{\binom{r}{2}}h_r$. We call ε_r and η_r the *dual odd elementary and complete symmetric functions*. Applying γ to (4.10) gives that

$$\varepsilon(t)\eta(t) = \eta(t)\varepsilon(t) = 1. \tag{4.42}$$

where $\varepsilon(t) := \sum_{r\geq 0}(-1)^r\varepsilon_r t^{-r}$ and $\eta(t) := \sum_{r\geq 0}\eta_r t^{-r}$. These should make it clear that e and h belong together as do ε and η. It is not so easy to relate e to η or h to ε in terms of generating functions; cf. (10.29).

Consider again the truncation $OSym_n$. Let $\varepsilon_r^{(n)}$ and $\eta_r^{(n)}$ be the images ε_r and η_r in $OSym_n$. From Theorem 4.4, it is clear that

$$e_r^{(n)} = \sum_{1\leq i_1<\cdots<i_r\leq n} x_{i_1}\cdots x_{i_r} \qquad h_r^{(n)} = \sum_{n\geq i_r\geq\cdots\geq i_1\geq 1} x_{i_r}\cdots x_{i_1} \tag{4.43}$$

$$\varepsilon_r^{(n)} = \sum_{n\geq i_r>\cdots>i_1\geq 1} x_{i_r}\cdots x_{i_1}, \qquad \eta_r^{(n)} = \sum_{1\leq i_1\leq\cdots\leq i_r\leq n} x_{i_1}\cdots x_{i_r}. \tag{4.44}$$

This gives an explanation for the existence of the four basic familes of odd symmetric functions e_r, h_r, ε_r and η_r. When working in $OSym_n$ with generating functions, we prefer to modify the definitions slightly, working not exactly with the images of $e(t), h(t), \varepsilon(t)$ and $\gamma(t)$ in $OSym_n[\![t^{-1}]\!]$, but incorporating a shift in t:

$$e^{(n)}(t) := \sum_{r=0}^{n}(-1)^r e_r^{(n)} t^{n-r}, \qquad h^{(n)}(t) := \sum_{r\geq 0} h_r^{(n)} t^{-n-r}, \tag{4.45}$$

$$\varepsilon^{(n)}(t) := \sum_{r=0}^{n}(-1)^r \varepsilon_r^{(n)} t^{n-r}, \qquad \eta^{(n)}(t) := \sum_{r\geq 0} \eta_r^{(n)} t^{-n-r}. \tag{4.46}$$

The advantage of this is that $e^{(n)}(t)$ and $\varepsilon^{(n)}(t)$ are *polynomials* $OSym_n[t]$, indeed, we have that

$$e^{(n)}(t) = (t-x_1)\cdots(t-x_n), \qquad \varepsilon^{(n)}(t) = (t-x_n)\cdots(t-x_1). \tag{4.47}$$

Also, noting that $(t-x)^{-1} = t^{-1} + xt^{-2} + x^2t^{-3} + \cdots \in \mathbb{F}[x][\![t^{-1}]\!]$, we have that

$$h^{(n)}(t) = (t-x_n)^{-1}\cdots(t-x_1)^{-1}, \qquad \eta^{(n)}(t) = (t-x_1)^{-1}\cdots(t-x_n)^{-1} \tag{4.48}$$

in $OSym_n[\![t^{-1}]\!]$. Now we have that

$$h^{(n)}(t)e^{(n)}(t) = e^{(n)}(t)h^{(n)}(t) = 1, \qquad \eta^{(n)}(t)\varepsilon^{(n)}(t) = \varepsilon^{(n)}(t)\eta^{(n)}(t) = 1, \tag{4.49}$$

equality in the ring $OSym_n(\!(t^{-1})\!)$ of formal Laurent series in t^{-1}.

The next result is elementary but does not appear in the existing literature. Observe by (4.3) that

$$z_{2r} := \sum_{s=0}^{r} e_{2s}h_{2r-2s} = \delta_{r,0} + \sum_{s=0}^{r-1} e_{2s+1}h_{2r-2s-1}. \tag{4.50}$$

The element z_{2r} is *central*: it commutes with all even e_t by the first form of the definition, and it commutes with all odd e_t by the second one. Also let omicron be the special element

$$o := e_1 = h_1, \tag{4.51}$$

noting that $z_2 = o^2$. The relations (4.2) and (4.5) imply that

$$e_{2r+1} = \tfrac{1}{2}(oe_{2r} + e_{2r}o), \qquad h_{2r+1} = \tfrac{1}{2}(oh_{2r} + h_{2r}o), \tag{4.52}$$

so that $OSym$ is generated already by o and all even e_{2r} $(r \geq 1)$.

Theorem 4.11. *The graded superalgebra $OSym$ is generated by o and e_{2r} $(r \geq 1)$ subject only to the relations*

$$[e_{2r}, e_{2s}] = 0 \tag{4.53}$$

$$[o^2, e_{2r}] = 0 \tag{4.54}$$

$$[o, e_{2r+2}] = \left[\tfrac{1}{2}(oe_{2r} + e_{2r}o), e_2\right] \tag{4.55}$$

for $r, s \geq 1$.

Proof. Let A be the graded superalgebra generated by an odd element o of degree 2 and even elements e_{2r} $(r \geq 1)$ of degree $4r$ subject to the relations (4.53) to (4.55). For $r \geq 0$, we set $e_1 := o$ and $e_{2r+1} := \frac{1}{2}(oe_{2r} + e_{2r}o) \in A$ for $r \geq 1$; cf. (4.52).

We first construct a homomorphism $\alpha : A \to OSym$ by mapping $o \mapsto o$ and $e_r \mapsto e_r$. To see that this makes sense, we need to check that the relations (4.53) to (4.55) hold in $OSym$. The first is immediate, and the second follows because we have observed already that $z_2 = o^2$ is central in $OSym$. For the third, in $OSym$, we have that $[o, e_{2r+2}] = [e_{2r+1}, e_2]$ by (4.5), and also $\frac{1}{2}(oe_{2r} + e_{2r}o) = e_{2r+1}$ by (4.52). Now the relation is clear.

Next we construct a homomorphism $\beta : OSym \to A$ in the other direction so that on generators it sends $e_r \mapsto e_r$ for each $r \geq 1$. To show this is well defined, we must again check relations, this time showing that (4.4) and (4.5) hold in A. The first one is immediate if r and s are both even. When r is odd and s is even, (4.5) is equivalent to the relation

$$[e_{2r-1}, e_{2s}] = [e_{2s-1}, e_{2r}] \tag{4.56}$$

for $r, s \geq 1$. To check it holds in A, we must show that the expression $[e_{2r-1}, e_{2s}] \in A$ is symmetric in r and s. We have that

$$\begin{aligned}
4[e_{2r-1}, e_{2s}] &= 2[oe_{2r-2} + e_{2r-2}o, e_{2s}] \\
&= 2oe_{2r-2}e_{2s} + 2e_{2r-2}oe_{2s} - 2e_{2s}oe_{2r-2} - 2e_{2s}e_{2r-2}o \\
&= 2(oe_{2s} - e_{2s}o)e_{2r-2} + 2e_{2r-2}(oe_{2s} - e_{2s}o) \\
&= 2[o, e_{2s}]e_{2r-2} + 2e_{2r-2}[o, e_{2s}] \\
&= [oe_{2s-2} + e_{2s-2}o, e_2]e_{2r-2} + e_{2r-2}[oe_{2s-2} + e_{2s-2}o, e_2] \\
&= [o, e_2]e_{2s-2}e_{2r-2} + e_{2s-2}[o, e_2]e_{2r-2} + e_{2r-2}[o, e_2]e_{2s-2} + e_{2r-2}e_{2s-2}[o, e_2],
\end{aligned}$$

which is indeed symmetric in r and s. When r is even and s is odd, (4.5) is equivalent to the relation

$$[e_{2r}, e_{2s+1}]_+ = [e_{2s}, e_{2r+1}]_+ \tag{4.57}$$

for $r, s \geq 0$, where $[x, y]_+$ here denotes $xy+yx$. So again we must show that $[e_{2r}, e_{2s+1}]_+ \in A$ is symmetric in r and s. We have that

$$2[e_{2r}, e_{2s+1}]_+ = e_{2r}oe_{2s} + e_{2r}e_{2s}o + oe_{2s}e_{2r} + e_{2s}oe_{2r}.$$

This is symmetric in r and s because $e_{2r}e_{2s} = e_{2s}e_{2r}$. It remains to check the relation (4.4) when r and s are both odd. Equivalently, we show that $[e_{2r+1}, e_{2s+1}] = 0$ for $r, s \geq 0$ by induction on s. The base case $s = 0$ follows because

$$2[e_{2r+1}, o] = [oe_{2r} + e_{2r}o, o] = oe_{2r}o + e_{2r}o^2 - o^2e_{2r} - oe_{2r}o = -\left[o^2, e_{2r}\right] = 0.$$

The following establishes the induction step: for $s > 0$ we have in A that

$$\begin{aligned}
2[e_{2r+1}, e_{2s+1}] &= [e_{2r+1}, oe_{2s} + e_{2s}o] = o[e_{2r+1}, e_{2s}] + [e_{2r+1}, e_{2s}]o \\
&= o[e_{2s-1}, e_{2r+2}] + [e_{2s-1}, e_{2r+2}]o = [e_{2s-1}, oe_{2r+2} + e_{2r+2}o] \\
&= 2[e_{2s-1}, e_{2r+3}] = -2[e_{2r+3}, e_{2s-1}] = 0,
\end{aligned}$$

using the $s = 0$ case for the second and fourth equalities, (4.56) for the third equality, and the induction hypothesis for the final equality.

It remains to observe that α and β are two-sided inverses. This is obviously the case on generators by the way we have defined the maps. □

In the corollaries, we use the following notation:

- *Sym* is the algebra of symmetric functions over $\mathbb{F}$ as in [Mac] viewed as a graded superalgebra so that the rth elementary and complete symmetric functions are even of degree $4r$;
- Sym_n is the usual algebra of symmetric polynomials in n variables, i.e., it is the quotient of *Sym* obtained by setting the rth even elementary symmetric polynomials to zero for all $r > n$;
- $Sym[c]$ and $Sym_n[c]$ are the supercommutative graded superalgebras obtained from *Sym* and Sym_n by adjoining an odd element c of degree 2 with $c^2 = 0$.

Corollary 4.12. *The largest supercommutative quotient of OSym is the graded superalgebra*

$$R := OSym/\langle o^2, [o, e_2]\rangle. \tag{4.58}$$

Writing $\dot{a}$ for the canonical image of $a \in OSym$ in R, we have that

$$\dot{e}_{2r+1} = \dot{e}_{2r}\dot{o}, \qquad \dot{h}_{2r+1} = \dot{h}_{2r}\dot{o}, \qquad \dot{\varepsilon}_{2r+1} = \dot{\varepsilon}_{2r}\dot{o}, \qquad \dot{\eta}_{2r+1} = \dot{\eta}_{2r}\dot{o}, \qquad \dot{z}_{2r} = \delta_{r,0} \tag{4.59}$$

for all $r \geq 0$. Moreover, there is an isomorphism of graded superalgebras $\theta : R \xrightarrow{\sim} Sym[c]$ taking $\dot{o}$ to c, $\dot{\varepsilon}_{2r} = (-1)^r \dot{e}_{2r}$ to the rth even elementary symmetric function and $\dot{h}_{2r} = (-1)^r \dot{\eta}_{2r}$ to the rth even complete symmetric function.

Proof. In any supercommutative quotient of $OSym$, we must have that $o^2 = 0$ and $[o, e_2] = 0$. Now we let I be the two-sided ideal of $OSym$ generated by o^2 and $[o, e_2]$ and show that $OSym/I$ is supercommutative. Since $OSym$ is generated by the elements e_{2r} $(r \geq 1)$ and o, the proof of this reduces at once to checking that $[o, e_{2r}] \in I$ for all $r \geq 1$, which holds because

$$2[o, e_{2r}] = 2[e_{2r-1}, e_2] = [e_{2r-2}o + oe_{2r-2}, e_2] = e_{2r-2}[o, e_2] + [o, e_2]e_{2r-2} \in I.$$

Thus, we have shown that $R := OSym/I$ is the largest supercommutative quotient of $OSym$. Next we observe that $\dot{e}_{2r+1} = \dot{e}_{2r}\dot{o}$, $\dot{h}_{2r+1} = \dot{h}_{2r}\dot{o}$ and $\dot{z}_{2r} = \delta_{r,0}$ for all $r \geq 0$. The first two of these follow from (4.52) and the supercommutativity of $OSym/I$, then the final equality follows using the first two together with the second form of the definition of z_{2r} in (4.50). The superalgebra anti-involution $* : OSym \to OSym$ induces an anti-involution of R. Since it fixes the generators $\dot{e}_r$ $(r \geq 1)$ and R is supercommutative, this induced anti-involution is actually the identity. So by (4.40) and (4.41), we have that $\dot{\varepsilon}_r = (-1)^{\binom{r}{2}} \dot{e}_r$ and $\dot{\eta}_r = (-1)^{\binom{r}{2}} \dot{h}_r$. The remaining identities in (4.59) follow using this.

Finally, we construct the isomorphism θ. We start by observing that there is a homomorphism $OSym \to Sym[c]$ taking $o \mapsto c$ and $\varepsilon_{2r} = (-1)^r e_{2r}$ to the rth even elementary symmetric function for $r \geq 1$. To see this, we apply Theorem 4.11 to reduce to checking that the relations (4.53) to (4.55) all hold in $Sym[c]$, which is clear because it is supercommutative. Now this homomorphism factors through the quotient to induce $\theta : R \to Sym[c]$. Moreover, R is spanned by the monomials $\dot{e}_\lambda$ and $\dot{e}_\lambda \dot{o}$ for partitions λ with all parts even. The images under θ of these elements give a linear basis for $Sym[c]$. This shows that θ is an isomorphism. It just remains to check that θ takes $\dot{h}_{2r} = (-1)^r \dot{\eta}_{2r}$ to the rth even complete symmetric function. This follows from the usual infinite Grassmannian relation relating even complete symmetric functions to even elementary symmetric functions in Sym providing we can show that

$$\sum_{s=0}^{r} \dot{e}_{2s} \dot{h}_{2r-2s} = \delta_{r,0} \tag{4.60}$$

for all $r \geq 0$. This is true because the sum on the left hand side is $\dot{z}_{2r}$ by the first form of the definition (4.50), which we have already shown is zero for $r \geq 1$. □

Corollary 4.13. *The largest supercommutative quotient of $OSym_n$ is*

$$R_n := OSym_n \Big/ \left\langle (o^{(n)})^2, [o^{(n)}, e_2^{(n)}] \right\rangle. \tag{4.61}$$

The isomorphism θ from Corollary 4.12 induces an isomorphism θ_n from R_n to $Sym_{(n-1)/2}[c]$ if n is odd, or to the quotient of $Sym_{n/2}[c]$ obtained by setting the product of c and the $(n/2)$th even elementary symmetric polynomial to zero if n is even.

Proof. This follows from Corollary 4.12 since a supercommutative quotient of $OSym_n$ is a supercommutative quotient of $OSym$ in which the images of all e_r $(r > n)$ are zero. □

5. Odd nil-Hecke algebras

This section is largely an exposition of results from [EKL]. The *odd nil-Hecke algebra* is the graded superalgebra ONH_n with generators x_i $(i = 1, \dots, n)$ and τ_j $(j = 1, \dots, n-1)$ which are odd of degrees 2 and -2, respectively, subject to the following relations:

$$x_i x_j = -x_j x_i \qquad (i \neq j) \tag{5.1}$$

$$\tau_i\tau_j = -\tau_j\tau_i \qquad (|i-j|>1) \tag{5.2}$$
$$x_i\tau_j = -\tau_j x_i \qquad (i \neq j, j+1) \tag{5.3}$$
$$\tau_j^2 = 0 \tag{5.4}$$
$$\tau_j\tau_{j+1}\tau_j = -\tau_{j+1}\tau_j\tau_{j+1} \tag{5.5}$$
$$x_i\tau_i - \tau_i x_{i+1} = 1 = \tau_i x_i - x_{i+1}\tau_i. \tag{5.6}$$

We warn the reader that the above is *not* the standard form of the presentation for this algebra which appears in all of the existing literature. The difference is in the relations (5.5) and (5.6), in which our minus signs become plus signs in the standard presentation. To obtain the above presentation from the standard one, note that our generators x_i and τ_j are equal to the elements denoted $(-1)^{i-1}x_i$ and $(-1)^{j-1}\tau_j$ elsewhere in the literature. This change certainly impacts many other formulae below, but it is usually straightforward to make the appropriate adaptation. One advantage of our modified sign convention can already be seen in the definitions (4.36) and (4.37) above—the corresponding formulae in [EKL] involve some additional signs.

Let S_n act on the left on $OPol_n$ by graded superalgebra automorphisms so that ${}^w x_i = (-1)^{\ell(w)+w(i)-i} x_{w(i)}$ for $w \in S_n$, $1 \leq i \leq n$. In particular:

$$ {}^{s_j}x_i = \begin{cases} x_{j+1} & \text{if } i = j \\ x_j & \text{if } i = j+1 \\ -x_i & \text{otherwise.} \end{cases} \tag{5.7}$$

The *odd Demazure operator* $\partial_j : OPol_n \to OPol_n$ is the linear map defined on $f \in OPol_n$ by

$$\partial_j(f) = \frac{(x_j + x_{j+1})f - ({}^{s_j}f)\,(x_j + x_{j+1})}{x_j^2 - x_{j+1}^2}, \tag{5.8}$$

which makes sense because the denominator is central. This formula first appeared in [KKO1, (4.10)] remembering, of course, our modified choice of signs. We actually never use this form of the definition of ∂_j, preferring the following recursive definition: ∂_j is the unique odd linear map of degree -2 such that

$$\partial_j(x_i) = \delta_{i,j} - \delta_{i,j+1}, \qquad \partial_j(fg) = \partial_j(f)g + ({}^{s_j}f)\,\partial_j(g) \tag{5.9}$$

for $f, g \in OPol_n$. Now we make the graded vector superspace $OPol_n$ into a left ONH_n-supermodule so that $x_i \in ONH_n$ acts on $f \in OPol_n$ by $x_i \cdot f := x_i f$, and $\tau_j \in ONH_n$ acts by $\tau_j \cdot f := \partial_j(f)$. A tedious relation check shows that this definition makes sense. It is straightforward to show by induction on r that

$$\tau_i \cdot x_i^{r+1} = \sum_{s=0}^{r} x_{i+1}^s x_i^{r-s}, \qquad \tau_i \cdot x_{i+1}^{r+1} = -\sum_{s=0}^{r} x_i^s x_{i+1}^{r-s}. \tag{5.10}$$

One can rewrite (5.10) as the generating function identities:

$$\tau_i \cdot (t-x_i)^{-1} = (t-x_{i+1})^{-1}(t-x_i)^{-1}, \qquad \tau_i \cdot (t-x_{i+1})^{-1} = -(t-x_i)^{-1}(t-x_{i+1})^{-1}, \tag{5.11}$$

equalities in $OPol_n[\![t^{-1}]\!]$. From the former, we get that

$$\tau_{n-1}\cdots\tau_1 \cdot (t-x_1)^{-1} = (t-x_n)^{-1}\cdots(t-x_1)^{-1}. \tag{5.12}$$

Hence, recalling (4.48), we get that

$$\tau_{n-1}\cdots\tau_1 \cdot x_1^{n+r-1} = h_r^{(n)} \tag{5.13}$$

on computing t^{-n-r}-coefficients.

By the relations (5.1) to (5.6), ONH_n admits an algebra involution γ_n and a superalgebra anti-involution $*$ defined by

$$\gamma_n : ONH_n \to ONH_n, \qquad x_i \mapsto x_{n+1-i}, \qquad \tau_j \mapsto -\tau_{n-j}, \tag{5.14}$$

$$* : ONH_n \to ONH_n, \qquad x_i \mapsto x_i, \qquad \tau_j \mapsto -\tau_j. \tag{5.15}$$

These definitions are consistent with (4.34) and (4.35). Note also that γ_n and $*$ commute with each other. Mirroring the notation for symmetric groups from *General conventions*, there is also a homomorphism

$$\mathrm{sh}_n : ONH_{n'} \to ONH_{n+n'}, \qquad x_i \mapsto x_{i+n}, \tau_j \mapsto \tau_{j+n}. \tag{5.16}$$

We will show shortly that this is injective, but this is not yet clear. Similarly, there is a homomorphism $\mathrm{sh}_n : OPol_{n'} \to OPol_{n+n'}, x_i \mapsto x_{i+n}$, which is obviously injective. We have that

$$\gamma_n(a) \cdot \gamma_n(f) = \gamma_n(a \cdot f), \qquad \mathrm{sh}_n(a) \cdot \mathrm{sh}_n(f) = \mathrm{sh}_n(a \cdot f), \tag{5.17}$$

for $a \in ONH_n$, $f \in OPol_n$, or $a \in ONH_{n'}$, $f \in OPol_{n'}$, respectively.

For each $w \in S_n$, we pick a reduced expression $w = s_{j_1} \cdots s_{j_l}$ then set $\tau_w := \tau_{j_1} \cdots \tau_{j_l}$. For the longest element w_n, we choose the reduced expression $(s_{n-1}s_{n-2} \cdots s_1)(s_{n-1}s_{n-2} \cdots s_2) \cdots (s_{n-1}s_{n-2})s_{n-1}$, and adopt the shorthands

$$\omega_n := \tau_{w_n} = \tau_{n-1}\tau_{n-2} \cdots \tau_1 \,\mathrm{sh}_1(\omega_{n-1}), \qquad \xi_n := x_{n-1}x_{n-2}^2 \cdots x_1^{n-1} = \mathrm{sh}_1(\xi_{n-1})x_1^{n-1}. \tag{5.18}$$

These elements have the following desirable property.

Lemma 5.1. $\omega_n \cdot \xi_n = 1$.

Proof. When $n = 1$, this is clear as $\omega_n = 1 = \xi_n$. The result for $n > 1$ follows by induction:

$$\begin{aligned}\omega_n \cdot \xi_n &= \tau_{n-1} \cdots \tau_1 \,\mathrm{sh}_1(\omega_{n-1}) \cdot \mathrm{sh}_1(\xi_{n-1})x_1^{n-1} \\ &= \tau_{n-1} \cdots \tau_1 \cdot \mathrm{sh}_1(\omega_{n-1} \cdot \xi_{n-1})x_1^{n-1} = \tau_{n-1} \cdots \tau_1 \cdot x_1^{n-1} = 1,\end{aligned}$$

using (5.13) for the final equality. □

It is also clear that

$$\gamma_n(\omega_n) = \zeta_n \omega_n \tag{5.19}$$

for some $\zeta_n \in \{\pm 1\}$. One can verify explicitly that $\zeta_n = (-1)^{\binom{n+1}{3}}$; the calculation is similar to the proof of [EKL, Lem. 3.2]. However, the only place this sign is used is in the proof of Theorem 6.12, and in that place we actually do not need to know its acual value.

An important role will be played by the *odd Schubert polynomials*

$$p_w^{(n)} := \tau_{w^{-1}w_n} \cdot \xi_n \in OPol_n. \tag{5.20}$$

For example, we have that $p_1^{(3)} = 1$, $p_{s_1}^{(3)} = -x_1$, $p_{s_2}^{(3)} = x_1 + x_2$, $p_{s_2s_1}^{(3)} = x_1^2$, $p_{s_1s_2}^{(3)} = -x_2x_1$ and $p_{s_2s_1s_2}^{(3)} = x_2x_1^2$. In general, $p_w^{(n)}$ depends up to sign on the choice of reduced expression for $w^{-1}w_n$, but we always have that $p_{w_n}^{(n)} = \xi_n$ and $p_1^{(n)} = 1$ thanks to Lemma 5.1. Note also that $\deg\left(p_w^{(n)}\right) = 2\ell(w)$ and $\mathrm{par}\left(p_w^{(n)}\right) \equiv \ell(w) \pmod 2$.

Theorem 5.2 ([EKL, Prop. 2.11])**.** *The elements* $\{x^\kappa \tau_w = x_1^{\kappa_1} \cdots x_n^{\kappa_n} \tau_w \mid w \in S_n, \kappa \in \mathbb{N}^n\}$ *give a basis for* ONH_n*. Moreover,* $OPol_n$ *is a faithful* ONH_n*-module.*

Proof. First one shows using (5.2), (5.4) and (5.5) that any word in τ_j $(j = 1, \ldots, n-1)$ can be reduced to 0 or $\pm\tau_w$ for some $w \in S_n$. It follows that the set in Theorem 5.2 spans ONH_n. Then to establish the linear independence, suppose that we have some non-trivial linear relation

$$\sum_{w \in S_n} \sum_{\kappa \in \mathbb{N}^n} c_{w,\kappa} x^\kappa \tau_w = 0$$

between the elements of this set. Pick w of minimal length such that $c_{w,\kappa} \neq 0$ for some κ. Then we act on $p_w^{(n)}$. For $w' \neq w$ with $\ell(w') \geq \ell(w)$, we have that $\tau_{w'} \cdot p_w^{(n)} = 0$ by the relations (5.2), (5.4) and (5.5), and $\tau_w \cdot p_w^{(n)} = \pm 1$ by Lemma 5.1. So we deduce that $\sum_{\kappa \in \mathbb{N}^n} c_{w,\kappa} x^\kappa = 0$, which is a contradiction. This also shows $OPol_n$ is faithful. □

Corollary 5.3. $\dim_{q,\pi} ONH_n = \dim_{q,\pi} OSym_n \times q^{\binom{n}{2}}[n]^!_{q,\pi} \times q^{-\binom{n}{2}}\overline{[n]}^!_{q,\pi}$,

Proof. The theorem gives that

$$\dim_{q,\pi} ONH_n = \dim_{q,\pi} OPol_n \times \sum_{w \in S_n} (\pi q^2)^{-\ell(w)}.$$

Now replace $\dim_{q,\pi} OPol_n$ by the first formula for it from Corollary 4.8, and replace the summation by a product using (3.3). □

The basis theorem just established implies that the obvious homomorphism $OPol_n \to ONH_n$ is injective. Henceforth, we identify $OPol_n$ with a subalgebra of ONH_n via this map. Another application of the basis theorem shows that the homomorphism $ONH_n \hookrightarrow ONH_{n+1}$ taking x_i to x_i and τ_j to τ_j is injective. Thus, we have a tower of graded superalgebras $ONH_0 \subset ONH_1 \subset ONH_2 \subset \cdots$. The basis theorem also shows that the homomorphism $\mathrm{sh}_n : ONH_{n'} \to ONH_{n+n'}$ from (5.16) is injective, as promised earlier. For $\alpha \in \Lambda(k,n)$, we let ONH_α be the subalgebra $\{x^\kappa \tau_w \bigm| w \in S_\alpha, \kappa \in \mathbb{N}^n\}$ of ONH_n. Finally, let ONH_n^{fin} be the subalgebra of ONH_n with basis $\{\tau_w \mid w \in S_n\}$. As an algebra, ONH_n^{fin} is generated by the elements τ_j ($j = 1, \dots, n-1$) subject just to (5.2), (5.4) and (5.5). There is a unique way to make the ground field $\mathbb{F}$ into a purely even graded left ONH_n^{fin}-supermodule concentrated in degree 0; each τ_j acts as zero. There is then a canonical isomorphism of graded ONH_n-supermodules

$$ONH_n \otimes_{ONH_n^{\mathrm{fin}}} \mathbb{F} \xrightarrow{\sim} OPol_n, \qquad x^\kappa \otimes 1 \mapsto x^\kappa. \tag{5.21}$$

This isomorphism explains the origin of the polynomial representation of ONH_n.

Now recall the subalgebra $OSym_n$ of $OPol_n$ which was defined just after Theorem 4.4—it is the subalgebra of $OPol_n$ generated by the odd symmetric polynomials $e_r^{(n)}$ from (4.36). A different formulation of the definition of $OSym_n$ was adopted in [EKL], where $OSym_n$ was defined from the outset to be $\bigcap_{i=1}^{n-1} \ker \partial_i$, which is a subalgebra of $OPol_n$. We will deduce the equality of $OSym_n$ with this subalgebra in Corollary 5.5, but one containment is obvious: we have that

$$OSym_n \subseteq \bigcap_{i=1}^{n-1} \ker \partial_i. \tag{5.22}$$

To see this, it suffices to check that $\partial_i(e_r^{(n)}) = 0$ for all i and $r = 1, \dots, n$, which follows from the definitions since $\partial_i(x_i + x_{i+1}) = \partial_i(x_i x_{i+1}) = 0$.

Theorem 5.4 ([EKL, Prop. 2.13, Cor. 2.14])**.** *The graded right $OSym_n$-supermodule $OPol_n$ is free of graded rank $q^{\binom{n}{2}}[n]^!_{q,\pi}$ with basis $\{p_w^{(n)} \mid w \in S_n\}$ given by the odd Schubert polynomials from (5.20). So we have that*

$$OPol_n = \bigoplus_{w \in S_n} p_w^{(n)} OSym_n \qquad \text{with} \qquad p_w^{(n)} OSym_n \simeq (\Pi Q^2)^{\ell(w)} OSym_n \tag{5.23}$$

as graded right $OSym_n$-supermodules. Moreover, the action of ONH_n on $OPol_n$ induces a graded superalgebra isomorphism

$$ONH_n \xrightarrow{\sim} \mathrm{End}_{\text{-}OSym_n}(OPol_n). \tag{5.24}$$

Proof. We claim that the polynomials $p^{(n)}_w$ $(w \in S_n)$ are linearly independent over $OSym_n$. To see this, take a non-trivial linear relation

$$\sum_{w\in S_n} p^{(n)}_w b_w = 0$$

for $b_w \in OSym_n$. Choose w of maximal length such that $b_w \neq 0$. Then we act with τ_w. We have that $\tau_w \cdot p^{(n)}_{w'} b_{w'} = 0$ for $w' \neq w$ by the relations (5.2), (5.4) and (5.5), and $\tau_w \cdot p^{(n)}_w b_w = \pm b_w$, so we deduce that $b_w = 0$, a contradiction. The claim implies that the $OSym_n$-submodule of $OPol_n$ generated by $p^{(n)}_w$ $(w \in S_n)$ is of graded superdimension $\dim_{q,\pi} OSym_n \times \sum_{w\in S_n} (\pi q^2)^{\ell(w)}$, which is equal to $\dim_{q,\pi} OPol_n$ by Corollary 4.8. Hence, the $p^{(n)}_w$ $(w \in S_n)$ also span $OPol_n$ as an $OSym_n$-module, and we have proved (5.23).

To establish (5.24), we first note by (5.23) that

$$\dim_{q,\pi} \operatorname{End}_{\text{-}OSym_n}(OPol_n) = \sum_{x,y\in S_n} (\pi q^2)^{\ell(x)-\ell(y)} = \left(\sum_{x\in S_n} (\pi q^2)^{\ell(x)}\right)\left(\sum_{y\in S_n} (\pi q^2)^{-\ell(y)}\right) = q^{\binom{n}{2}}[n]^!_{q,\pi} \times q^{-\binom{n}{2}}\overline{[n]}^!_{q,\pi},$$

applying (3.3). The homomorphism $\rho : ONH_n \to \operatorname{End}_{\text{-}OSym_n}(OPol_n)$ is injective by Theorem 5.2. Therefore it is an isomorphism because the graded superdimensions are the same thanks to Corollary 5.3. □

Corollary 5.5. *We have that* $OSym_n = \bigcap_{i=1}^{n-1} \ker \partial_i = \bigcap_{i=1}^{n-1} \operatorname{im} \partial_i$.

Proof. It is easy to see that $\ker \partial_i = \operatorname{im} \partial_i$ for each i, hence, the second equality holds. For the first one, we have already noted in (5.22) that $OSym_n \subseteq \bigcap_{i=1}^{n-1} \ker \partial_i$. Conversely, take $f \in \bigcap_{i=1}^{n-1} \ker \partial_i$ and write it as $f = \sum_{w\in S_n} p^{(n)}_w b_w$ for $b_w \in OSym_n$. We need to show that $b_w = 0$ except when $w = 1$. Suppose for a contradiction that this is not the case, and pick w of maximal length such that $b_w \neq 0$. Then we act on f with τ_w to see that $b_w = 0$, contradiction. □

Remark 5.6. As well as the basis $F := \{p^{(n)}_w \mid w \in S_n\}$ of odd Schubert polynomials from Theorem 5.4, the monomials $G := \{x_n^{\kappa_n} \cdots x_1^{\kappa_1} \mid \kappa \in \mathbb{N}^n \text{ with } 0 \le \kappa_i \le n-i\}$ form a basis for $OPol_n$ as a free right $OSym_n$-module. To see this, it suffices to show that $\mathbb{F}F = \mathbb{F}G$. The elements of F are linearly independent over $\mathbb{F}$ by Theorem 5.4, so $\dim \mathbb{F}F = n!$. Also $\dim \mathbb{F}G = n!$ obviously. So we are reduced to checking that $\mathbb{F}F \subseteq \mathbb{F}G$. To see this, we note first that $\mathbb{F}G$ is invariant under the action of each τ_i, as may be seen directly using (5.10) plus $\tau_i \cdot x_{i+1}^r x_i^r = 0$. Since $\xi_n \in \mathbb{F}G$, it follows that $p^{(n)}_w = \tau_{w^{-1}w_n} \cdot \xi_n \in \mathbb{F}G$ for each $w \in S_n$ as claimed.

Let $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}(OSym_n)$ denote the usual algebra of matrices $A = (a_{w,w'})_{w,w'\in S_n}$ with entries in $OSym_n$ viewed as a graded superalgebra so that the matrix with $a \in (OSym_n)_{i,p}$ in its (w,w')-entry and zeros elsewhere is of degree $i + 2\ell(w) - 2\ell(w')$ and parity $p + \ell(w) - \ell(w') \pmod 2$. This graded superalgebra may be identified with $\operatorname{End}_{\text{-}OSym_n}(OPol_n)$ so that the matrix A just described corresponds to the unique right $OSym_n$-supermodule endomorphism of $OPol_n$ taking $p^{(n)}_{w'}$ to $\sum_{w\in S_n} p^{(n)}_w a_{w,w'}$ for every $w' \in S_n$. Thus, Theorem 5.4 shows that $ONH_n \cong M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}(OSym_n)$. It follows that the graded superfunctors

$$- \otimes_{ONH_n} OPol_n : \text{gsMod-}ONH_n \to \text{gsMod-}OSym_n, \tag{5.25}$$

$$\operatorname{Hom}_{ONH_n}(OPol_n, -) : ONH_n\text{-gsMod} \to OSym_n\text{-gsMod} \tag{5.26}$$

are equivalences of graded (Q,Π)-supercategories.

Theorem 5.7 ([EKL, Prop. 2.15])**.** *The even center $Z(ONH_n)_{\bar 0}$ of ONH_n is the graded algebra consisting of symmetric polynomials in $x_1^2, \dots, x_n^2$. This coincides with the even center $Z(OSym_n)_{\bar 0}$ of $OSym_n$ embedded into ONH_n in the natural way.*

Proof. This is proved in [EKL] but we give a slightly different argument since there are some minor issues in the first paragraph of the original proof, which does not restrict attention to the *even* center. Take $z \in Z(ONH_n)$. Using Theorem 5.2, we have that

$$z = \sum_{w \in S_n} f_w \tau_w$$

for unique $f_w \in OPol_n$. The first step is to show that $f_w = 0$ unless $w = 1$. To see this, suppose for a contradiction that it is not the case. Let w be of maximal length such that $f_w \neq 0$. Pick $i \in \{1, \dots, n\}$ such that $j := w(i) \neq i$. We have that

$$x_i z = \sum_{w \in S_n} x_i f_w \tau_w.$$

Now we use the relations to express zx_i as a linear combination $\sum_{v \in S_n} g_v \tau_v$ for $g_v \in OPol_n$. Since $\tau_w x_i = \pm x_j \tau_w$ plus a linear combination of $\tau_{w'}$ for w' with $\ell(w') < \ell(w)$, we see that $g_w = \pm f_w x_j$. Thus, we must have that $x_i f_w = \pm f_w x_j$. Since $i \neq j$, it is easy to see that this implies that $f_w = 0$. So now we have proved that $z \in OPol_n$. Next, *assuming also that z is even*, we show that z is in fact in $\mathbb{F}[x_1^2, \dots, x_n^2]$ essentially following the idea from the proof in [EKL]. Take any $1 \leq i \leq n$ and suppose that $z = \sum_{k \geq 0} f_k x_i^k$ for f_k belonging to the subalgebra of $OPol_n$ generated by $x_1, \dots, x_{i-1}, x_{i+1}, \dots, x_n$. Since $x_i z = z x_i$, we get that each f_k must be even. Since z is even too it follows that $f_k = 0$ unless k is even. This shows that z only involves even powers of x_i. This is true for each i, so $z \in \mathbb{F}[x_1^2, \dots, x_n^2]$ as claimed. To complete the proof that z is actually a symmetric polynomial in $x_1^2, \dots, x_n^2$, and to show that any such polynomial is central, we can now refer the reader to the argument given in the second two paragraphs of the proof of [EKL, Prop. 2.15].

Finally we explain how to see that $Z(ONH_n)_{\bar{0}}$ coincides with $Z(OSym_n)_{\bar{0}}$. The supercenter of the matrix algebra $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}(OSym_n)$ is isomorphic to $Z(OSym_n)$ via the map taking $z \in Z(OSym_n)$ to the matrix $\operatorname{diag}(z, \dots, z)$. It follows that the even centers are isomorphic too. Given z in the even center of ONH_n, we have just shown that it is a polynomial in $x_1^2, \dots, x_n^2$, so we have that $zp_w^{(n)} = p_w^{(n)} z$ for all $w \in S_n$. It follows that z acts on $OPol_n$ in the same way as the matrix $\operatorname{diag}(z, \dots, z)$ under the identification of ONH_n with matrices described above. This shows that the natural embedding of $OSym_n$ into ONH_n restricts to give an isomorphism between the even centers of $OSym_n$ and ONH_n. □

The idempotents in ONH_n corresponding to the diagonal matrix units $e_{w,w} \in M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}(OSym_n)$, that is, the elements which act on $OPol_n$ as the projections onto the indecomposable summands in (5.23), give a complete set of primitive idempotents in ONH_n. It is clear from Theorem 5.2 that the component of ONH_n of smallest degree is 1-dimensional spanned by ω_n. Since $\omega_n \xi_n \omega_n$ is of the same degree as ω_n, it follows that $\omega_n \xi_n \omega_n$ is a scalar multiple of ω_n. Moreover, both $\omega_n \xi_n \omega_n$ and ω_n map ξ_n to 1 by Lemma 5.1, hence, we actually have that

$$\omega_n \xi_n \omega_n = \omega_n. \tag{5.27}$$

From this it follows that the following are both idempotents:

$$(\xi\omega)_n := \xi_n \omega_n, \qquad (\omega\xi)_n := \omega_n \xi_n. \tag{5.28}$$

The first of these, $(\xi\omega)_n$, is exactly the matrix unit e_{w_n,w_n} which projects $OPol_n$ onto the top degree component $p_{w_n}^{(n)} OSym_n$. This follows almost immediately since Lemma 5.1 shows that $\xi_n \omega_n \cdot p_{w_n}^{(n)} = p_{w_n}^{(n)}$ and $\xi_n \omega_n \cdot p_w^{(n)} = 0$ for all other $w \in S_n$ as $\omega_n \cdot p_w^{(n)} = 0$ by degree considerations. In particular, this shows that $(\xi\omega)_n$ is a primitive idempotent. The second one, $(\omega\xi)_n$, is also primitive since we have that

$$(\omega\xi)_n = (\xi\omega)_n^*. \tag{5.29}$$

To see this, it is clear from the definitions that $(\xi\omega)_n^* = \pm(\omega\xi)_n$, and the sign must be plus since $(\omega\xi)_n$ is an idempotent. Note also that $(\omega\xi)_n OPol_n = OSym_n$. To see this, every ∂_i annihilates $(\omega\xi_n) \cdot OPol_n$, so

$(\omega\xi)_n \cdot OPol_n \subseteq OSym_n$ thanks to Corollary 5.5, and it is easy to see directly that $(\omega\xi)_n \cdot f = f$ for any $f \in OSym_n$ giving the other containment. Thus, we have shown that

$$(\xi\omega)_n \cdot OPol_n = \xi_n OSym_n, \qquad (\omega\xi)_n \cdot OPol_n = OSym_n. \tag{5.30}$$

For $n \geq 2$, $(\omega\xi)_n$ is *not* the idempotent corresponding to the matrix unit $e_{1,1}$ in the matrix algebra $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}(OSym_n)$, i.e., it is a projection of $OPol_n$ onto $OSym_n$, but along a different direct sum decomposition to (5.23). It is convenient to work with since left multiplication by ω_n defines a homogeneous isomorphism $(\xi\omega)_n \cdot OPol_n \xrightarrow{\sim} (\omega\xi)_n \cdot OPol_n$, with inverse defined by left multiplication by ξ_n.

Lemma 5.8. *We have that* $ONH_n \simeq \bigoplus_{w\in S_n}(\Pi Q^2)^{\ell(w)}\,(\omega\xi)_n ONH_n \simeq \bigoplus_{w\in S_n}(\Pi Q^2)^{-\ell(w)}\,(\xi\omega)_n ONH_n$ *as a graded right* ONH_n*-supermodule.*

Proof. Left multiplication by ω_n defines an isomorphism $(\xi\omega)_n ONH_n \simeq (\Pi Q^2)^{\binom{n}{2}}(\omega\xi)_n ONH_n$ with inverse given by left multiplication by ξ_n. Therefore it suffices to prove the first isomorphism. Since (5.25) is a graded superequivalence, we can apply it to reduce the problem to proving that

$$OPol_n \simeq \bigoplus_{w\in S_n}(\Pi Q^2)^{\ell(w)}\,OSym_n$$

as graded right $OSym_n$-supermodules, where we have used that $(\omega\xi)_n OPol_n = OSym_n$ by (5.30). This follows from (5.23). □

Lemma 5.9. *The map* $\imath : OPol_n \to ONH_n(\omega\xi)_n, f \mapsto f(\omega\xi)_n$ *is an even degree 0 isomorphism of graded left* ONH_n*-supermodules. The map* $\jmath : OSym_n \to (\omega\xi)_n ONH_n(\omega\xi)_n$ *defined by the composition of the natural inclusion of* $OSym_n$ *into* ONH_n *followed by the projection* $a \mapsto (\omega\xi)_n a(\omega\xi)_n$ *is a graded superalgebra isomorphism. Moreover, we have that* $\imath(fa) = \imath(f)\jmath(a)$ *for all* $f \in OPol_n$ *and* $a \in OSym_n$*.*

Proof. Since $\tau_j(\omega\xi)_n = 0$ by degree considerations, there is a unique graded left ONH_n-supermodule homomorphism $OPol_n \to ONH_n(\omega\xi)_n$ taking 1 to $(\omega\xi)_n$ thanks to (5.21). This is $\imath$. Also $(\omega\xi)_n \cdot 1 = 1$, so there is a supermodule homomorphism $ONH_n(\omega\xi)_n \to OPol_n, a \mapsto a \cdot 1$. These two maps are mutual inverses, hence, $\imath$ is an isomorphism.

The restriction of $\imath$ gives an isomorphism $(\omega\xi)_n \cdot OPol_n \xrightarrow{\sim} (\omega\xi)_n OPol_n(\omega\xi)_n$. Since $(\omega\xi)_n a(\omega\xi)_n = a(\omega\xi)_n$ for $a \in OSym_n$ and $(\omega\xi)_n \cdot OPol_n = OSym_n$ by (5.30), this restriction is the isomorphism $\jmath$ from the statement of the lemma. □

Corollary 5.10. *Using* $\jmath$ *to identify* $OSym_n$ *with* $(\omega\xi)_n ONH_n(\omega\xi)_n$*, the superfunctors* $-\otimes_{ONH_n} OPol_n$ *and* $\mathrm{Hom}_{ONH_n}(OPol_n, -)$ *from (5.25) and (5.26) are isomorphic to the idempotent truncation functors defined by right and left multiplication by the idempotent* $(\omega\xi)_n$*, respectively.*

We note finally that there is also a *right* action of ONH_n on $OPol_n$, and everything in this section could be reformulated in terms of this viewed as an $(OSym_n, ONH_n)$-superbimdule. This right action may be defined succinctly from

$$f \cdot a := (-1)^{\mathrm{par}(a)\,\mathrm{par}(f)}(a^* \cdot f^*)^* \tag{5.31}$$

for $f \in OPol_n, a \in ONH_n$. The following more explicit description similar to (5.9) can easily be derived from this:

$$x_i \cdot \tau_j = \delta_{i,j} - \delta_{i,j+1}, \qquad (fg)\cdot\tau_j = f(g\cdot\tau_j) + (f\cdot\tau_j)\,({}^{s_j}g)\,. \tag{5.32}$$

for $f, g \in OPol_n$. The right action of ONH_n on $OPol_n$ obviously commutes with the natural action of $OSym_n$ by left multiplication. Theorem 5.4 and (5.29) imply that

$$OPol_n \simeq \bigoplus_{w\in S_n}(\Pi Q^2)^{\ell(w)} OSym_n \tag{5.33}$$

as a graded left $OSym_n$-supermodule, with the "bottom" summand that is $OSym_n$ itself being the image of the idempotent $(\xi\omega)_n$ acting on the right. We stress that the action (5.31) is different from the right action defined via $f \cdot a := (-1)^{\mathrm{par}(a)\,\mathrm{par}(f)} a^* \cdot f$; the latter action does not commute with the left action of $OSym_n$. When we talk about $OPol_n$ as a right ONH_n-supermodule, we always mean the action defined via (5.31).

Lemma 5.11. $(t + x_2)^{-1} x_1^r \cdot \tau_1 = (t + x_2)^{-1} x_2^r \big(t + (-1)^r x_1\big)^{-1} + \sum_{q=0}^{r-1} (-1)^{qr} (t + x_2)^{-1} x_2^q x_1^{r-q-1}$.

Proof. Similarly to (5.10) and (5.11), one shows that $(t + x_2)^{-1} \cdot \tau_1 = (t + x_2)^{-1}(t + x_1)^{-1}$ and $x_1^r \cdot \tau_1 = \sum_{q=0}^{r-1} x_1^{r-q-1} x_2^q$. These combine using (5.32) to give the final formula; one also needs to commute all x_2 to the left of all x_1 producing some additional signs. □

6. Odd Schur polynomials

Another important basis of $OSym$ is introduced in [EK, Sec. 3.3]: the basis of *odd Schur functions* $\{s_\lambda \mid \lambda \in \Lambda^+\}$. As explained after [EK, Cor. 3.9], this is the basis of $OSym$ characterized uniquely by the properties that $(s_\lambda, h_\mu)^- = 0$ if $\mu >_{\text{lex}} \lambda$ and $s_\lambda = h_\lambda$+(a $\mathbb{Z}$-linear combination of other h_μ for $\mu >_{\text{lex}} \lambda$). Some examples can be found in the appendix of [EK]. The key property of odd Schur functions is that they are signed-orthonormal:

Theorem 6.1 ([EK, Cor. 3.9])**.** *For $\lambda, \mu \in \Lambda^+$, we have that $(s_\lambda, s_\mu)^- = (-1)^{dN(\lambda)} \delta_{\lambda,\mu}$.*

The *odd Kostka matrix* $(K_{\lambda,\mu})_{\lambda,\mu\in\Lambda^+}$ is the transition matrix defined from

$$h_\mu = \sum_{\lambda\in\Lambda^+} K_{\lambda,\mu} s_\lambda. \tag{6.1}$$

There is an explicit formula for the entries of this matrix derived in [EK, (3.7)], as follows. For a λ-tableau T (=a function from the Young diagram of λ to $\mathbb{Z}$), we let $N(T)$ be the number of pairs of boxes (A, B) such that B is strictly north of A and also $T(B) \geq T(A)$. For example, if T is the unique semistandard λ-tableau of content λ (so all entries on row i are equal to i) then $N(T) = 0$. Then

$$K_{\lambda,\mu} = \sum_T (-1)^{N(T)} \tag{6.2}$$

summing over semistandard λ-tableaux T of content μ. Note from this description that $K_{\lambda,\mu} = 0$ unless $\lambda \geq \mu$ in the dominance order. So we actually have that

$$s_\lambda = h_\lambda + (\text{a } \mathbb{Z}\text{-linear combination of other } h_\mu \text{ for } \mu > \lambda) \tag{6.3}$$

in the dominance rather than merely lexicographic ordering.

Since the involution $\psi_1\psi_2$ in [EK] is our ψ by Remark 4.3, [EK, Lem. 3.11] shows that

$$\psi(s_\lambda) = (-1)^{NE(\lambda)+|\lambda|} s_{\lambda^t} \tag{6.4}$$

for any $\lambda \in \Lambda^+$. Hence, applying ψ to (6.3), we have that

$$s_{\lambda^t} = (-1)^{NE(\lambda)} e_\lambda + (\text{a } \mathbb{Z}\text{-linear combination of other } e_\mu \text{ for } \mu > \lambda). \tag{6.5}$$

From (6.3) and (6.5), we see in particular that

$$s_{(r)} = h_r, \qquad s_{(1^r)} = e_r. \tag{6.6}$$

Using also (4.30) and (6.4), Theorem 6.1 implies:

Corollary 6.2. *For $\lambda, \mu \in \Lambda^+$, we have that $(s_\lambda, s_\mu)^+ = (-1)^{dE(\lambda)} \delta_{\lambda,\mu}$.*

Applying (6.4) one more time, this time combined with the first identity from (4.30), it follows that s_λ can also be characterized as the unique element of $OSym$ such that $(s_\lambda, e_\mu)^+ = 0$ for $\mu >_{\text{lex}} \lambda^{\mathtt{t}}$ and $s_\lambda = (-1)^{NE(\lambda)} e_{\lambda^{\mathtt{t}}} +$(a $\mathbb{Z}$-linear combination of e_μ for $\mu >_{\text{lex}} \lambda^{\mathtt{t}}$). This characterization plus Lemma 4.2 and the second identity from (4.30) implies that

$$\gamma(s_\lambda)^* = (-1)^{dN(\lambda)+dE(\lambda)} s_\lambda. \tag{6.7}$$

We define the *dual odd Schur function*

$$\sigma_\lambda := \gamma(s_\lambda) = (-1)^{dN(\lambda)+dE(\lambda)} s_\lambda^*. \tag{6.8}$$

In particular, applying γ to (6.6) and using the definitions (4.40) and (4.41), we have that

$$\sigma_{(r)} = \eta_r, \qquad \sigma_{(1^r)} = \varepsilon_r. \tag{6.9}$$

The *odd Schur polynomial* $s_\lambda^{(n)}$ and the *dual odd Schur polynomial* $\sigma_\lambda^{(n)}$ are the images of s_λ and σ_λ under the quotient map $\pi_n : OSym \to OSym_n$, respectively. The dual odd Schur polynomials coincide with the polynomials introduced in [EKL, Def. 4.10] and play an important role in Theorem 6.12 below.

Theorem 6.3. *The set $\{s_\lambda^{(n)} \mid \lambda \in \Lambda_n^+\}$ is a basis for $OSym_n$. Moreover, for any $\lambda \in \Lambda^+$, we have that*

$$s_\lambda^{(n)} = \begin{cases} x^\lambda + (\text{a } \mathbb{Z}\text{-linear combination of } x^\kappa \text{ for } \kappa \in \mathbb{N}^n \text{ with } \kappa < \lambda) & \text{if } \operatorname{ht}(\lambda) \leq n \\ 0 & \text{if } \operatorname{ht}(\lambda) > n. \end{cases} \tag{6.10}$$

Proof. This follows from (6.5) and Theorem 4.5 plus (4.39). □

Corollary 6.4. *The set $\{\sigma_\lambda^{(n)} \mid \lambda \in \Lambda_n^+\}$ is a basis for $OSym_n$. Moreover, $\sigma_\lambda^{(n)} = 0$ for $\lambda \in \Lambda^+$ with $\operatorname{ht}(\lambda) > n$.*

Proof. Apply γ_n to the results established in the theorem. □

Corollary 6.5. *The set $\{h_\lambda^{(n)} \mid \lambda \in \Lambda_n^+\}$ is a basis for $OSym_n$.*

Proof. By graded dimension considerations, it suffices to show that $\{h_\lambda^{(n)} \mid \lambda \in \Lambda_n^+\}$ spans $OSym_n$. This follows from the theorem using (6.3) and also the observation that $\mu > \lambda \in \Lambda_n^+ \Rightarrow \mu \in \Lambda_n^+$. □

The next result was originally formulated as a conjecture in [EKL, Conj. 5.3], and the conjecture was proved in [E, Th. 3.8]. However, we also need to reformulate it using our sign conventions, and for this we need a preliminary lemma.

Lemma 6.6. *For $f \in OSym_{n-1}$, $m \geq 0$ and $k = 1, \dots, n$, we have that*

$$\tau_{k-1} \cdots \tau_1 x_1^{m+k-1} \cdot \operatorname{sh}_1(f) = \sum_{i=0}^{k-1} \operatorname{sh}_i (h_{m+i}^{(k-i)}) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f),$$

equality in the ONH_n-supermodule $OPol_n$.

Proof. We prove this by induction on k, the case $k = 1$ being trivial. For the induction step, we have by induction that $\tau_{k-1} \cdots \tau_1 x_1^{m+k} \cdot \operatorname{sh}_1(f) = \sum_{i=0}^{k-1} \operatorname{sh}_i (h_{m+i+1}^{(k-i)}) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f)$. Applying τ_k to both sides, we deduce that

$$\tau_k \cdots \tau_1 x_1^{m+k} \cdot \operatorname{sh}_1(f) = \sum_{i=0}^{k-1} \left(\tau_k \cdot \operatorname{sh}_i (h_{m+i+1}^{(k-i)})\right) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f) + \sum_{i=0}^{k-1} {}^{s_k} \operatorname{sh}_i (h_{m+1+i}^{(k-i)}) \tau_k \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f).$$

By (5.10), we have that $\tau_{k-i} \cdot h_{m+i+1}^{(k-i)} = h_{m+i}^{(k-i+1)}$, so the ith term in the first summation becomes

$$\operatorname{sh}_i (\tau_{k-i} \cdot h_{m+i+1}^{(k-i)}) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f) = \operatorname{sh}_i (h_{m+i}^{(k+1-i)}) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f).$$

The second summation gives zero except when $i = k - 1$, when it gives $\operatorname{sh}_k (h_{m+k}^{(1)}) \tau_k \cdots \tau_1 \cdot \operatorname{sh}_1(f)$. In total, we obtain the desired $\sum_{i=0}^{k} \operatorname{sh}_i (h_{m+i}^{(k+1-i)}) \tau_i \cdots \tau_1 \cdot \operatorname{sh}_1(f)$. □

Recall for $\lambda \in \mathbb{N}^n$ that x^λ denotes $x_1^{\lambda_1} \cdots x_n^{\lambda_n}$.

Theorem 6.7 ([E, Th. 3.8])**.** *For $\lambda \in \Lambda_n^+$, we have that $s_\lambda^{(n)} = (\omega\xi)_n \cdot x^\lambda$.*

Proof. The original formula from [EKL] took the form

$$s_\lambda = (-1)^{\binom{n}{3}} \left[\partial_{w_0}\left(x_1^{\lambda_1} \cdots x_n^{\lambda_n} x_1^{n-1} \cdots x_{n-2}^2 x_{n-1}\right)\right]^{w_0}, \tag{6.11}$$

using their notation everywhere. The result was proved in [E] with exactly this in place of our $(\omega\xi)_n \cdot x^\lambda$. In (6.11), the conjugation by w_0 corresponds up to a sign to an application of our involution γ_n, which commutes with the action of ∂_{w_0}, again up to a sign, due to (5.17) and (5.19). This shows that the right hand side of (6.11) is equal to $(\omega\xi)_n \cdot x^\lambda$ up to a sign. Hence, $(\omega\xi)_n \cdot x^\lambda = \pm s_\lambda^{(n)}$. Presumably, one could see that the sign is actually a plus by carefully keeping track of all of the sign changes in this translation. However, this is rather prone to error, so we give an alternative approach. It suffices by (6.10) to check that the x^λ-coefficient of $(\omega\xi)_n \cdot x^\lambda$ is 1. From (5.18), we have that $\omega_n = \tau_{n-1} \cdots \tau_1 \operatorname{sh}_1(\omega_{n-1})$ and $\xi_n = \operatorname{sh}_1(\xi_{n-1}) x_1^{n-1}$. Also $x^\lambda = x_1^{\lambda_1} \operatorname{sh}_1(x^\mu)$ where $\mu = (\lambda_2, \dots, \lambda_n)$. Using these and induction on n, we get that

$$\begin{aligned}(\omega\xi)_n \cdot x^\lambda &= \tau_{n-1} \cdots \tau_1 x_1^{\lambda_1+n-1} \cdot \operatorname{sh}_1\left(\omega_{n-1}\xi_{n-1} \cdot x^\mu\right) \\ &= \tau_{n-1} \cdots \tau_1 x_1^{\lambda_1+n-1} \cdot \operatorname{sh}_1\left(s_\mu^{(n-1)}\right) = \sum_{i=0}^{n-1} \operatorname{sh}_i\left(h_{\lambda_1+i}^{(n-i)}\right)\tau_i \cdots \tau_1 \cdot \operatorname{sh}_1\left(s_\mu^{(n-1)}\right),\end{aligned}$$

the last equality being an application of Lemma 6.6. Now we express this in terms of the monomial basis for $OPol_n$. The only place a monomial whose x_1-exponent is $\geq \lambda_1$ can arise is from the $i = 0$ term, which is $h_{\lambda_1}^{(n)} \operatorname{sh}_1\left(s_\mu^{(n-1)}\right)$. This has leading term exactly x^λ, as required. □

Now we are going to discuss a graded superalgebra which may be interpreted as the odd analog of the equivariant cohomology algebra of the Grassmannian. We set things up initially in greater generality. Switching our default choice of variable from n to ℓ for reasons that will become clear shortly, suppose that $\alpha \in \Lambda(k, \ell)$. This represents the "shape" of a partial flag variety, Grassmannians being the special case that $k = 2$. Let

$$OSym_\alpha := \bigcap_{\substack{i \in \{1,\dots,\ell\} \\ i \notin \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\cdots+\alpha_k\}}} \ker \partial_i = \bigcap_{\substack{i \in \{1,\dots,\ell\} \\ i \notin \{\alpha_1, \alpha_1+\alpha_2, \dots, \alpha_1+\cdots+\alpha_k\}}} \operatorname{im} \partial_i, \tag{6.12}$$

which is a subalgebra of $OPol_\ell$ containing $OSym_\ell$. We think of $OSym_\alpha$ as being the odd analog of the ring of "partial" invariants $\mathbb{F}[x_1, \dots, x_\ell]^{S_\alpha}$. For example, we have that $OSym_\alpha = OSym_\ell$ if $\alpha = (\ell)$, and $OSym_\alpha = OPol_\ell$ if $\alpha = (1^\ell)$. Note also that the superalgebra anti-involution $*$ of $OPol_\ell$ leaves $OSym_\alpha$ invariant, whereas the involution γ_ℓ takes $OSym_\alpha$ to $OSym_{w_k(\alpha)}$ where $w_k(\alpha) = (\alpha_k, \dots, \alpha_1)$ is the reversed composition.

Consider the following diagram:

$$\begin{array}{ccccc} OSym & \xrightarrow{\qquad\Delta_k^+\qquad} & & & \overbrace{OSym \otimes \cdots \otimes OSym}^{k \text{ times}} \\ \downarrow{\scriptstyle \pi_\ell} & & & & \downarrow{\scriptstyle \pi_{\alpha_1} \otimes \cdots \otimes \pi_{\alpha_k}} \\ OSym_\ell & \hookrightarrow & OSym_\alpha & = & OSym_{\alpha_1} \otimes \cdots \otimes OSym_{\alpha_k} \\ \downarrow & & & & \downarrow \\ OPol_\ell & & = & & OPol_{\alpha_1} \otimes \cdots \otimes OPol_{\alpha_k} \end{array} \tag{6.13}$$

The top horizontal map Δ_k^+ is the $(k-1)$th iteration of the comultiplication $\Delta^+ : OSym \to OSym \otimes OSym$. The bottom equality is the canonical identification explained just after (2.3), and the outside square commutes thanks to (4.38). In view of Corollary 5.5, the subalgebra $OSym_\alpha$ of $OPol_\ell$ is identified with the subalgebra $OSym_{\alpha_1} \otimes \cdots \otimes OSym_{\alpha_k}$ of $OPol_{\alpha_1} \otimes \cdots \otimes OPol_{\alpha_k}$. This shows that the natural inclusion of $OSym_\ell$ into $OSym_\alpha$ is induced by the comultiplication Δ^+.

Recall that w_α is the longest element of S_α and w^α is the longest element of $[S_\ell/S_\alpha]_{\min}$, so that $w_\ell = w^\alpha w_\alpha$. Noting that $w_\alpha = w_{\alpha_1}\,\mathrm{sh}_{\alpha_1}(w_\beta)$ where $\beta := (\alpha_2, \dots, \alpha_k)$, we recursively define

$$\omega_\alpha := \omega_{\alpha_1}\,\mathrm{sh}_{\alpha_1}(\omega_\beta)\ \ (= \pm\tau_{w_\alpha}), \qquad\qquad \xi_\alpha := \mathrm{sh}_{\alpha_1}(\xi_\beta)\xi_{\alpha_1}. \tag{6.14}$$

We get from Lemma 5.1 and induction on k that

$$\omega_\alpha \cdot \xi_\alpha = 1 \tag{6.15}$$

for any α. The following identity is proved in the same way as (5.27):

$$\omega_\alpha \xi_\alpha \omega_\alpha = \omega_\alpha. \tag{6.16}$$

Similarly to (5.28), it follows that the elements

$$(\xi\omega)_\alpha := \xi_\alpha\omega_\alpha, \qquad\qquad (\omega\xi)_\alpha := \omega_\alpha\xi_\alpha \tag{6.17}$$

are primitive idempotents in ONH_α such that

$$(\xi\omega)_\alpha \cdot OPol_\ell = \xi_\alpha OSym_\alpha, \qquad\qquad (\omega\xi)_\alpha \cdot OPol_\ell = OSym_\alpha. \tag{6.18}$$

Also let $\omega^\alpha := \pm\tau_{w^\alpha}$ for the particular sign chosen so that

$$\omega_\ell = \omega^\alpha \omega_\alpha \tag{6.19}$$

and let

$$\xi^\alpha := \omega_\alpha \cdot \xi_\ell \in OSym_\alpha. \tag{6.20}$$

We have that

$$\omega^\alpha \cdot \xi^\alpha = 1, \qquad\qquad \xi_\ell = \xi_\alpha\,\xi^\alpha. \tag{6.21}$$

The first of these equalities follows because $\omega^\alpha \cdot \xi^\alpha = \omega^\alpha\omega_\alpha \cdot \xi_\ell = \omega_\ell \cdot \xi_\ell = 1$ by (6.19) and Lemma 5.1. To establish the second, one first checks that $\xi_\alpha\xi^\alpha = \pm\xi_\ell$ for some choice of sign, and the sign is plus because $\omega_\ell \cdot \xi_\alpha\xi^\alpha = \omega^\alpha\omega_\alpha \cdot \xi_\alpha\xi^\alpha = \omega^\alpha \cdot \xi^\alpha = 1 = \omega_\ell \cdot \xi_\ell$. Finally, we have that

$$\omega_\alpha\xi_\alpha\omega_\ell = \omega_\ell = \omega_\ell\xi_\alpha\omega_\alpha. \tag{6.22}$$

This follows because all three expressions act in the same way on $\xi_\ell \in OPol_\ell$ due to Lemma 5.1 and (6.15), (6.20) and (6.21).

Theorem 6.8. *For $\alpha \in \Lambda(k, \ell)$, the graded superalgebra $OSym_\alpha$ is free as a right $OSym_\ell$-supermodule with basis $\{p_w^{(\ell)} \mid w \in [S_\ell/S_\alpha]_{\min}\}$. Each $p_w^{(\ell)}$ in this basis belongs to the subalgebra $OSym_\alpha \cap OPol_{\ell-\alpha_k}$.*

Proof. By Corollary 4.8 and (3.8), we have that

$$\frac{\dim_{q,\pi} OSym_\alpha}{\dim_{q,\pi} OSym_\ell} = \frac{q^{\binom{\ell}{2}}[\ell]^!_{q,\pi}}{\prod_{i=1}^k q^{\binom{\alpha_i}{2}}[\alpha_i]^!_{q,\pi}} = q^{N(\alpha)}\begin{bmatrix}\ell\\ \alpha\end{bmatrix}_{q,\pi} = \sum_{w\in[S_n/S_\alpha]_{\min}} (\pi q^2)^{\ell(w)}. \tag{6.23}$$

This is the graded rank of a free graded right $OSym_\ell$-supermodule with basis $\{p_w^{(\ell)} \mid w \in [S_\ell/S_\alpha]_{\min}\}$. So, to prove the theorem, it just remains to show that the elements $p_w^{(\ell)}$ $(w \in [S_\ell/S_\alpha]_{\min})$ belong to $OSym_\alpha \cap OPol_{\ell-\alpha_k}$ and are linearly independent over $OSym_\ell$. The linear independence is immediate from Theorem 5.4.

To show that $p_w^{(\ell)} \in OSym_\alpha$, we need to show that $\partial_i(p_w^{(\ell)}) = 0$ for all i such that $s_i \in S_\alpha$. We have that $w^{-1}w_\ell = w_\alpha w'$ for some $w' \in [S_\alpha\backslash S_\ell]_{\min}$. So $p_w^{(\ell)} = \tau_{w^{-1}w_\ell} \cdot \xi_\ell = \pm\omega_\alpha\tau_{w'} \cdot \xi_\ell$. Since $\ell(s_iw_\alpha) < \ell(w_\alpha)$ when $s_i \in S_\alpha$, the relations in ONH_ℓ now imply that $\tau_i \cdot p_w^{(\ell)} = 0$.

To show that $p_w^{(\ell)} \in OPol_{\ell-\alpha_k}$, we again use $w^{-1}w_\ell = w_\alpha w'$ to deduce that $\tau_{w^{-1}w_\ell} = \pm\, \mathrm{sh}_{\ell-\alpha_k}(\omega_{\alpha_k})\tau_{w''}$ for some $w'' \in S_\ell$. By the argument explained in the last sentence of Remark 5.6, it follows that $p_w^{(\ell)}$ is a linear combination of terms of the form $\mathrm{sh}_{\ell-\alpha_k}(\omega_{\alpha_k}) \cdot x^\kappa$ for $\kappa \in \mathbb{N}^\ell$ with $0 \le \kappa_i \le \ell - i$ for all i. It is now clear that $p_w^{(\ell)} \in OPol_{\ell-\alpha_k}$ since $\mathrm{sh}_{\ell-\alpha_k}(\omega_{\alpha_k}) \cdot x_\ell^{\kappa_\ell} \cdots x_{\ell-\alpha_k+1}^{\kappa_{\ell-\alpha_k+1}}$ is a scalar by Lemma 5.1 and degree considerations. □

Corollary 6.9. *Suppose that $\alpha \in \Lambda(k,\ell)$ for $k \ge 1$.*

(1) The graded superalgebra $OSym_\alpha$ is a free as a graded right $OSym_{(\alpha_1,\ell-\alpha_1)}$-supermodule with basis $\left\{ \mathrm{sh}_{\alpha_1}(p_w^{(\ell-\alpha_1)}) \;\middle|\; w \in (S_{\ell-\alpha_1}/S_{(\alpha_2,\dots,\alpha_k)})_{\min}\right\}$.

(2) The graded superalgebra $OSym_\alpha$ is free as a graded right $OSym_{(\ell-\alpha_k,\alpha_k)}$-supermodule with basis $\left\{\gamma_{\ell-\alpha_k}(p_w^{(\ell-\alpha_k)}) \;\middle|\; w \in (S_{\ell-\alpha_k}/S_{(\alpha_{k-1},\dots,\alpha_1)})_{\min}\right\}$.

All vectors in the bases described in (1)–(2) belong to the subalgebra $\mathrm{sh}_{\alpha_1}(OSym_{(\alpha_2,\dots,\alpha_{k-1})})$.

Proof. (1) This follows immediately from the theorem.

(2) This follows by applying the involution γ_ℓ to the the result from (1) with α replaced by the reverse composition α^{r}. □

Continuing with $\alpha \in \Lambda(k,\ell)$, we need a few more pieces of notation. For $i = 1,\dots,k$, we define

$$h_r^{(\alpha;i)} := \mathrm{sh}_{\alpha_1+\cdots+\alpha_{i-1}}(h_r^{(\alpha_i)}), \qquad e_r^{(\alpha;i)} := \mathrm{sh}_{\alpha_1+\cdots+\alpha_{i-1}}(e_r^{(\alpha_i)}). \tag{6.24}$$

Under the identication of $OSym_\alpha$ with $OSym_{\alpha_1}\otimes\cdots\otimes OSym_{\alpha_k}$ from (6.13), these are $1^{\otimes(i-1)}\otimes h_r^{(\alpha_i)}\otimes 1^{\otimes(k-i)}$ and $1^{\otimes(i-1)} \otimes e_r^{(\alpha_i)} \otimes 1^{\otimes(k-i)}$, respectively. We use similar notation for other elements of $OSym_\alpha$ such as $e_\lambda^{(\alpha;i)}$, $h_\lambda^{(\alpha;i)}$ and $s_\lambda^{(\alpha;i)}$ for $\lambda \in \Lambda^+$. From (4.36) and (4.37), we get that

$$e_r^{(\ell)} = \sum_{\substack{r_1,\dots,r_k\ge 0\\ r_1+\cdots+r_k=r}} e_{r_1}^{(\alpha;1)}\cdots e_{r_k}^{(\alpha;k)}, \qquad h_r^{(\ell)} = \sum_{\substack{r_1,\dots,r_k\ge 0\\ r_1+\cdots+r_k=r}} h_{r_k}^{(\alpha;k)}\cdots h_{r_1}^{(\alpha;1)}. \tag{6.25}$$

These are more convenient when written in terms of the generating functions

$$e^{(\alpha;i)}(t) := \sum_{r=0}^{\alpha_i}(-1)^r e_r^{(\alpha;i)} t^{\alpha_i-r}, \qquad h^{(\alpha;i)}(t) := \sum_{r\ge 0} h_r^{(\alpha;i)} t^{-\alpha_i-r}. \tag{6.26}$$

Now the identities (6.25) become

$$e^{(\ell)}(t) := e^{(\alpha;1)}(t)e^{(\alpha;2)}(t)\cdots e^{(\alpha;k)}(t), \qquad h^{(\ell)}(t) := h^{(\alpha;k)}(t)\cdots h^{(\alpha;2)}(t)h^{(\alpha;1)}(t). \tag{6.27}$$

These identities, which generalize (4.47) and (4.48), together with the infinite Grassmannian relation (4.49) are useful when moving between different families of generators, as illustrated by the following lemma.

Lemma 6.10. *Suppose that $\ell = n + n'$ and $r \ge 0$.*

(1) We have that $\sum_{s=0}^{r}(-1)^s h_{r-s}^{(n)} e_s^{(\ell)} = (-1)^r\, \mathrm{sh}_n(e_r^{(n')})$, which is zero for $r > n'$.

(2) We have that $\sum_{s=0}^{r}(-1)^s e_s^{(\ell)}\, \mathrm{sh}_n(h_{r-s}^{(n')}) = (-1)^r e_r^{(n)}$, which is zero for $r > n$.

Proof. (1) The first identity from (6.27) when $\alpha = (n,n')$ plus (4.49) gives that

$$\mathrm{sh}_n(e^{(n')}(t)) = h^{(n)}(t)e^{(\ell)}(t).$$

Now equate the coefficients of $t^{n'-r}$ on both sides.

(2) Similar. □

Lemma 6.11. *Suppose that $n \geq 0$. The following hold in $OPol_{n+1}$ for any $m \geq 0$:*

(1) $x_{n+1}^{m+n+1} = -\sum_{q=0}^{n}\left(\sum_{r=0}^{m}(-1)^{m+n+1-q-r}h_r^{(n+1)}e_{m+n+1-q-r}^{(n+1)}\right)x_{n+1}^q$;

(2) $x_1^{m+n+1} = -\sum_{p=0}^{n}x_1^p\left(\sum_{s=0}^{m}(-1)^{m+n+1-p-s}e_{m+n+1-p-s}^{(n+1)}h_s^{(n+1)}\right)$.

Proof. (1) Induction on m. The base case $m = 0$ follows as $\sum_{q=0}^{n+1}(-1)^{n+1-q}e_{n+1-q}^{(n+1)}x_{n+1}^q = 0$ due to Lemma 6.10(2) with ℓ, n, n' and r replaced with $n+1, n, 1$ and $n+1$, respectively. Now take the identity we are trying to prove for some $m \geq 0$ and multiply on the right by x_{n+1} to obtain

$$x_n^{m+1+n+1} = -\sum_{q=0}^{n}\left(\sum_{r=0}^{m}(-1)^{m+n+1-q-r}h_r^{(n+1)}e_{m+n+1-q-r}^{(n+1)}\right)x_{n+1}^{q+1}. \tag{6.28}$$

The $q = n$ term of the summation here is $-\sum_{r=0}^{m}(-1)^{m+1-r}h_r^{(n+1)}e_{m+1-r}^{(n+1)}x_{n+1}^{n+1}$, which equals $h_{m+1}^{(n+1)}x_{n+1}^{n+1}$ by the infinite Grassmannian relation (4.11). Using the $m = 0$ case of the identity we are proving this can then be rewritten as

$$-\sum_{q=0}^{n}(-1)^{n+1-q}h_{m+1}^{(n+1)}e_{n+1-q}^{(n+1)}x_{n+1}^q.$$

For the terms of (6.28) with $0 \leq q \leq n-1$, we reindex the summation replacing q by $q-1$ to obtain

$$-\sum_{q=1}^{n}\left(\sum_{r=0}^{m}(-1)^{m+1+n+1-q-r}h_r^{(n+1)}e_{m+1+n+1-q-r}^{(n+1)}\right)x_{n+1}^q.$$

The expression in brackets is zero for $q = 0$ since $e_{m+1+n+1-r}^{(n+1)} = 0$ for all $0 \leq r \leq m$, so we can sum instead from $q = 0$ to n. Thus, we have shown that

$$\begin{aligned}
x_{n+1}^{m+1+n+1} &= -\sum_{q=0}^{n}(-1)^{n+1-q}h_{m+1}^{(n+1)}e_{n+1-q}^{(n+1)}x_{n+1}^q - \sum_{q=0}^{n}\left(\sum_{r=0}^{m}(-1)^{m+1+n+1-q-r}h_r^{(n+1)}e_{m+1+n+1-q-r}^{(n+1)}\right)x_{n+1}^q\\
&= -\sum_{q=0}^{n}\left((-1)^{n+1-q}h_{m+1}^{(n+1)}e_{n+1-q}^{(n+1)} + \sum_{r=0}^{m}(-1)^{m+1+n+1-q-r}h_r^{(n+1)}e_{m+1+n+1-q-r}^{(n+1)}\right)x_{n+1}^q\\
&= -\sum_{q=1}^{n+1}\left(\sum_{r=0}^{m+1}(-1)^{m+1+n+1-q-r}h_r^{(n+1)}e_{m+1+n+1-q-r}^{(n+1)}\right)x_{n+1}^q.
\end{aligned}$$

This is just what is needed for the induction step.

(2) This is similar, or may be proved by applying $\gamma_{n+1} \circ *$ to (1). □

Now we focus on the most important case $k = 2$, so $\alpha = (n, n') \in \Lambda(2, \ell)$ for some $n, n' \geq 0$. Then $\xi_\alpha = \xi_{(n,n')} = \mathrm{sh}_n(\xi_{n'})\xi_n$.

Theorem 6.12. *Suppose that $\ell = n + n'$ for $n, n' \geq 0$. Then $OSym_{(n,n')}$ has the following two bases as a free graded right $OSym_\ell$-supermodule:*

(1) $\left\{s_\lambda^{(n)} \,\middle|\, \lambda \in \Lambda_{n\times n'}^+\right\}$;

(2) $\left\{\mathrm{sh}_n\left(\sigma_\mu^{(n')}\right) \,\middle|\, \mu \in \Lambda_{n'\times n}^+\right\}$.

Also let Tr : $OSym_{(n,n')} \to OSym_\ell$ *be the linear map* $a \mapsto \omega_\ell\, \xi_{(n,n')} \cdot a$. *This map is a homogeneous homomorphism of graded right* $OSym_\ell$*-supermodules of degree* $-2nn'$ *and parity* $nn' \pmod 2$*, and the bases (1)–(2) satisfy*

$$\mathrm{Tr}\left(s^{(n)}_\lambda\, \mathrm{sh}_n\,(\sigma^{(n')}_\mu)\right) = \begin{cases} \mathrm{sgn}(\mu) & \text{if } \mu^{\mathrm{t}}_i = n' - \lambda_{n+1-i} \text{ for } i = 1,\dots,n \\ 0 & \text{otherwise} \end{cases} \tag{6.29}$$

for $\mathrm{sgn}(\mu) \in \{\pm 1\}$ *with* $\mathrm{sgn}(\varnothing) = 1$*; see Corollary 7.6 below for a formula for* $\mathrm{sgn}(\mu)$ *for general* μ.

Proof. The main work here is to prove (6.29). This turns out to be significantly harder than the analogous formula in the ordinary even theory; see [EKL, Rem. 4.12] for an illuminating example. Fortunately, the details are already worked out in [EKL, Prop. 4.11] up to an undetermined sign since our conventions are different. To keep track of this sign, we repeat the first few steps of the proof in [EKL] in our set up. By Theorem 6.7 and (5.17), (5.19), (6.8) and (6.22), we have that

$$\begin{aligned}\mathrm{Tr}\left(s^{(n)}_\lambda\, \mathrm{sh}_n\,(\sigma^{(n')}_\mu)\right) &= \omega_\ell\, \xi_{(n,n')} \cdot s^{(n)}_\lambda\, \mathrm{sh}_n\left(\gamma_{n'}(s^{(n')}_\mu)\right)\\ &= \omega_\ell\, \xi_{(n,n')} \cdot (\omega_n \xi_n \cdot x^\lambda)\ \mathrm{sh}_n\,(\gamma_{n'}(\omega_{n'}\xi_{n'} \cdot x^\mu))\\ &= \zeta_{n'}\omega_\ell\, \xi_{(n,n')}\omega_{(n,n')} \cdot\ \mathrm{sh}_n\,(\gamma_{n'}(\xi_{n'}))\xi_n x^\lambda\, \mathrm{sh}_n\,(\gamma_{n'}(x^\mu))\\ &= \zeta_{n'}\omega_\ell \cdot \mathrm{sh}_n\,(\gamma_{n'}(\xi_{n'}))\xi_n x^\lambda\, \mathrm{sh}_n\,(\gamma_{n'}(x^\mu)).\end{aligned} \tag{6.30}$$

Up to another sign, the monomial appearing after the $\cdot$ in (6.30) is as considered in [EKL, Lem. 4.9], so applying that lemma gives that $\mathrm{Tr}\left(s^{(n)}_\lambda\, \mathrm{sh}_n\,(\sigma^{(n')}_\mu)\right)$ is ± 1 if $\mu^{\mathrm{t}}_i = n' - \lambda_{n+1-i}$ for $i = 1,\dots,n$, and it is zero otherwise. It remains to check that (6.30) equals $+1$ in the special case that $\lambda = (n'^n)$ and $\mu = \varnothing$. To see this, one first checks that $\xi_n x_1^{n'} \cdots x_n^{n'} = x_n^{n'} x_{n-1}^{n'+1} \cdots x_1^{n+n'-1}$. Hence, letting $\omega_\ell = \tau\, \mathrm{sh}_n(\omega_{n'})$ for $\tau \in ONH_\ell$, (6.30) simplifies in this case to give

$$\begin{aligned}\mathrm{Tr}\left(s^{(n)}_{(n'^n)}\right) &= \zeta_{n'}\tau\, \mathrm{sh}_n(\omega_{n'}) \cdot \mathrm{sh}_n\,(\gamma_{n'}(\xi_{n'}))x_n^{n'} x_{n-1}^{n'+1} \cdots x_1^{n+n'-1} = \tau\, \mathrm{sh}_n\,(\gamma_{n'}(\omega_{n'} \cdot \xi_{n'})) \cdot x_n^{n'} x_{n-1}^{n'+1} \cdots x_1^{n'+n-1}\\ &= \tau \cdot x_n^{n'} x_{n-1}^{n'+1} \cdots x_1^{n+n'-1} = \tau\, \mathrm{sh}_n(\omega_{n'}) \cdot \mathrm{sh}_n(\xi_{n'})x_n^{n'} x_{n-1}^{n'+1} \cdots x_1^{n+n'-1} = \omega_\ell \cdot \xi_\ell = 1.\end{aligned}$$

Now (6.29) is proved.

It is clear from the definition that Tr is a homogeneous homomorphism of graded right $OSym_\ell$-supermodules of degree $-2nn'$ and parity $nn' \pmod 2$. It remains to show that the elements (1) and (2) are bases. To see that the elements (1) are linearly independent over $OSym_\ell$, take a linear relation $\sum_\lambda s^{(n)}_\lambda a_\lambda$ for $a_\lambda \in OSym_\ell$. To see that $a_\lambda = 0$ for any given λ, let $\mu \in \Lambda^+$ be defined so that $\mu^{\mathrm{t}}_i = n' - \lambda_{n+1-i}$ for $i = 1,\dots,n$ and $\mu^{\mathrm{t}}_i = 0$ for $i > n$. Then we have using (6.29) that

$$0 = \mathrm{Tr}\Big(\mathrm{sh}_n\,(\sigma^{(n')}_\mu) \sum_{\lambda'} s^{(n)}_{\lambda'} a_{\lambda'}\Big) = \sum_{\lambda'} (-1)^{|\lambda'||\mu|}\, \mathrm{Tr}\left(s^{(n)}_{\lambda'}\, \mathrm{sh}_n\,(\sigma^{(n')}_\mu)\right) a_{\lambda'} = (-1)^{|\lambda||\mu|}\, \mathrm{sgn}(\mu) a_\lambda.$$

This establishes the linear independence. As $s^{(n)}_\lambda$ is of degree $2|\lambda|$ and parity $|\lambda| \pmod 2$, we deduce from Corollary 3.2 that the elements (1) generate a free graded right $OSym_\ell$-supermodule of graded rank $q^{nn'}\begin{bmatrix}\ell\\ n\end{bmatrix}_{q,\pi}$. In view of Theorem 6.8, it follows that this submodule is all of $OSym_{(n,n')}$. This proves that (1) is a basis. A similar argument gives that (2) is a basis too. ∎

Corollary 6.13. *Suppose that* $\ell = n+d+n'$ *for* $n, d, n' \geq 0$. *Then* $OSym_{(n,d,n')}$ *is a free right* $OSym_{(n,n'+d)}$*-supermodule with basis* $\left\{ \mathrm{sh}_n\,(s^{(d)}_\lambda) \,\middle|\, \lambda \in \Lambda^+_{d\times n'} \right\}$, *and it is a free right* $OSym_{(n+d,n')}$*-supermodule with basis* $\left\{ \mathrm{sh}_n\,(\sigma^{(d)}_\lambda) \,\middle|\, \lambda \in \Lambda^+_{d\times n} \right\}$.

7. The odd analog of cohomology of Grassmannians

Continue with $\ell = n + n'$. In the purely even theory, when all of the algebras involved are commutative, the analog of the map Tr from Theorem 6.12 is actually a graded bimodule homomorphism, so that it gives a trace making $Sym_{(n,n')}$ into a graded Frobenius algebra over Sym_ℓ. However, in the odd case, $OSym_{(n,n')}$ is usually *not* a Frobenius extension of $OSym_\ell$, e.g., it is already false in the case $n = n' = 1$ since one can check directly that $OSym_2 < OPol_2$ has no complement as a graded $(OSym_2, OSym_2)$-superbimodule. This is a significant obstruction to the development of the odd theory. At this point in [EKL, Sec. 5], the obstruction is avoided by passing to the finite-dimensional graded superalgebra

$$\overline{OH}_n^\ell := OSym_n \,/\, \langle h_r^{(n)} \,\big|\, r > n' \rangle \;\Big(= OSym \,/\, \langle h_r, e_s \,\big|\, r > n', s > n \rangle \Big). \tag{7.1}$$

This is called the *odd Grassmannian cohomology algebra* since it is an odd analog of the cohomology algebra $H^*(\mathrm{Gr}_n^\ell; \mathbb{F})$ of the Grassmannian Gr_n^ℓ of n-dimensional subspaces of $\mathbb{C}^\ell$.

We denote the image of $a \in OSym_n$ in $\overline{OH}_n^\ell$ by $\bar{a}$. The first part of following theorem is [EKL, Prop. 5.4], but we give a different argument which gives extra information.

Theorem 7.1. *Suppose that $\ell = n + n'$. The odd Schur polynomials $\bar{s}_\lambda^{(n)}$ for $\lambda \in \Lambda^+_{n\times n'}$ give a linear basis for $\overline{OH}_n^\ell$, and all other $\bar{s}_\lambda^{(n)}$ are zero. Moreover, viewing $\mathbb{F}$ as a graded $OSym_\ell$-supermodule in the obvious way, there is a commuting diagram*

$$\begin{array}{ccc}
\overline{OH}_n^\ell & \xrightarrow{\;\bar{a}\mapsto a\otimes 1\;} & OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F} \\
{\scriptstyle \overline{\psi}_n^\ell}\big\downarrow & & \big\uparrow{\scriptstyle 1\otimes a\mapsto a^*\otimes 1} \\
\overline{OH}_{n'}^\ell & \xrightarrow[\;\bar{a}\mapsto(-1)^{n\,\mathrm{par}(a)}1\otimes \mathrm{sh}_n(a)\;]{} & \mathbb{F}\otimes_{OSym_\ell} OSym_{(n,n')}
\end{array} \tag{7.2}$$

of isomorphisms in which

(1) *the top map is an even degree 0 isomorphism of graded left $OSym_n$-supermodules;*
(2) *the bottom map is an even degree 0 isomorphism of graded right $OSym_{n'}$-supermodules for the action on $\mathbb{F} \otimes_{OSym_\ell} OSym_{(n,n')}$ defined by restriction along $\mathrm{sh}_n \circ \mathrm{p}^n : OSym_{n'} \to OSym_{(n,n')}$;*
(3) *the right hand map is an even linear isomorphism of degree 0;*
(4) *the left hand map $\overline{\psi}_n^\ell$ is the graded superalgebra isomorphism induced by the involution $\psi \circ \mathrm{p}^n$ of OSym.*

Remark 7.2. The inclusion of the parity involution p^n in the definition of the left and bottom maps in (7.2) is hard to justify at this point—it could simply be omitted in both places and the simplified result is also true. The signs have been included for consistency with Theorem 8.5 below, in which they are essential.

Proof of Theorem 7.1. (1) To construct the top map so that it is a homomorphism of graded left $OSym_n$-supermodules, we must show that $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ can be made into a graded left $\overline{OH}_n^\ell$-supermodule so that $\bar{a} \cdot (b \otimes 1) = ab \otimes 1$ for all $a \in OSym_n, b \in OSym_{(n,n')}$. Since it is already a graded left $OSym_n$-supermodule, and $\overline{OH}_n^\ell$ is the quotient of $OSym_n$ by the relations $h_r^{(n)} = 0$ for $r > n'$, it suffices to check that $h_r^{(n)}$ acts as zero on $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ for all $r > n'$. By Theorem 6.12(2), any homogeneous element of $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ can be written as $\mathrm{sh}_n(b) \otimes 1$ for $b \in OSym_{n'}$. Now we must show that $h_r^{(n)}\, \mathrm{sh}_n(b) \otimes 1 = 0$ for $r > n'$. This follows from the calculation

$$h_r^{(n)}\, \mathrm{sh}_n(b) \otimes 1 = (-1)^{r\,\mathrm{par}(b)}\, \mathrm{sh}_n(b) h_r^{(n)} \otimes 1 = (-1)^{r\,\mathrm{par}(b)}\, \mathrm{sh}_n(b) \sum_{s=0}^{r} (-1)^s h_{r-s}^{(n)} e_s^{(\ell)} \otimes 1$$

$$= (-1)^{r\,\mathrm{par}(b)} \mathrm{sh}_n(b)\, \mathrm{sh}_n\left((-1)^r e_r^{(n')}\right) \otimes 1 = 0.$$

The second equality here is just the observation that $e_s^{(\ell)} \otimes 1$ is zero in $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ for $s > 0$. The penultimate equality follows from Lemma 6.10(1).

Now consider $\bar{s}_\lambda^{(n)} \in \overline{OH}_n^\ell$. If $\mathrm{ht}(\lambda) > n$, we already know this is zero by Theorem 6.3. We have by (6.1) that $\bar{s}_\lambda^{(n)} = \bar{h}_\lambda^{(n)} +$ (a linear combination of $\bar{h}_\mu^{(n)}$ for $\mu > \lambda$). We deduce that $\bar{s}_\lambda^{(n)} = 0$ if $\lambda_1 > n'$ since $\bar{h}_\lambda^{(n)}$ and all of the $\bar{h}_\mu^{(n)}$ appearing in this expansion are zero by the defining relations of $\overline{OH}_n^\ell$. This shows that $\overline{OH}_n^\ell$ is spanned by the elements $\bar{s}_\lambda^{(n)}$ $(\lambda \in \Lambda^+_{n\times n'})$. To see that these elements are linearly independent, hence, a basis for $\overline{OH}_n^\ell$, we act on the vector $1 \otimes 1 \in OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ to obtain the vectors $s_\lambda^{(n)} \otimes 1$ $(\lambda \in \Lambda^+_{n\times n'})$ which constitute a basis for $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ by Theorem 6.12(1). This argument also shows that the map (1) is an isomorphism.

(2) Similarly, to construct the bottom map, we must make $\mathbb{F} \otimes_{OSym_\ell} OSym_{(n,n')}$ into a graded right $\overline{OH}_{n'}^\ell$-supermodule so that $(1 \otimes b) \cdot \bar{a} = (-1)^{n\,\mathrm{par}(a)} 1 \otimes b\, \mathrm{sh}_n(a)$ for $b \in OSym_{(n,n')}$ and $a \in OSym_{n'}$. To do this, one first applies $*$ to Theorem 6.12 to deduce that $\mathbb{F} \otimes_{OSym_\ell} OSym_{(n,n')}$ is spanned by vectors of the form $1 \otimes a$ for $a \in OSym_n$. This plus Lemma 6.10(2) are then used establish the well-definedness of the action. The fact that the bottom map is an isomorphism could be deduced using Theorem 6.12(2) like in the previous paragraph, but it also follows once we have checked the commutativity of the diagram using that the other three maps (1), (3) and (4) are all isomorphisms.

(3) To obtain the map (3), we start with the isomorphism $OSym_{(n,n')} \xrightarrow{\sim} OSym_{(n,n')}, a \mapsto a^*$ where $*$ here is the restriction of the superalgebra anti-involution $* : OPol_\ell \to OPol_\ell$. Since we have that $(ab)^* = (-1)^{\mathrm{par}(a)\,\mathrm{par}(b)} b^* a^*$ for any $b \in OSym_{(n,n')}$ and $a \in OSym_\ell$ with $a^* \in OSym_\ell$ again, this induces the desired isomorphism $\mathbb{F} \otimes_{OSym_\ell} OSym_{(n,n')} \xrightarrow{\sim} OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$.

(4) By definition, $\overline{OH}_n^\ell$ is the quotient of $OSym$ by the two-sided ideal generated by $\{e_r | r > n\} \cup \{h_r | r > n'\}$ and $\overline{OH}_{n'}^\ell$ is the quotient of $OSym$ by the two-sided ideal generated by $\{h_r \mid r > n\} \cup \{e_r \mid r > n'\}$. The involution $\psi \circ \mathrm{p}^n$ interchanges these two ideals so it factors through the quotients to induce an isomorphism $\bar{\psi}_n^\ell : \overline{OH}_n^\ell \xrightarrow{\sim} \overline{OH}_{n'}^\ell$. This gives the graded superalgebra isomorphism (4).

To complete the proof, it just remains to show that the diagram commutes. Consider $\bar{h}_{r_1}^{(n)} \cdots \bar{h}_{r_k}^{(n)} \in \overline{OH}_n^\ell$ for $r_1, \dots, r_k > 0$ and $k \geq 0$. The map (1) takes it to $h_{r_1}^{(n)} \cdots h_{r_k}^{(n)} \otimes 1$. As in the opening paragraph of the proof, we have that

$$f h_r^{(n)} \otimes 1 = (-1)^r f\, \mathrm{sh}_n\,(e_r^{(n')}) \otimes 1 = (-1)^{r+\mathrm{par}(f) r}\, \mathrm{sh}_n\,(e_r^{(n')}) f \otimes 1$$

for any r and $f \in OSym_n$. By induction, it follows that

$$h_{r_1}^{(n)} \cdots h_{r_k}^{(n)} \otimes 1 = (-1)^{r_1+\cdots+r_k+\sum_{i<j} r_i r_j}\, \mathrm{sh}_n\left(e_{r_k}^{(n')} \cdots e_{r_1}^{(n')}\right) \otimes 1 = (-1)^{r_1+\cdots+r_k}\, \mathrm{sh}_n\,(e_{r_1}^{(n')} \cdots e_{r_k}^{(n')})^* \otimes 1.$$

This is the same as the image of $\bar{h}_{r_1}^{(n)} \cdots \bar{h}_{r_k}^{(n)}$ going around the other three sides of the square. □

Corollary 7.3. *For $\ell = n + n'$, $\overline{OH}_n^\ell$ is of graded superdimension $q^{nn'} \begin{bmatrix} \ell \\ n \end{bmatrix}_{q,\pi}$.*

Proof. This follows from the basis described in Theorem 7.1 plus Corollary 3.2. □

Corollary 7.4. *In $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$, we have that $\mathrm{sh}_n\,(\sigma_\mu^{(n')}) \otimes 1 = (-1)^{\overline{NE}(\mu)} s_{\mu^t}^{(n)} \otimes 1$ for every $\mu \in \Lambda^+$.*

Proof. Note that $(-1)^{\overline{NE}(\mu)} s_{\mu^t}^{(n)} \otimes 1$ is the image of $(-1)^{\overline{NE}(\mu)} \bar{s}_{\mu^t}^{(n)}$ under the top map in the commuting square (7.2). Now we compute the image of $(-1)^{\overline{NE}(\mu)} \bar{s}_{\mu^t}^{(n)}$ around the other three edges of this square.

Using that $\overline{NE}(\mu) = |\mu| + dN(\mu) + dE(\mu) + NE(\mu)$, it maps first to $(-1)^{dN(\mu)+dE(\mu)+n|\mu|}\bar{s}_\mu^{(n')}$ thanks to (6.4), then to $(-1)^{dN(\mu)+dE(\mu)} 1 \otimes \mathrm{sh}_n(s_\mu^{(n')})$, then to $\mathrm{sh}_n(\sigma_\mu^{(n')}) \otimes 1$ thanks to (6.8). □

Corollary 7.5. *For $\ell = n + n'$, there is a unique (up to scalars) trace map $\overline{\mathrm{tr}} : \overline{OH}_n^\ell \to \mathbb{F}$ making $\overline{OH}_n^\ell$ into a graded Frobenius superalgebra over $\mathbb{F}$ of degree $2nn'$ and parity $nn' \pmod 2$. Moreover, normalizing $\overline{\mathrm{tr}}$ so that $\overline{\mathrm{tr}}(\bar{s}^{(n)}_{(n'^n)}) = 1$ and recalling the definition of Tr from Theorem 6.12, we have that $\mathrm{Tr}(a) \otimes 1 = 1 \otimes \overline{\mathrm{tr}}(\bar{a})$ in $OSym_{(n,n')} \otimes_{OSym_\ell} \mathbb{F}$ for any $a \in OSym_n$.*

Proof. If it exists, the trace map is unique up to a non-zero scalar; cf. the discussion after (2.10). Now we *define* $\overline{\mathrm{tr}} : \overline{OH}_n^\ell \to \mathbb{F}$ so that $\mathrm{Tr}(a) \otimes 1 = 1 \otimes \overline{\mathrm{tr}}(\bar{a})$ and check that is a trace map sending $\bar{s}^{(n)}_{(n'^n)}$ to 1. The latter statement follows because we know in (6.29) that $\mathrm{sgn}(\varnothing) = 1$. To show that $\overline{\mathrm{tr}}$ is a trace map, we need to show that there exist linear bases $b_1, \dots, b_m$ and $b_1^\vee, \dots, b_m^\vee$ for $\overline{OH}_n^\ell$ satisfying (2.9) and (2.12). We take $b_1^\vee, \dots, b_m^\vee$ and $b_1, \dots, b_m$ to be the elements $\bar{s}_\lambda^{(n)}$ and $(-1)^{\overline{NE}(\mu)}\,\mathrm{sgn}(\mu)\bar{s}^{(n)}_{\mu^t}$ for $\lambda \in \Lambda^+_{n\times n'}$ and $\mu \in \Lambda^+_{n'\times n}$, respectively, enumerated so that $b_r^\vee = \bar{s}_\lambda^{(n)}$ and $b_r = (-1)^{\overline{NE}(\mu)}\,\mathrm{sgn}(\mu)\bar{s}^{(n)}_{\mu^t}$ if and only if $\mu_i^t = n' - \lambda_{n+1-i}$ for each $i = 1, \dots, n$. The properties (2.9) obviously hold, and we have a pair of dual bases as in (2.12) thanks to Corollary 7.4 and Theorem 6.12. □

Corollary 7.6. *Let $\lambda, \mu \in \Lambda^+$ be related as in the first case of (6.29). Then $\mathrm{sgn}(\mu) = (-1)^{\overline{NE}(\mu)} LR^\nu_{\lambda,\mu^t}$ where $\nu := (n'^n)$ and LR^ν_{λ,μ^t} denotes the odd Littlewood-Richardson coefficient, that is, the coefficient of s_ν when $s_\lambda s_{\mu^t}$ is expanded in terms of odd Schur functions.*

Proof. By the previous two corollaries and the definition (6.29), we have that

$$\mathrm{sgn}(\mu) = (-1)^{\overline{NE}(\mu)}\overline{\mathrm{tr}}\Big(\bar{s}_\lambda^{(n)}\bar{s}_{\mu^t}^{(n)}\Big) = (-1)^{\overline{NE}(\mu)} LR^\nu_{\lambda,\mu^t}.$$

□

Some very special odd Littlewood-Richardson coefficients arise in the odd analog of the *Pieri formula* proved in [EKL, (2.72)]:

$$s_\lambda h_r = \sum_\mu (-1)^{NE(\lambda)+NE(\mu)+S(\lambda,\mu)} s_\mu. \tag{7.3}$$

The sum here is over all partitions μ whose Young diagram is obtained by adding one box to the bottom of r different columns of the Young diagram of λ, and $S(\lambda,\mu) := \sum_{1\le j\le r}\sum_{k=i_j+1}^{\lambda_1} \lambda_k^t$ assuming these columns are indexed by $i_1 < \cdots < i_r$. The ghastly signs appearing in (7.3) and in the next lemma fortunately play no significant role.

Lemma 7.7. *The inclusion $OSym_n \hookrightarrow OSym_{(n-1,1)}$ maps*

$$s_\mu^{(n)} \mapsto \sum_{r,\lambda} (-1)^{NE(\lambda)+dE(\lambda)+NE(\mu)+dE(\mu)+S(\lambda,\mu)+\binom{r}{2}} s_\lambda^{(n-1)} x_n^r$$

where the sum is over all $r \geq 0$ and partitions λ whose Young diagram is obtained by removing one box from the bottom of r different columns of the Young diagram of μ, including all boxes from its nth row since $s_\lambda^{(n-1)} = 0$ if $\lambda_n > 0$.

Proof. From (6.13), it follows that the coefficient of $s_\lambda^{(n-1)} x_n^r$ when $s_\mu^{(n)}$ is expanded in terms of the Schur basis for $OSym_{(n-1,1)}$ is equal to the $s_\lambda \otimes h_r$-coefficient of $\Delta^+(s_\mu)$. Using Corollary 6.2 and (6.6), this is

$$(-1)^{dE(\lambda)+\binom{r}{2}}(s_\lambda \otimes h_r, \Delta^+(s_\mu))^+ \overset{(4.29)}{=} (-1)^{dE(\lambda)+\binom{r}{2}}(s_\lambda h_r, s_\mu)^+.$$

Now use (7.3) plus Corollary 6.2 once again. □

Remark 7.8. A general formula for odd Littlewood-Richardson coefficients is derived in [E, Th. 4.8], showing that they can be computed by counting the same set of semi-standard skew tableaux that appear in the ordinary Littlewood-Richardson rule, but counting each one with a sign ±. A useful consequence of this is that if an ordinary even Littlewood-Richardson coefficient is zero then so is the corresponding odd Littlewood-Richardson coefficient. This, together with the odd Pieri rule, is all that we actually use below.

8. Equivariant odd Grassmannian cohomology algebras

Recall from Corollary 4.13 that R_ℓ denotes the largest supercommutative quotient of $OSym_\ell$. We are now going to work over this as our base ring. We use the notation $\dot{c}$ to denote the image of $c \in OSym_\ell$ in R_ℓ. Note that $OSym_n \otimes R_\ell$ is a graded R_ℓ-superalgebra with structure map $\eta : R_\ell \to OSym_n \otimes R_\ell, \dot{c} \mapsto 1 \otimes \dot{c}$.

Definition 8.1. For $\ell = n + n'$, the *equivariant odd Grassmannian cohomology algebra* is the graded R_ℓ-superalgebra

$$OH_n^\ell := OSym_n \otimes R_\ell \bigg/ \bigg\langle \sum_{s=0}^{r} (-1)^s h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \,\bigg|\, r > n' \bigg\rangle. \tag{8.1}$$

For $a \in OSym_n$ and $c \in OSym_\ell$, we denote the canonical image of $a \otimes \dot{c} \in OSym_n \otimes R_\ell$ in the quotient OH_n^ℓ by $a \,\bar{\otimes}\, \dot{c}$.

As the name suggests, this is an odd analog of the $GL_\ell(\mathbb{C})$-equivariant cohomology algebra of the Grassmannian of n-dimensional subspaces of $\mathbb{C}^\ell$. Given any graded supercommutative R_ℓ-superalgebra A, one can specialize to obtain the graded A-superalgebra $OH_n^\ell \otimes_{R_\ell} A$. In particular, the ordinary odd Grassmannian cohomology algebra $\overline{OH}_n^\ell$ from (7.1) is naturally identified with the specialization $OH_n^\ell \otimes_{R_\ell} \mathbb{F}$.

Example 8.2. We have that $OH_1^2 \cong OPol_2 / \langle x_1^2 + x_2^2, x_1^3 + x_1^2 x_2 \rangle$ via the isomorphism $h_r^{(1)} \,\bar{\otimes}\, 1 \mapsto x_1^r$, $1 \,\bar{\otimes}\, \dot{e}_1^{(2)} \mapsto x_1 + x_2$, $1 \,\bar{\otimes}\, \dot{e}_2^{(2)} \mapsto x_1 x_2$. A linear basis is given by the elements $\{x_1^r, x_2, x_1 x_2 \mid r \geq 0\}$.

Lemma 8.3. *For $\ell = n + n'$, there is a surjective graded superalgebra homomorphism $\alpha_n^\ell : OH_n^\ell \twoheadrightarrow R_{n'}$ taking $a \,\bar{\otimes}\, 1$ to zero for $a \in OSym_n$ of positive degree and $1 \,\bar{\otimes}\, \dot{c}^{(\ell)}$ to $\dot{c}^{(n')}$ for $c \in OSym$.*

Proof. This is clear from the nature of the defining relations (8.1). □

Lemma 8.4. *The two-sided ideal $\left\langle \sum_{s=0}^{r} (-1)^s h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \,\middle|\, r > n' \right\rangle$ of $OSym_n \otimes R_\ell$ contains the elements $\sum_{s=0}^{r} h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ for all $r > n' + 1$. Also the two-sided ideal $\left\langle \sum_{s=0}^{r} (-1)^{(r-n')s} h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \,\middle|\, r > n' \right\rangle$ contains the elements $\sum_{s=0}^{r} (-1)^{(r-n')s+s} h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ for all $r > n' + 1$. Hence, these two ideals are equal.*

Proof. By (4.16), we have that $h_1 h_r = h_r h_1$ if r is odd and $h_1 h_r = -h_r h_1 + 2h_{r+1}$ if r is even. Suppose that $r > n' + 1$. The ideal contains $a := \sum_{s=0}^{r-1} (-1)^s h_{r-1-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$. Hence, it contains

$$(-1)^r (h_1 \otimes 1) a - a (h_1 \otimes 1) = \sum_{s=0}^{r} (1 - (-1)^{r-s}) h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}.$$

Adding this to $\sum_{s=0}^{r} (-1)^{r-s} h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$, which is also in the ideal, gives the claimed elements for the first assertion of the lemma. The second assertion is proved similarly. To deduce that these ideals are the same, the facts established so far show that both are generated by the lowest degree generator $\sum_{s=0}^{n'+1} (-1)^s h_{n'+1-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ together with the higher degree generators $\sum_{s=0}^{r} (-1)^s h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ and $\sum_{s=0}^{r} h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ for all $r > n' + 1$. □

For any $\alpha \in \Lambda(k,\ell)$, $OSym_\alpha \otimes_{OSym_\ell} R_\ell$ is a graded left $OSym_\alpha$-supermodule, and it is a graded R_ℓ-supermodule for the left action that is induced by the natural right action. Thus, for $a \in OSym_\alpha$ and $c \in OSym_\ell$, we have that

$$ac \otimes 1 = a \otimes \dot{c} = (a \otimes 1) \cdot \dot{c} = (-1)^{\mathrm{par}(a)\,\mathrm{par}(c)} \dot{c} \cdot (a \otimes 1). \tag{8.2}$$

However this is in general *not* equal to $ca \otimes 1$. Note also that $OSym_\alpha \otimes_{OSym_\ell} R_\ell$ is not itself a graded superalgebra in any apparent way. Indeed, R_ℓ is the quotient of $OSym_\ell$ by the two-sided ideal I_ℓ generated by $(o^{(\ell)})^2$ and $[o^{(\ell)}, e_2^{(\ell)}]$, so

$$OSym_\alpha \otimes_{OSym_\ell} R_\ell = OSym_\alpha \otimes_{OSym_\ell} OSym_\ell / I_\ell \simeq OSym_\alpha / OSym_\alpha I_\ell.$$

However, in general, $OSym_\alpha I_\ell$ is merely a left ideal, not a two-sided ideal of $OSym_\alpha$. Similar remarks apply to $R_\ell \otimes_{OSym_\ell} OSym_\alpha$, which is a graded right $OSym_\alpha$-supermodule, and a graded R_ℓ-supermodule for the right action that is induced by the natural left action.

Theorem 8.5. *For $\ell = n + n'$, OH_n^ℓ is free as a graded R_ℓ-supermodule with basis given by the odd Schur polynomials $s_\lambda^{(n)} \bar{\otimes} 1$ for $\lambda \in \Lambda^+_{n \times n'}$. Moreover, there is a commuting diagram*

$$\begin{array}{ccc} OH_n^\ell & \xrightarrow{\;a \bar{\otimes} \dot{c} \mapsto a \otimes \dot{c}\;} & OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell \\ {\scriptstyle \psi_n^\ell}\big\downarrow & & \big\uparrow {\scriptstyle \dot{c} \otimes a \mapsto (-1)^{\mathrm{par}(a)\,\mathrm{par}(c)} a^* \otimes \dot{c}} \\ OH_{n'}^\ell & \xrightarrow[\;a \bar{\otimes} \dot{c} \mapsto (-1)^{\mathrm{par}(a)(n + \mathrm{par}(c))} \dot{c} \otimes \mathrm{sh}_n(a)\;]{} & R_\ell \otimes_{OSym_\ell} OSym_{(n,n')} \end{array} \tag{8.3}$$

of isomorphisms in which

(1) the top map is an even degree 0 isomorphism of graded $(OSym_n, R_\ell)$-superbimodules;
(2) the bottom map is an even degree 0 isomorphism of graded $(R_\ell, OSym_{n'})$-superbimodules for the right action of $OSym_{n'}$ on $R_\ell \otimes_{OSym_\ell} OSym_{(n,n')}$ defined by restricting the natural right action of $OSym_{(n,n')}$ along $\mathrm{sh}_n \circ \mathrm{p}^n : OSym_{n'} \to OSym_{(n,n')}$;
(3) the right hand map is an even degree 0 graded R_ℓ-supermodule isomorphism;
(4) the left hand map ψ_n^ℓ is a graded R_ℓ-superalgebra isomorphism such that

$$\psi_n^\ell(a \bar{\otimes} 1) = \sum_{i=1}^p a_i \bar{\otimes} \dot{c}_i \tag{8.4}$$

for $a \in OSym_n$ such that $a = \sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)} \mathrm{sh}_n(a_i)^* c_i$ for $a_i \in OSym_{n'}, c_i \in OSym_\ell$.

Proof. (1) To construct the top map, we must show that $OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell$ can be made into a graded left OH_n^ℓ-supermodule so that

$$a \bar{\otimes} \dot{c} \cdot (b \otimes 1) = (-1)^{\mathrm{par}(c)\,\mathrm{par}(b)} ab \otimes \dot{c}$$

for $a \in OSym_n, b \in OSym_{(n,n')}$ and $c \in OSym_\ell$. To see this is well defined, we know by Theorem 6.12(2) that $OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell$ is generated as a R_ℓ-supermodule by vectors of the form $\mathrm{sh}_n(b) \otimes 1$ for $b \in OSym_{n'}$, so it suffices to check that $\sum_{s=0}^r (-1)^s h_{r-s}^{(n)} \bar{\otimes} \dot{e}_s^{(\ell)}$ acts as zero on $\mathrm{sh}_n(b) \otimes 1$ for $b \in OSym_{n'}$ and $r > n'$. This follows by Lemma 6.10(1):

$$\begin{aligned} \sum_{s=0}^r (-1)^s h_{r-s}^{(n)} \bar{\otimes} \dot{e}_s^{(\ell)} \cdot (\mathrm{sh}_n(b) \otimes 1) &= \sum_{s=0}^r (-1)^{s + s\,\mathrm{par}(b)} h_{r-s}^{(n)} \mathrm{sh}_n(b) \bar{\otimes} \dot{e}_s^{(\ell)} \\ &= (-1)^{r\,\mathrm{par}(b)} \mathrm{sh}_n(b) \sum_{s=0}^r (-1)^s h_{r-s}^{(n)} e_s^{(\ell)} \otimes 1 = 0. \end{aligned}$$

Thus, we have defined the top map.

Next, we show that the elements $\{s_\lambda^{(n)} \bar\otimes 1 \mid \lambda \in \Lambda^+_{n\times n'}\}$ generate OH_n^ℓ as a graded R_ℓ-supermodule. This is not quite as easy as before since it is no longer the case that $s_\lambda^{(n)} \bar\otimes 1 = 0$ in OH_n^ℓ when $\lambda_1 > n$. Instead, one shows by induction on $|\lambda|$ that any $s_\lambda^{(n)} \bar\otimes 1$ for λ with $\lambda_1 > n$ can be written as an R_ℓ-linear combination of other $s_\mu^{(n)} \bar\otimes 1$ for μ with $\mu_1 \le n$ and $|\mu| < |\lambda|$.

Now the proof of the first part of the theorem can be completed. The spanning set for OH_n^ℓ just constructed is also linearly independent because it becomes the basis for $OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell$ arising from Theorem 6.12(1) when we act on the vector $1 \otimes 1$. This also shows that the top map (1) is an isomorphism.

(2) We need to make $R_\ell \otimes_{OSym_\ell} OSym_{(n,n')}$ into a graded right $OH_{n'}^\ell$-supermodule so that

$$(1 \otimes b) \cdot a \bar\otimes \dot c = (-1)^{(\mathrm{par}(a)+\mathrm{par}(b))\,\mathrm{par}(c)+n\,\mathrm{par}(a)} \dot c \otimes b\, \mathrm{sh}_n(a)$$

for $a \in OSym_{n'}$, $b \in OSym_{(n,n')}$ and $c \in OSym_\ell$. To check that this action is well defined, we know from Lemma 8.4 that the ideal defining $OH_{n'}^\ell$ is generated by the elements $\sum_{s=0}^r (-1)^{(r-n)s} h_{r-s}^{(n')} \otimes \dot e_s^{(\ell)}$ for $r > n$. It suffices to check that the image of each such element in $OH_{n'}^\ell$ acts as zero on $1 \otimes b$ for $b \in OSym_n$. This follows by Lemma 6.10(2):

$$\begin{aligned}(1 \otimes b) \cdot \sum_{s=0}^r (-1)^{(r-n)s} h_{r-s}^{(n')} \bar\otimes \dot e_s^{(\ell)} &= (-1)^{rn} \sum_{s=0}^r (-1)^{s+s\,\mathrm{par}(b)} \dot e_s^{(\ell)} \otimes b\, \mathrm{sh}_n\,(h_{r-s}^{(n')}) \\ &= (-1)^{r(\mathrm{par}(b)+n)} 1 \otimes \sum_{s=0}^r (-1)^s e_s^{(\ell)}\, \mathrm{sh}_n\,(h_{r-s}^{(n')}) b = 0.\end{aligned}$$

Applying $*$ to the basis from Theorem 6.12(2), we deduce that $R_\ell \otimes_{OSym_\ell} OSym_{(n,n')}$ is a free graded R_ℓ-supermodule with basis $\left\{1 \otimes \mathrm{sh}_n(s_\mu^{(n')}) \,\middle|\, \mu \in \Lambda^+_{n'\times n}\right\}$. Also the elements $\{s_\mu^{(n')} \bar\otimes 1 \mid \mu \in \Lambda^+_{n'\times n}\}$ span $OH_{n'}^\ell$ as in the previous paragraph. Acting on $1 \otimes 1$ shows finally that these elements form a basis for $OH_{n'}^\ell$ and that the bottom map is an isomorphism.

(3) To construct the right hand map, we start with the map

$$R_\ell \otimes OSym_{(n,n')} \to OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell, \qquad \dot c \otimes a \mapsto (-1)^{\mathrm{par}(a)\,\mathrm{par}(c)} a^* \otimes \dot c$$

for $c \in OSym_\ell, a \in OSym_{(n,n')}$. Using that R_ℓ is supercommutative, this is easily checked to be a graded R_ℓ-supermodule homomorphism. Also this map is balanced. To see this, we need to show that the images of $\dot c_1 \dot c_2 \otimes a$ and $\dot c_1 \otimes c_2 a$ are the same for $a \in OSym_{(n,n')}$ and $c_1, c_2 \in OSym_\ell$. Note that the superalgebra anti-involution $*$ of $OSym_\ell$ descends to a superalgebra anti-involution $*$ of R_ℓ. Since R_ℓ is supercommutative and each of its generators $\dot e_r^{(\ell)}$ $(r \ge 1)$ is fixed by $*$, this is induced anti-involution is equal to the identity. So $\dot c_2 = \dot c_2^*$, and the image of $\dot c_1 \dot c_2 \otimes a$ is

$$\begin{aligned}(-1)^{\mathrm{par}(a)(\mathrm{par}(c_1)+\mathrm{par}(c_2))} a^* \otimes \dot c_1 \dot c_2 &= (-1)^{\mathrm{par}(a)\,\mathrm{par}(c_1)+\mathrm{par}(a)\,\mathrm{par}(c_2)+\mathrm{par}(c_1)\,\mathrm{par}(c_2)} a^* \otimes \dot c_2^* \dot c_1 \\ &= (-1)^{\mathrm{par}(a)\,\mathrm{par}(c_1)+\mathrm{par}(a)\,\mathrm{par}(c_2)+\mathrm{par}(c_1)\,\mathrm{par}(c_2)} a^* c_2^* \otimes \dot c_1 \\ &= (-1)^{(\mathrm{par}(a)+\mathrm{par}(c_2))\,\mathrm{par}(c_1)} (c_2 a)^* \otimes \dot c_1\end{aligned}$$

which is equal to the image of $\dot c_1 \otimes c_2 a$. So this map induces the required graded R_ℓ-supermodule isomorphism.

(4) We define ψ_n^ℓ to be the unique graded R_ℓ-supermodule isomorphism making the diagram commute. If $a = \sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)} \mathrm{sh}_n(a_i)^* c_i$ for $a_i \in OSym_{n'}, c_i \in OSym_\ell$ then the image of $a \bar\otimes 1$ under the top map is equal to $\sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)} \mathrm{sh}_n(a_i)^* \otimes \dot c_i$. It follows that $\psi_n^\ell(a \bar\otimes 1) = \sum_{i=1}^p a_i \otimes \dot c_i$ because the latter expression also maps to $\sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)} \mathrm{sh}_n(a_i)^* \otimes \dot c_i$ when the bottom map followed by the right hand map is applied.

We still need to show that ψ_n^ℓ is actually a graded R_ℓ-superalgebra isomorphism. For this, we take $a, b \in OSym_n$ such that $a = \sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)} \mathrm{sh}_n(a_i)^* c_i$ and $b = \sum_{j=1}^q (-1)^{n\,\mathrm{par}(b_j)} \mathrm{sh}_n(b_j)^* d_j$ for $a_i, b_j \in OSym_{n'}$ and $c_i, d_j \in OSym_\ell$. We have that

$$\begin{aligned} ab &= \sum_{j=1}^q (-1)^{n\,\mathrm{par}(b_j)} a\, \mathrm{sh}_n(b_j)^* d_j = \sum_{j=1}^q (-1)^{(\mathrm{par}(a)+n)\,\mathrm{par}(b_j)} \mathrm{sh}_n(b_j)^* a d_j \\ &= \sum_{i=1}^p \sum_{j=1}^q (-1)^{n\,\mathrm{par}(a_i)+(\mathrm{par}(a_i)+\mathrm{par}(c_i)+n)\,\mathrm{par}(b_j)} \mathrm{sh}_n(b_j)^* \,\mathrm{sh}_n(a_i)^* c_i d_j \\ &= \sum_{i=1}^p \sum_{j=1}^q (-1)^{n(\mathrm{par}(a_i)+\mathrm{par}(b_j))+\mathrm{par}(c_i)\,\mathrm{par}(b_j)} \mathrm{sh}_n(a_i b_j)^* c_i d_j. \end{aligned}$$

So

$$\psi_n^\ell(ab) = \sum_{i=1}^p \sum_{j=1}^q (-1)^{\mathrm{par}(c_i)\,\mathrm{par}(b_j)} a_i b_j \otimes \dot c_i \dot d_j = \Big(\sum_{i=1}^p a_i \bar\otimes \dot c_i\Big)\Big(\sum_{j=1}^q b_j \bar\otimes \dot d_j\Big) = \psi_n^\ell(a)\psi_n^\ell(b).$$

□

Corollary 8.6. *For $\ell = n + n'$, OH_n^ℓ is a free graded R_ℓ-supermodule of graded rank $q^{nn'}\begin{bmatrix}\ell\\ n\end{bmatrix}_{q,\pi}$.*

Proof. This follows by the basis theorem that is the first assertion of Theorem 8.5 plus Corollary 3.2. □

Corollary 8.7. *For $\ell = n + n'$, there is a unique (up to scalars) trace map* $\mathrm{tr} : OH_n^\ell \to R_\ell$ *making OH_n^ℓ into a graded Frobenius superalgebra over R_ℓ of degree $2nn'$ and parity nn'* (mod 2)*. Moreover, normalizing* tr *so that* $\mathrm{tr}\,(s^{(n)}_{(n'^n)} \bar\otimes 1) = 1$ *and recalling the definition of* Tr *from Theorem 6.12, we have that* $\mathrm{Tr}(a) \otimes 1 = 1 \otimes \mathrm{tr}(a \bar\otimes 1)$ *in $OSym_{(n,n')} \otimes_{OSym_\ell} R_\ell$ for any $a \in OSym_n$.*

Proof. The uniqueness follows from the general principles discussed after (2.10) since $(OH_n^\ell)_0 = \mathbb{F}$. For existence, define $\mathrm{tr} : OH_n^\ell \to R_\ell$ to be the unique graded R_ℓ-supermodule homomorphism such that $\mathrm{Tr}(a)\otimes 1 = 1\otimes \mathrm{tr}(a \bar\otimes 1)$ for $a \in OSym_n$. Noting that $\binom{n}{2}+\binom{n'}{2}-\binom{\ell}{2} = -nn'$, the definition of Tr implies that this is a homogeneous linear map of degree $-2nn'$ and parity nn' (mod 2) such that $\mathrm{tr}\,(s^{(n)}_{(n'^n)} \bar\otimes 1) = 1$. It remains to check (2.9) and (2.12). We take $b_1^\vee, \dots, b_m^\vee, b_1, \dots, b_m \in OH_n^\ell$ to be the elements $s_\lambda^{(n)} \bar\otimes 1$ and $(\psi_n^\ell)^{-1}\big((-1)^{dN(\mu)+dE(\mu)+n|\mu|} \mathrm{sgn}(\mu) s_\mu^{(n')} \bar\otimes 1\big)$ for $\lambda \in \Lambda^+_{n\times n'}$ and $\mu \in \Lambda^+_{n'\times n}$, respectively, enumerated in such a way that $b_r^\vee = s_\lambda^{(n)} \bar\otimes 1$ and $\psi_n^\ell(b_r) = (-1)^{dN(\mu)+dE(\mu)+n|\mu|} \mathrm{sgn}(\mu) s_\mu^{(n')} \bar\otimes 1$ if and only if $\mu_i^\dagger = n' - \lambda_{n+1-i}$ for each $i = 1, \dots, n$. By (6.8) and the commutativity of the diagram (8.3), the top horizontal map in the diagram takes $b_r^\vee$ to $s_\lambda^{(n)} \otimes 1$ and b_r to $(-1)^{dN(\mu)+dE(\mu)} \mathrm{sgn}(\mu)\, \mathrm{sh}_n\big(s_\mu^{(n')}\big)^* \otimes 1 = \mathrm{sgn}(\mu)\, \mathrm{sh}_n\,(\sigma_\mu^{(n')}) \otimes 1$. Now Theorem 6.12 implies that $b_1^\vee, \dots, b_m^\vee$ and $b_1, \dots, b_m$ give dual bases for OH_n^ℓ as a free graded R_ℓ-supermodule, as required for (2.12). □

The next lemma investigates the graded R_ℓ-superalgebra isomorphism $\psi_n^\ell : OH_n^\ell \xrightarrow{\sim} OH_{n'}^\ell$ constructed in Theorem 8.5(4).

Lemma 8.8. *For $\ell = n + n'$, the isomorphism $\psi_n^\ell : OH_n^\ell \to OH_{n'}^\ell$ maps*

$$h_r^{(n)} \bar\otimes 1 \mapsto \sum_{s=0}^r (-1)^{(n+1)(r-s)} e_{r-s}^{(n')} \bar\otimes \dot h_s^{(\ell)}, \qquad e_r^{(n)} \bar\otimes 1 \mapsto \sum_{s=0}^r (-1)^{(n+r)(r-s)} h_{r-s}^{(n')} \bar\otimes \dot e_s^{(\ell)} \tag{8.5}$$

for $r \geq 0$. The inverse isomorphism $(\psi_n^\ell)^{-1} : OH_{n'}^\ell \to OH_n^\ell$ maps

$$h_r^{(n')} \bar\otimes 1 \mapsto \sum_{s=0}^r (-1)^{(n+r)(r-s)+ns} e_{r-s}^{(n)} \bar\otimes \dot h_s^{(\ell)}, \qquad e_r^{(n')} \bar\otimes 1 \mapsto \sum_{s=0}^r (-1)^{(n+1)(r-s)+ns} h_{r-s}^{(n)} \bar\otimes \dot e_s^{(\ell)} \tag{8.6}$$

for $r \geq 0$.

Proof. We first observe that $h^{(n)}(t) = \mathrm{sh}_n\left(e^{(n')}(t)\right)h^{(\ell)}(t)$ and $\varepsilon^{(n)}(t) = \mathrm{sh}_n\left(\eta^{(n')}(t)\right)\varepsilon^{(\ell)}(t)$ in $OSym_{(n,n')}$. This follows from (4.49) using the identities $h^{(\ell)}(t) = \mathrm{sh}_n\,(h^{(n')}(t))h^{(n)}(t)$ and $\varepsilon^{(\ell)}(t) = \mathrm{sh}_n\,(\varepsilon^{(n')}(t))\varepsilon^{(n)}(t)$. Hence, using (8.4), we deduce that ψ_n^ℓ maps

$$h^{(n)}(t)\,\bar{\otimes}\,1 \mapsto (-1)^{nn'}e^{(n')}((-1)^n t)^*\,\bar{\otimes}\,\dot{h}^{(\ell)}(t), \qquad \varepsilon^{(n)}(t)\,\bar{\otimes}\,1 \mapsto (-1)^{nn'}\eta^{(n')}((-1)^n t)^*\,\bar{\otimes}\,\dot{\varepsilon}^{(\ell)}(t).$$

The first formula in (8.5) follows from the first of these identities on equating t^{-n-r}-coefficients, remembering that $e_r^{(n')}$ is fixed by $*$. Similarly, equating t^{n-r}-coefficients in the second identity gives that

$$\psi_n^\ell\left(\varepsilon_r^{(n)}\,\bar{\otimes}\,1\right) = \sum_{s=0}^{r}(-1)^{(n+1)(r-s)}(\eta_{r-s}^{(n')})^*\,\bar{\otimes}\,\dot{\varepsilon}_s^{(\ell)}.$$

Replacing $\varepsilon_r^{(n)}$ with $(-1)^{\binom{r}{2}}e_r^{(n)}$, $(\eta_{r-s}^{(n')})^*$ with $(-1)^{\binom{r-s}{2}}h_{r-s}^{(n')}$ and $\dot{\varepsilon}_s^{(\ell)}$ with $(-1)^{\binom{s}{2}}\dot{e}_s^{(\ell)}$ using (4.40) and (4.41) gives the second formula in (8.5). The formulae in (8.6) are easily deduced from (8.5) together with the infinite Grassmannian relation in R_ℓ. □

The following composition defines a graded R_ℓ-superalgebra automorphism:

$$\delta_n^\ell := \psi_{n'}^\ell \circ \psi_n^\ell : OH_n^\ell \xrightarrow{\sim} OH_n^\ell. \tag{8.7}$$

Using Lemma 8.8, this can be described explicitly on generators, as follows.

Corollary 8.9. *The automorphism δ_n^ℓ maps*

$$h_r^{(n)}\,\bar{\otimes}\,1 \mapsto (-1)^{\ell r}h_r^{(n)}\,\bar{\otimes}\,1 + (-1)^{\ell(r-1)}(1+(-1)^{n+r})h_{r-1}^{(n)}\,\bar{\otimes}\,\dot{o}^{(\ell)}, \tag{8.8}$$

$$e_r^{(n)}\,\bar{\otimes}\,1 \mapsto (-1)^{\ell r}e_r^{(n)}\,\bar{\otimes}\,1 + (-1)^{(\ell+1)(r-1)}(1+(-1)^{n+r})e_{r-1}^{(n)}\,\bar{\otimes}\,\dot{o}^{(\ell)}, \tag{8.9}$$

$$\varepsilon_r^{(n)}\,\bar{\otimes}\,1 \mapsto (-1)^{\ell r}\varepsilon_r^{(n)}\,\bar{\otimes}\,1 + (-1)^{\ell(r-1)}(1+(-1)^{n+r})\varepsilon_{r-1}^{(n)}\,\bar{\otimes}\,\dot{o}^{(\ell)}, \tag{8.10}$$

$$\eta_r^{(n)}\,\bar{\otimes}\,1 \mapsto (-1)^{\ell r}\eta_r^{(n)}\,\bar{\otimes}\,1 + (-1)^{(\ell+1)(r-1)}(1+(-1)^{n+r})\eta_{r-1}^{(n)}\,\bar{\otimes}\,\dot{o}^{(\ell)} \tag{8.11}$$

for any $r \geq 1$. In particular, $\delta_n^\ell(o^{(n)}\,\bar{\otimes}\,1) = (-1)^\ell o^{(n)}\,\bar{\otimes}\,1 + (1-(-1)^n)1\,\bar{\otimes}\,\dot{o}^{(\ell)}$.

Proof. We first prove (8.8). Using (8.5), we compute $(\psi_{n'}^\ell \circ \psi_n^\ell)(h_r^{(n)}\,\bar{\otimes}\,1)$: it equals

$$\sum_{p=0}^{r}\sum_{q=0}^{r-p}(-1)^{(n+1)(r-p)+(n'+r-p)(r-p-q)}h_{r-p-q}^{(n)}\,\bar{\otimes}\,\dot{e}_q^{(\ell)}\dot{h}_p^{(\ell)} = \sum_{s=0}^{r}(-1)^{\ell r+(n'+r)s}h_{r-s}^{(n)}\,\bar{\otimes}\left(\sum_{t=0}^{s}(-1)^{(n+r-s+1)t}\dot{e}_{s-t}^{(\ell)}\dot{h}_t^{(\ell)}\right).$$

Now we claim that the expression in the brackets here is equal to $\delta_{s,0}$ if $n+r-s+1$ is odd, and it is equal to $\delta_{s,0} + 2\delta_{s,1}\dot{o}^{(\ell)}$ is $n+r-s+1$ is even. The formula (8.8) is easily deduced using this claim. To prove the claim, it follows immediately for odd $n+r-s+1$ by the infinite Grassmannian relation. When $n+r-s+1$ is even, it follows from the relation

$$\sum_{t=0}^{s}\dot{e}_{s-t}\dot{h}_t = \delta_{s,0} + 2\delta_{s,1}\dot{o} \tag{8.12}$$

in R, which is easily checked in the cases $s=0$ and $s=1$ and follows for $s \geq 2$ using (4.59) and (4.60).

The proof of (8.9) is a similar calculation.

Then (8.10) follows easily since $\varepsilon_r^{(n)} = (-1)^{\binom{r}{2}}e_r^{(n)}$ and $\varepsilon_{r-1}^{(n)} = (-1)^{\binom{r}{2}+r+1}e_{r-1}^{(n)}$.

Finally, to deduce (8.11), we first write (8.10) in terms of generating functions:

$$\delta_n^\ell(\varepsilon^{(n)}(t)\,\bar{\otimes}\,1) = (-1)^{\ell n}\varepsilon^{(n)}((-1)^\ell t)\,\bar{\otimes}\,1 - (-1)^{\ell n}t^{-1}\left[\varepsilon^{(n)}((-1)^\ell t) - \varepsilon^{(n)}((-1)^{\ell+1}t)\right]\bar{\otimes}\,\dot{o}^{(\ell)}. \tag{8.13}$$

This can be checked by equating coefficients of t^{n-r} on both sides. Then we formally invert both sides of (8.13) using (4.49) to obtain the identity

$$\delta_n^\ell(\eta^{(n)}(t)\bar{\otimes}1) = (-1)^{\ell n}\eta^{(n)}((-1)^\ell t)\bar{\otimes}1 + (-1)^{(\ell-1)n}t^{-1}\Big[\eta^{(n)}((-1)^{\ell+1}t) - \eta^{(n)}((-1)^\ell t)\Big]\bar{\otimes}\dot{o}^{(\ell)}. \tag{8.14}$$

To see that the right hand of this is indeed the inverse of the right hand side of (8.13), one just has to multiply together using $(\dot{o}^{(\ell)})^2 = 0$ then simplify the result to see that it equals 1, which is straightforward. Finally, we equate coefficients of t^{-n-r} on both sides of (8.14) to obtain (8.11). □

9. Deformed odd cyclotomic nil-Hecke algebras

The *odd cyclotomic nil-Hecke algebra* $\overline{ONH}_n^\ell$ is the quotient of ONH_n by the two-sided ideal generated by the element x_1^ℓ. This algebra was introduced originally in [EKL, Sec. 5]. In particular, in [EKL, Prop. 5.2], it is shown that $\overline{ONH}_n^\ell$ is zero unless $0 \leq n \leq \ell$, in which case it is isomorphic to the graded matrix superalgebra $M_{q^{n/2}[n]^!_{q,\pi}}(\overline{OH}_n^\ell)$, notation as explained after Remark 5.6.

Definition 9.1. The *deformed odd cyclotomic nil-Hecke algebra* is the quotient algebra

$$ONH_n^\ell := ONH_n \otimes R_\ell \Big/ \left\langle \sum_{r=0}^{\ell} (-1)^r x_1^{\ell-r} \otimes \dot{e}_r^{(\ell)} \right\rangle \tag{9.1}$$

for $n > 0$. We also let $ONH_0^\ell := ONH_0 \otimes R_\ell \cong R_\ell$ so that ONH_n^ℓ makes sense for all $n \geq 0$. We denote the image of $a \otimes \dot{c}$ in ONH_n^ℓ by $a \bar{\otimes} \dot{c}$.

The graded R_ℓ-superalgebra ONH_n^ℓ is the odd analog of the algebras defined for $\mathfrak{sl}_2$ in [R1, Sec. 5.2.1] and for other Cartan types in [R2, Sec. 4.4.1]; our definition (9.1) looks more like the latter formulation.

Theorem 9.2. *The deformed odd cyclotomic nil-Hecke algebra ONH_n^ℓ is zero unless $0 \leq n \leq \ell$. Assuming $0 \leq n \leq \ell$, the natural left action of $ONH_n \otimes R_\ell$ on $OPol_n \otimes_{OSym_n} OH_n^\ell$ factors through the quotient ONH_n^ℓ to make $OPol_n \otimes_{OSym_n} OH_n^\ell$ into a graded (ONH_n^ℓ, OH_n^ℓ)-superbimodule. The associated representation*

$$\rho : ONH_n^\ell \to \operatorname{End}_{\text{-}OH_n^\ell}\left(OPol_n \otimes_{OSym_n} OH_n^\ell\right)$$

is an isomorphism of graded superalgebras. Moreover, $OPol_n \otimes_{OSym_n} OH_n^\ell$ is free as a graded right OH_n^ℓ-supermodule with basis $\{x_n^{\kappa_n} \cdots x_1^{\kappa_1} \bar{\otimes} 1 \mid \kappa \in K_n\}$ where

$$K_n := \{\kappa = (\kappa_1, \dots, \kappa_n) \in \mathbb{N}^n \mid 0 \leq \kappa_i \leq n - i \text{ for } i = 1, \dots, n\}.$$

Theorem 9.2 shows that ONH_n^ℓ is isomorphic to the graded matrix superalgebra $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}\left(OH_n^\ell\right)$. We will prove the theorem later in the section, ultimately deducing it from Theorem 5.4 which showed that ONH_n is isomorphic to $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}\left(OSym_n\right)$. First, we state some corollaries. The first is the analog of Corollary 5.10 for ONH_n^ℓ.

Corollary 9.3. *The element $(\overline{\omega\xi})_n := (\omega\xi)_n \bar{\otimes} 1$ is a primitive idempotent in ONH_n^ℓ. Moreover, the maps $\imath$ and $\jmath$ from Lemma 5.9 induce an isomorphism $OH_n^\ell \cong (\overline{\omega\xi})_n ONH_n^\ell (\overline{\omega\xi})_n$, of graded superalgebras and an isomorphism $OPol_n \otimes_{OSym_n} OH_n^\ell \simeq ONH_n^\ell(\overline{\omega\xi})_n$ of graded (ONH_n^ℓ, OH_n^ℓ)-superbimodules. Making these identifications, the idempotent truncation functor*

$$(\overline{\omega\xi})_n - : ONH_n^\ell\text{-gsMod} \to OH_n^\ell\text{-gsMod}$$

is an equivalence of graded (Q, Π)-supercategories.

Corollary 9.4. *For $0 \leq n \leq \ell$, ONH_n^ℓ is a free graded R_ℓ-supermodule of graded rank*

$$q^{n(\ell-n)}[\ell]^!_{q,\pi}\overline{[n]^!_{q,\pi}}/[\ell-n]^!_{q,\pi}.$$

Proof. This follows using the final part of the theorem and Corollary 8.6. □

Corollary 9.5. *For $0 \leq n \leq \ell$, the monomials*

$$\{x^\kappa \tau_w \bar{\otimes} 1 \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \dots, n\}$$

form a basis for ONH_n^ℓ as a free R_ℓ-supermodule.

Proof. The free graded R_ℓ-supermodule with basis given by elements of the same degrees and parities as these monomials is graded rank $q^{n(\ell-n)}[\ell]^!_{q,\pi}\overline{[n]}^!_{q,\pi}/[\ell-n]^!_{q,\pi}$, which is the same as the graded rank of ONH_n^ℓ according to Corollary 9.4. Therefore it suffices to show that the monomials $\{x^\kappa \tau_w \bar{\otimes} 1 \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \dots, n\}$ are linearly independent over R_ℓ.

We first prove this linear independence in the special case that $n = \ell$, in which case we have simply that $OH_\ell^\ell = R_\ell$. Suppose we have a linear relation $\sum_{w,\kappa} x^\kappa \tau_w \bar{\otimes} \dot{c}_{w,\kappa} = 0$ in ONH_ℓ^ℓ for $\dot{c}_{w,\kappa} \in OSym_\ell$ that are not all zero, summing over $w \in S_\ell, \kappa \in \mathbb{N}^\ell$ with $0 \leq \kappa_i \leq \ell - i$ for all $i = 1, \dots, \ell$. Pick w of minimal length such that $\dot{c}_{w,\kappa} \neq 0$ for some κ. Then we act on the vector $p_w^{(\ell)} \bar{\otimes} 1 \in OPol_\ell \otimes_{OSym_\ell} R_\ell$ to deduce as in the proof of Theorem 5.2 that $\sum_\kappa (-1)^{\mathrm{par}(c_{w,\kappa})\ell(w)} x^\kappa \bar{\otimes} \dot{c}_{w,\kappa} = 0$. The elements $x^\kappa \bar{\otimes} 1$ for $\kappa \in \mathbb{N}^\ell$ with $0 \leq \kappa_i \leq \ell - i$ for all $i = 1, \dots, \ell$ are linearly independent over R_ℓ thanks to Remark 5.6. So this implies that $\dot{c}_{w,\kappa} = 0$ for all κ, which is a contradiction.

Now we treat the general case. The inclusion $ONH_n \otimes R_\ell \hookrightarrow ONH_\ell \otimes R_\ell$ induces an R_ℓ-superalgebra homomorphism $\imath : ONH_n^\ell \to ONH_\ell^\ell$. The monomials in ONH_n^ℓ which we are trying to show are linearly independent map to a subset of the monomials shown to be linearly independent in the previous paragraph. This completes the proof (and also shows that $\imath$ is injective). □

Corollary 9.6. *For $0 \leq n < \ell$, the graded superalgebra homomorphism $ONH_n^\ell \to ONH_{n+1}^\ell$ induced by the natural embedding $ONH_n \otimes R_\ell \hookrightarrow ONH_{n+1} \otimes R_\ell$ is injective.*

Proof. This follows immediately from the basis theorem in the previous corollary. □

Corollary 9.7. *The graded superalgebra ONH_n^ℓ is a graded Frobenius superalgebra over R_ℓ of degree $2n(\ell - n)$ and parity $n(\ell - n) \pmod 2$.*

Proof. This follows from Theorem 9.2 and Corollary 8.7. □

Remark 9.8. Since $\overline{ONH}_n^\ell \cong ONH_n^\ell \otimes_{R_\ell} \mathbb{F}$ and $\overline{OH}_n^\ell \cong OH_n^\ell \otimes_{R_\ell} \mathbb{F}$, Corollary 9.4 implies that $\overline{ONH}_n^\ell$ is isomorphic to the graded matrix superalgebra $M_{q^{\binom{n}{2}}[n]^!_{q,\pi}}\big(\overline{OH}_n^\ell\big)$, recovering [EKL, Prop. 5.2]. Also Corollary 9.5 implies that $\overline{ONH}_n^\ell$ has basis given by the canonical images of the monomials

$$\{x^\kappa \tau_w \mid w \in S_n, \kappa \in \mathbb{N}^n \text{ with } 0 \leq \kappa_i \leq \ell - i \text{ for } i = 1, \dots, n\},$$

recovering [HS, Th. 4.10]. The proof that these monomials span given in [HS] gives an explicit algorithm to "straighten" arbitrary monomials into this form.

In the remainder of the section, we prove Theorem 9.2. The approach is based on the proof of [EKL, Prop. 5.2] (the result which we are generalizing). First we record some preliminary lemmas.

Lemma 9.9. *Suppose that $n \geq 1$. Let y be a non-zero homogeneous element of $\mathrm{sh}_1(OPol_{n-1})$ and consider the (free) right $OSym_n$-submodule of $OPol_n$ with basis $v_1, \dots, v_n$ defined from $v_i := yx_1^{i-1}$. The matrix of the endomorphism of this subspace defined by the left action of $(-1)^{\mathrm{par}(y)}x_1$ is equal to the* *(non-commutative) companion matrix*

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & (-1)^{n-1}e_n^{(n)} \\ 1 & 0 & \cdots & 0 & (-1)^{n-2}e_{n-1}^{(n)} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 & -e_2^{(n)} \\ 0 & \cdots & 0 & 1 & e_1^{(n)} \end{pmatrix} \tag{9.2}$$

of the polynomial $(t - x_1) \cdots (t - x_n) \in OPol_n[t]$.

Proof. We have that $(-1)^{\mathrm{par}(y)} x_1 y x_1^{i-1} = y x_1^i$. This gives all but the last column of the matrix already. For the last column, use Lemma 6.10(1), taking n and n' there to be 1 and $n-1$ in the current setup, noting that $h_{r-s}^{(1)} = x_1^{r-s}$. □

Lemma 9.10. *Let C be the $n \times n$ companion matrix from (9.2). For any $1 \leq i, j \leq n$ and $k \geq 0$, the (i, j)-entry of C^k is equal to*

$$c_{i,j;k} := \sum_{t=0}^{\min(k+j-i,n-i)} (-1)^t e_t^{(n)} h_{k+j-i-t}^{(n)}, \tag{9.3}$$

which is zero if $k < i - j$ and 1 if $k = i - j$.

Proof. This goes by induction on $k = 0, 1, \ldots$. When $k = 0$, we have that

$$c_{i,j;k} = \sum_{t=0}^{j-i} (-1)^t e_t^{(n)} h_{j-i-t}^{(n)}$$

which is zero if $i > j$ as it is the empty sum, and it is $\delta_{i,j}$ if $i \leq j$ by the infinite Grassmannian relation. This checks the induction base. Then we take $k \geq 0$ and consider the (i, j)-entry of $C^{k+1} = CC^k$. Since C has at most two non-zero entries in its ith row, namely, its $(i, i-1)$-entry 1 if $i > 1$ and its (i, n)-entry $(-1)^{n-i} e_{n+1-i}^{(n)}$, we get by induction that the (i, j)-entry of C^{k+1} is equal to $c_{i-1,j;k} + (-1)^{n-i} e_{n+1-i}^{(n)} c_{n,j;k}$ where the first term should be omitted in the case that $i = 1$. This is equal to

$$\sum_{t=0}^{\min(k+1+j-i,n+1-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} + (-1)^{n-i} e_{n+1-i}^{(n)} h_{k+j-n}^{(n)},$$

interpreting $h_{k+j-n}^{(n)}$ as zero if $k < n - j$ and noting in the case that $i = 1$ that the first term here is zero by the infinite Grassmannian relation (so there is no need to omit it). To complete the proof, we need to show that

$$\sum_{t=0}^{\min(k+1+j-i,n-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} = \sum_{t=0}^{\min(k+1+j-i,n+1-i)} (-1)^t e_t^{(n)} h_{k+1+j-i-t}^{(n)} + (-1)^{n-i} e_{n+1-i}^{(n)} h_{k+j-n}^{(n)}.$$

If $k + 1 + j - i \leq n - i$ both sums are over $0 \leq t \leq k + 1 + j - i$ and the second term on the right hand side is zero by convention since $k + j - n < 0$, so the equality is true. If $k + 1 + j - i > n - i$ then the sum on the right hand side has one extra term when $t = n + 1 - i$ compared to the sum on the left hand side. But this extra term is $(-1)^{n+1-i} e_{n+1-i}^{(n)} h_{k+j-n}^{(n)}$ which cancels with the final term on the right hand side. □

Proof of Theorem 9.2. Let $A := ONH_n \otimes R_\ell$, $B := OSym_n \otimes R_\ell$ and $V := OPol_n \otimes R_\ell$, which is a graded (A, B)-superbimodule. By Remark 5.6, V is free as a graded right B-supermodule with basis $\{x_n^{\kappa_n} \cdots x_1^{\kappa_1} \otimes 1 \mid \kappa \in K_n\}$. Also, by Theorem 5.4, $A \cong \mathrm{End}_{\text{-}B}(V)$ so that A can be identified with the graded matrix superalgebra consisting of matrices $(a_{\kappa,\kappa'})_{\kappa,\kappa' \in K_n}$, for $a_{\kappa,\kappa'} \in B$, this matrix representing the endomorphism $x^{\kappa'} \otimes 1 \mapsto \sum_{\kappa \in K_n} (x^\kappa \otimes 1) a_{\kappa,\kappa'}$. In this situation, Morita theory implies that there are bijections between the sets of graded superideals of A, graded sub-superbimodules of V and graded superideals of B so that $I \leftrightarrow IV = VJ \leftrightarrow J$. For $I \trianglelefteq A$ corresponding to $J \trianglelefteq B$ in this way, a set of generators for J is given by the matrix entries of a set of generators of I, and we have that $A/I \cong \mathrm{End}_{\text{-}B/J}(V/IV)$. Thus, to prove the theorem, we start from the two-sided ideal I of A from (9.1), which may be described equivalently as the two-sided ideal generated by the elements $\sum_{s=0}^{r}(-1)^s x_1^{r-s} \otimes \dot{e}_s^{(\ell)}$ $(r \geq \ell)$. We must show that the two-sided ideal J of B generated by the matrix entries of the generators of I is equal to B if $n > \ell$ and it is equal to the two-sided ideal of B from (8.1) if $n \leq \ell$.

Consider the matrix associated to the generator $\sum_{s=0}^{r}(-1)^s x_1^{r-s} \otimes \dot{e}_s^{(\ell)}$ of I for $r \geq \ell$. Lemma 9.9 implies that it is a block diagonal matrix with $n \times n$ blocks parametrized by all $\kappa \in K_n$ with $\kappa_1 = 0$, the (i, j)-entry of this block being the $x_n^{\kappa_n} \cdots x_2^{\kappa_2} x_1^{i-1} \otimes 1$-coefficient of $\sum_{s=0}^{r}(-1)^s x_1^{r-s} \otimes \dot{e}_s^{(\ell)} \cdot (x_n^{\kappa_n} \cdots x_2^{\kappa_2} x_1^{j-1} \otimes 1)$ for $1 \leq i, j \leq n$. Denote this matrix entry by $f_{i,j;r}^{(\kappa)}$. Using Lemmas 9.9 and 9.10, we have that

$$f_{i,j;r}^{(\kappa)} = (-1)^{r|\kappa|} \sum_{s=0}^{r} (-1)^{sj} \sum_{t=0}^{\min(r+j-i-s,n-i)} (-1)^t e_t^{(n)} h_{r+j-i-s-t}^{(n)} \otimes \dot{e}_s^{(\ell)}. \tag{9.4}$$

Apart from the leading sign which is irrelevant for the problem in hand, this does not depend on κ, so we may as well assume from now on that $\kappa = \mathbf{0} = (0, \ldots, 0)$. If $n > \ell$, we take $r = \ell$, $j = 1$ and $i = \ell + 1$, in which case the summations collapse and we deduce that $f_{\ell+1,1;\ell}^{(\mathbf{0})} = 1 \in J$. So $J = B$ as required in this case.

Now assume that $n \leq \ell$ and set $n' := \ell - n$. It remains to show that the elements

$$F := \left\{ f_{i,j;r}^{(\mathbf{0})} \,\middle|\, 1 \leq i, j \leq n, r \geq \ell \right\}, \qquad G := \left\{ \sum_{s=0}^{r} (-1)^s h_{r-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \,\middle|\, r > n' \right\}$$

generate the same two-sided ideal of B. We switch the summations in (9.4) to deduce that

$$f_{i,j;r}^{(\mathbf{0})} = \sum_{t=0}^{\min(r+j-i,n-i)} (-1)^t e_t^{(n)} \left(\sum_{s=0}^{\min(r,r+j-i-t)} (-1)^{sj} h_{r+j-i-t-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \right).$$

We have that $r + j - i \geq \ell + 1 - i > n - i$ so the first summation is over $0 \leq t \leq n - i$. Since $\dot{e}_s^{(\ell)} = 0$ for $s > \ell$ we can change the second summation so that it is over $0 \leq s \leq r + j - i - t$. Taking $i = n$ and $j = 1$ gives us the elements $\sum_{s=0}^{r-n+1}(-1)^s h_{r-n+1-s}^{(n)} \otimes \dot{e}_s^{(\ell)}$ for all $r \geq \ell$. Since we have all r so that $r - n + 1 > n'$, these already give us all of the elements of G, demonstrating one containment. It remains to show that all $f_{i,j;r}^{(\mathbf{0})}$ for $1 \leq i, j \leq n$ and $r \geq \ell$ also lie in $\langle G \rangle$. In fact, given that $t \leq n - i$, we have that

$$\sum_{s=0}^{r+j-i-t} (-1)^{sj} h_{r+j-i-t-s}^{(n)} \otimes \dot{e}_s^{(\ell)} \in \langle G \rangle$$

because $r + j - i - t \geq n' + j$, the sign is $(-1)^s$ if $j \geq 1$ is odd or 1 if $j \geq 2$ is even, and we know all of these elements lie in $\langle G \rangle$ by Lemma 8.4. ∎

10. The 2-category $\mathcal{OGB}im_\ell$ of odd Grassmannian bimodules

Throughout the section, ℓ is fixed. We will work always over the ground ring R_ℓ, but note that everything in this section also makes sense on base change from R_ℓ to any graded supercommutative R_ℓ-superalgebra, including the most important case when the ground ring is the field $\mathbb{F}$. For $0 \leq n \leq \ell$, we denote the element of OH_n^ℓ formerly denoted by $a \bar{\otimes} 1$ simply by $\bar{a}$ $(a \in OSym_n)$, and identify R_ℓ with a subalgebra of OH_n^ℓ via the embedding $R_\ell \hookrightarrow Z(OH_n^\ell)$, $\dot{c} \mapsto 1 \bar{\otimes} \dot{c}$ $(c \in OSym_\ell)$.

Suppose that $n, d, n' \geq 0$ with $n + d + n' = \ell$ and $\alpha \in \Lambda(k, d)$. The cases $\alpha = (d)$ and $\alpha = (1^d)$ will be particularly important. Let

$$\widetilde{V}_{n;\alpha}^\ell := OSym_{(n,\alpha_1,\ldots,\alpha_k,n')} \otimes_{OSym_\ell} R_\ell, \qquad U_{\alpha;n}^\ell := R_\ell \otimes_{OSym_\ell} OSym_{(n',\alpha_1,\ldots,\alpha_k,n)}. \tag{10.1}$$

These are graded R_ℓ-supermodules, with the action of R_ℓ on $\widetilde{V}^\ell_{n;\alpha}$ coming from the natural right action and the action of R_ℓ on $U^\ell_{\alpha;n}$ coming from the natural left action. We refer to $\widetilde{V}^\ell_{n;\alpha}$ and $U^\ell_{\alpha;n}$ as *odd Grassmannian bimodules*. According to the following lemma, they are graded superbimodules over equivariant odd Grassmannian cohomology algebras.

Lemma 10.1. *Let $\ell = n + d + n'$ and $\alpha \in \Lambda(k, d)$ be fixed as above.*

(1) There is a unique way to make $\widetilde{V}^\ell_{n;\alpha}$ into a graded $(OH^\ell_n, OH^\ell_{n+d})$-superbimodule so that the left action of $\bar{a}$ ($a \in OSym_n$) is defined by $\bar{a}(b \otimes 1) := ab \otimes 1$ for $b \in OSym_{(n,\alpha_1,\dots,\alpha_k,n')}$, and the right action of $\bar{a}$ ($a \in OSym_{n+d}$) is defined by $(b \otimes 1)\bar{a} := ba \otimes 1$ for $b \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$. Moreover:

(a) $\widetilde{V}^\ell_{n;\alpha}$ has basis $\left\{\gamma_{n+d}(p^{(n+d)}_w) \otimes 1 \;\middle|\; w \in [S_{n+d}/S_{(\alpha_k,\dots,\alpha_1,n)}]_{\min}\right\}$ as a right OH^ℓ_{n+d}-supermodule;

(b) $\widetilde{V}^\ell_{n;\alpha}$ has basis $\left\{\mathrm{sh}_n\,(p^{(n'+d)}_w) \otimes 1 \;\middle|\; w \in [S_{n'+d}/S_{(\alpha_1,\dots,\alpha_k,n')}]_{\min}\right\}$ as a left OH^ℓ_n-supermodule.

(2) There is a unique way to make $U^\ell_{\alpha;n}$ into a graded $(OH^\ell_{n+d}, OH^\ell_n)$-superbimodule so that the right action of $\bar{a}$ ($a \in OSym_n$) is defined by $(1 \otimes b)\bar{a} := (-1)^{(n'+d)\,\mathrm{par}(a)} 1 \otimes b\,\mathrm{sh}_{n'+d}(a)$ for $b \in OSym_{(n',\alpha_1,\dots,\alpha_k,n)}$, and the left action of $\bar{a}$ ($a \in OSym_{n+d}$) is defined by $\bar{a}(1 \otimes \mathrm{sh}_{n'}(b)) := (-1)^{n'\,\mathrm{par}(a)} 1 \otimes \mathrm{sh}_{n'}(ab)$ for $b \in OSym_{(\alpha_1,\dots,\alpha_k,n)}$. Moreover:

(a) $U^\ell_{\alpha;n}$ has basis $\left\{1 \otimes \gamma_{n'+d}(p^{(n'+d)}_w)^* \;\middle|\; w \in [S_{n'+d}/S_{(\alpha_k,\dots,\alpha_1,n')}]_{\min}\right\}$ as a right OH^ℓ_n-supermodule;

(b) $U^\ell_{\alpha;n}$ has basis $\left\{1 \otimes \mathrm{sh}_{n'}\,(p^{(n+d)}_w)^* \;\middle|\; w \in [S_{n+d}/S_{(\alpha_1,\dots,\alpha_k,n)}]_{\min}\right\}$ as a left OH^ℓ_{n+d}-supermodule.

Proof. (1) By Theorem 6.8, $OSym_{(n,\alpha_1,\dots,\alpha_k,n')}$ is generated by the elements of $OSym_{(n,\alpha_1,\dots,\alpha_k)}$ as a right $OSym_\ell$-supermodule. Hence, $\widetilde{V}^\ell_{n;\alpha}$ is generated as an R_ℓ-supermodule by elements of the form $b \otimes 1$ for $b \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$. In view of this, provided that it is well defined, there is a unique way to make $\widetilde{V}^\ell_{n;\alpha}$ into a graded right OH^ℓ_{n+d}-supermodule such that $(b \otimes 1)\bar{a} = ba \otimes 1$ for all $b \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$ and $a \in OSym_{n+d}$. It is also clear that the right action of OH^ℓ_{n+d} defined in this way and the left action of OH^ℓ_n from the statement of the lemma commute with each other, again assuming that both actions are well defined.

To see that the right action is well defined, we have that

$$\widetilde{V}^\ell_{n;\alpha} \simeq OSym_{(n,\alpha_1,\dots,\alpha_k,n')} \otimes_{OSym_{(n+d,n')}} OSym_{(n+d,n')} \otimes_{OSym_\ell} R_\ell.$$

By Corollary 6.9(2), we deduce that

$$\widetilde{V}^\ell_{n;\alpha} = \bigoplus_{w \in \left[S_{n+d}/S_{(\alpha_k,\dots,\alpha_1,n)}\right]_{\min}} \gamma_{n+d}(p^{(n+d)}_w) \otimes (OSym_{(n+d,n')} \otimes_{OSym_\ell} R_\ell)$$

with each summand being a copy of $OSym_{(n+d,n')} \otimes_{OSym_\ell} R_\ell$ shifted in degree and parity. Hence, each of these subspaces is isomorphic to OH^ℓ_{n+d} via the isomorphism from Theorem 8.5(1). The right action of OH^ℓ_{n+d} we are defining is just the natural right action of OH^ℓ_{n+d} on itself transported through these isomorphisms. So it is well defined. We have also proved (1a).

For the left action, we have that

$$\widetilde{V}^\ell_{n;\alpha} \simeq OSym_{(n,\alpha_1,\dots,\alpha_k,n')} \otimes_{OSym_{(n,n'+d)}} OSym_{(n,n'+d)} \otimes_{OSym_\ell} R_\ell.$$

By Corollary 6.9(1), we deduce that

$$\widetilde{V}^\ell_{n;\alpha} = \bigoplus_{w \in \left[S_{n'+d}/S_{(\alpha_1,\dots,\alpha_k,n')}\right]_{\min}} \mathrm{sh}_n\left(p^{(n'+d)}_w\right) \otimes (OSym_{(n,n'+d)} \otimes_{OSym_\ell} R_\ell)$$

with each summand being a graded left $OSym_n \otimes R_\ell$-submodule isomorphic to $OSym_{(n,n'+d)} \otimes_{OSym_\ell} R_\ell$ (shifted in parity and degree). It remains to apply Theorem 8.5(1) to see that the left action of OH^ℓ_n is well defined. This also establishes (1b).

(2) This is similar to the proof of (1), using the isomorphism from Theorem 8.5(2) in place of the one from Theorem 8.5(1), and the left supermodule analogs of Theorem 6.8 and Corollary 6.9 obtained by applying $\gamma_\ell \circ *$ to those assertions. □

More often than not, we will work with a degree- and parity-shifted version of $\widetilde{V}^\ell_{n;\alpha}$ and, very occasionally, of $U^\ell_{\alpha;n}$. These have been chosen to ensure that the adjunctions in Theorems 11.3 and 11.5 below are even of degree 0, and to eliminate any additional shifts in the definition of the singular Rouquier complex in the next section. Recall that $n\#d$ denotes $n+(n+1)+\cdots+(n+d-1)$. For $\ell = n+d+n'$ and $\alpha \in \Lambda(k,d)$ as before, we define

$$V^\ell_{n;\alpha} := (\Pi Q^{-2})^{n\#d}\widetilde{V}^\ell_{n;\alpha}, \qquad \widetilde{U}^\ell_{\alpha;n} := (\Pi Q^{-2})^{n'\#d}U^\ell_{\alpha;n}, \tag{10.2}$$

recalling (2.5). We also refer to these as odd Grassmannian bimodules. Of course, the bases for $\widetilde{V}^\ell_{n;\alpha}$ and $U^\ell_{\alpha;n}$ from Lemma 10.1 are also bases for $V^\ell_{n;\alpha}$ and $\widetilde{U}^\ell_{\alpha;n}$.

We proceed to develop the properties of odd Grassmannian bimodules in a systematic way. Take $\ell = n+d+n'$ and $\alpha \in \Lambda(k,d)$. There are even degree 0 isomorphisms of graded R_ℓ-supermodules

$$*: \widetilde{V}^\ell_{n;\alpha} \xrightarrow{\sim} U^\ell_{\alpha;n'} \quad\text{and}\quad *: V^\ell_{n;\alpha} \xrightarrow{\sim} \widetilde{U}^\ell_{\alpha;n'}, \qquad a\otimes\dot{c} \mapsto (-1)^{\mathrm{par}(a)\,\mathrm{par}(c)}\dot{c}\otimes a^*, \tag{10.3}$$

$$*: U^\ell_{\alpha;n} \xrightarrow{\sim} \widetilde{V}^\ell_{n';\alpha} \quad\text{and}\quad *: \widetilde{U}^\ell_{\alpha;n} \xrightarrow{\sim} V^\ell_{n';\alpha}, \qquad \dot{c}\otimes a \mapsto (-1)^{\mathrm{par}(a)\,\mathrm{par}(c)}a^*\otimes\dot{c}. \tag{10.4}$$

The first pair of these with the roles of n and n' switched are two-sided inverses of the second pair.

Lemma 10.2. *Continue with $\ell = n+d+n'$ and $\alpha \in \Lambda(k,d)$.*

(1) The isomorphisms from (10.3) satisfy

$$(\bar{a}_1 v\bar{a}_2)^* = \psi^\ell_n(\bar{a}_1)v^*\psi^\ell_{n+d}(\bar{a}_2)$$

for $a_1 \in OSym_n$, $a_2 \in OSym_{n+d}$ and $v \in \widetilde{V}^\ell_{n;\alpha}$ or $v \in V^\ell_{n;\alpha}$.

(2) The isomorphisms from (10.4) satisfy

$$(\bar{a}_1 u\bar{a}_2)^* = (\psi^\ell_{n'})^{-1}(\bar{a}_1)u^*(\psi^\ell_{n'+d})^{-1}(\bar{a}_2)$$

for $a_1 \in OSym_n$, $a_2 \in OSym_{n+d}$ and $u \in U^\ell_{\alpha;n}$ or $u \in \widetilde{U}^\ell_{\alpha;n}$.

Proof. (1) It suffices to prove this for $v \in \widetilde{V}^\ell_{n;\alpha}$, then the identity for v viewed instead as a vector in $V^\ell_{n;\alpha}$ follows as the same parity shifts are used to define $V^\ell_{n;\alpha}$ from $\widetilde{V}^\ell_{n;\alpha}$ as $\widetilde{U}^\ell_{\alpha;n'}$ from $U^\ell_{\alpha;n'}$. We first consider right actions. Take $a \in OSym_{n+d}$. By (8.4), we have that $\psi^\ell_{n+d}(\bar{a}) = \sum_{i=1}^p \bar{a}_i\dot{c}_i$ where $a = \sum_{1=1}^p (-1)^{(n+d)\,\mathrm{par}(a_i)}\,\mathrm{sh}_{n+d}(a_i)^*c_i$. We saw in the proof of Lemma 10.1(1) that $\widetilde{V}^\ell_{n;\alpha}$ is spanned as an R_ℓ-supermodule by vectors of the form $b\otimes 1$ for $b \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$. So we may assume that $v = b\otimes 1$ for such a b. We have that

$$(v\bar{a})^* = (ba\otimes 1)^* = \left(\sum_{i=1}^p (-1)^{(n+d)\,\mathrm{par}(a_i)} b\,\mathrm{sh}_{n+d}(a_i)^*c_i\otimes 1\right)^* = \left(\sum_{i=1}^p (-1)^{(n+d+\mathrm{par}(b))\,\mathrm{par}(a_i)}\,\mathrm{sh}_{n+d}(a_i)^*b\otimes\dot{c}_i\right)^*$$

$$= \sum_{i=1}^p (-1)^{(n+d)\,\mathrm{par}(a_i)+(\mathrm{par}(b)+\mathrm{par}(a_i))\,\mathrm{par}(c_i)}\dot{c}_i\otimes b^*\,\mathrm{sh}_{n+d}(a_i) = (1\otimes b^*)\cdot\sum_{i=1}^p \bar{a}_i\dot{c}_i = v^*\psi^\ell_{n+d}(\bar{a}).$$

For left actions, take $a \in OSym_n$ with $a = \sum_{i=1}^p (-1)^{n\,\mathrm{par}(a_i)}\,\mathrm{sh}_n(a_i)^*c_i$, so that $\psi^\ell_n(\bar{a}) = \sum_{i=1}^p \bar{a}_i\dot{c}_i$. We may assume that $v = \mathrm{sh}_n(b)\otimes 1$ for $b \in OSym_{(\alpha_1,\dots,\alpha_k,n')}$. Then we have that

$$(\bar{a}v)^* = (a\,\mathrm{sh}_n(b)\otimes 1)^* = (-1)^{\mathrm{par}(a)\,\mathrm{par}(b)}(\mathrm{sh}_n(b)a\otimes 1)^*$$

$$= \left(\sum_{i=1}^p (-1)^{(\mathrm{par}(a_i)+\mathrm{par}(c_i))\,\mathrm{par}(b)+n\,\mathrm{par}(a_i)}\,\mathrm{sh}_n(ba_i^*)\otimes\dot{c}_i\right)^* = \sum_{i=1}^p (-1)^{\mathrm{par}(a_i)(n+\mathrm{par}(c_i))}\dot{c}_i\otimes\mathrm{sh}_n(a_ib^*)$$

$$= \left(\sum_{i=1}^{p} a_i \dot{c}_i \right) \cdot (1 \otimes \mathrm{sh}_n(b^*)) = \psi_n^\ell(\bar{a}) v^*.$$

(2) By (1), we have that $\left[\left((\phi_{n'}^\ell)^{-1}(\bar{a}_1)\right)u^*\left((\phi_{n'+d}^\ell)^{-1}(\bar{a}_2)\right)\right]^* = \bar{a}_1 u \bar{a}_2$ for $a_1 \in OSym_n, a_2 \in OSym_{n+d}$ and $u \in U_{\alpha;n}^\ell$ or $u \in \widetilde{U}_{\alpha;n}^\ell$. Now apply $*$ to both sides. □

Assuming still that $\ell = n + d + n'$ and $\alpha \in \Lambda(k, d)$, we next introduce a convenient shorthand for special elements of odd Grassmannian bimodules. Recall that α^{r} denotes the reversed composition $(\alpha_k, \dots, \alpha_1)$, and note that the involution $\gamma_d : OSym_d \to OSym_d$ interchanges the subalgebras $OSym_\alpha$ and $OSym_{\alpha^{\mathrm{r}}}$. For $f \in OSym_\alpha$ and $g \in OSym_{\alpha^{\mathrm{r}}}$, we let

$$\tilde{v}_{n;\alpha}(f) := \mathrm{sh}_n(f) \otimes 1 \in \widetilde{V}_{n;\alpha}^\ell, \qquad v_{n;\alpha}(f) := \mathrm{sh}_n(f) \otimes 1 \in V_{n;\alpha}^\ell, \tag{10.5}$$

$$u_{\alpha;n}(g) := (-1)^{n' \,\mathrm{par}(g)} 1 \otimes \mathrm{sh}_{n'}(\gamma_d(g)) \in U_{\alpha;n}^\ell, \quad \tilde{u}_{\alpha;n}(g) := (-1)^{n' \,\mathrm{par}(g)} 1 \otimes \mathrm{sh}_{n'}(\gamma_d(g)) \in \widetilde{U}_{\alpha;n}^\ell. \tag{10.6}$$

The vectors $\tilde{v}_{n;\alpha}(f)$ and $u_{\alpha;n}(g)$ are of the same degrees and parities as f and g, respectively. The vector $v_{n;\alpha}(f)$ is equal to $\tilde{v}_{n;\alpha}(f)$ but it is being viewed as an element of a different superbimodule—the left action of OH_n^ℓ is different due to the parity shift in (10.2). Also, $v_{n;\alpha}(f)$ is of degree $\deg(f) - 2(n\#d)$ and parity $\mathrm{par}(f) + n\#d \pmod 2$. Similarly, $\tilde{u}_{\alpha;n}(g)$ is equal to $u_{\alpha;n}(g)$ but with a different left action of OH_{n+d}^ℓ, and $\tilde{u}_{\alpha;n}(g)$ is of degree $\deg(g) - 2(n'\#d)$ and parity $\mathrm{par}(g) + n'\#d \pmod 2$. The isomorphisms $*$ from (10.3) and (10.4) satisfy

$$\tilde{v}_{n;\alpha}(f)^* = (-1)^{n \,\mathrm{par}(f)} u_{\alpha;n'}(\gamma_d(f)^*), \qquad v_{n;\alpha}(f)^* = (-1)^{n \,\mathrm{par}(f)} \tilde{u}_{\alpha;n'}(\gamma_d(f)^*), \tag{10.7}$$

$$u_{\alpha;n}(g)^* = (-1)^{n' \,\mathrm{par}(g)} \tilde{v}_{n';\alpha}(\gamma_d(g)^*), \qquad \tilde{u}_{\alpha;n}(g)^* = (-1)^{n' \,\mathrm{par}(g)} v_{n';\alpha}(\gamma_d(g)^*). \tag{10.8}$$

We point out also that the basis vectors in both of the bases constructed in Lemma 10.1(1) are of the form $\tilde{v}_n(f)$ for $f \in OSym_\alpha$. So the vectors $\tilde{v}_n(f)$ (resp., $v_n(f)$) for all $f \in OSym_\alpha$ generate $\widetilde{V}_{n;\alpha}^\ell$ (resp., $V_{n;\alpha}^\ell$) either as a left OH_n^ℓ-supermodule or as a right OH_{n+d}^ℓ-supermodule. Similarly, the bases vectors in Lemma 10.1(2) are of the form $u_n(g)$ for $g \in OSym_{\alpha^{\mathrm{rev}}}$. So the vectors $u_n(g)$ (resp., $\tilde{u}_n(g)$) for all $g \in OSym_{\alpha^{\mathrm{rev}}}$ generate $U_{\alpha;n}^\ell$ (resp., $\widetilde{U}_{\alpha;n}^\ell$) either as a left OH_{n+d}^ℓ-supermodule or as a right OH_n^ℓ-supermodule.

Lemma 10.3. *Let $\ell = n + d + d' + n'$, $\alpha \in \Lambda(k, d)$ and $\alpha' \in \Lambda(k', d')$.*

(1) There are unique isomorphisms of graded $(OH_n^\ell, OH_{n+d+d'}^\ell)$-superbimodules

$$\tilde{c}_{\alpha,\alpha'} : \widetilde{V}_{n;\alpha\sqcup\alpha'}^\ell \xrightarrow{\sim} \widetilde{V}_{n;\alpha}^\ell \otimes_{OH_{n+d}^\ell} \widetilde{V}_{n+d;\alpha'}^\ell, \tag{10.9}$$
$$\tilde{v}_{n;\alpha\sqcup\alpha'}(f\,\mathrm{sh}_d(f')) \mapsto \tilde{v}_{n;\alpha}(f) \otimes \tilde{v}_{n+d;\alpha'}(f')$$

and

$$c_{\alpha,\alpha'} : V_{n;\alpha\sqcup\alpha'}^\ell \xrightarrow{\sim} V_{n;\alpha}^\ell \otimes_{OH_{n+d}^\ell} V_{n+d;\alpha'}^\ell, \tag{10.10}$$
$$v_{n;\alpha\sqcup\alpha'}(f\,\mathrm{sh}_d(f')) \mapsto (-1)^{((n+d)\#d')\,\mathrm{par}(f)} v_{n;\alpha}(f) \otimes v_{n+d;\alpha'}(f')$$

for $f \in OSym_\alpha, f' \in OSym_{\alpha'}$.

(2) There are unique isomorphisms of graded $(OH_{n+d+d'}^\ell, OH_n^\ell)$-superbimodules

$$b_{\alpha',\alpha} : U_{\alpha'\sqcup\alpha;n}^\ell \xrightarrow{\sim} U_{\alpha';n+d}^\ell \otimes_{OH_{n+d}^\ell} U_{\alpha;n}^\ell \tag{10.11}$$
$$u_{\alpha'\sqcup\alpha;n}(\mathrm{sh}_d(f')f) \mapsto (-1)^{d' \,\mathrm{par}(f)} u_{\alpha';n+d}(f') \otimes u_{\alpha;n}(f)$$

and

$$\tilde{b}_{\alpha',\alpha} : \widetilde{U}_{\alpha'\sqcup\alpha;n}^\ell \xrightarrow{\sim} \widetilde{U}_{\alpha';n+d}^\ell \otimes_{OH_{n+d}^\ell} \widetilde{U}_{\alpha;n}^\ell \tag{10.12}$$
$$\tilde{u}_{\alpha'\sqcup\alpha;n}(\mathrm{sh}_d(f')f) \mapsto (-1)^{d' \,\mathrm{par}(f)+((n'+d')\#d)\,\mathrm{par}(f')} \tilde{u}_{\alpha';n+d}(f') \otimes \tilde{u}_{\alpha;n}(f)$$

for $f \in OSym_{\alpha^{\mathtt{r}}}, f' \in OSym_{(\alpha')^{\mathtt{r}}}$.

Proof. In each case, the uniqueness is clear since the vectors specified are superbimodule generators.
(1) Let $\gamma := (n, \alpha_1, \dots, \alpha_k, \alpha'_1, \dots, \alpha'_{k'}, n')$. Consider the surjective $\mathbb{F}$-linear map

$$OSym_\gamma \to \widetilde{V}^\ell_{n;\alpha} \otimes_{OH^\ell_{n+d}} \widetilde{V}^\ell_{n+d;\alpha'}, \qquad b_1\, \mathrm{sh}_{n+d}(b_2) \mapsto (b_1 \otimes 1) \otimes (\mathrm{sh}_{n+d}(b_2) \otimes 1)$$

for $b_1 \in OSym_{(n,\alpha_1,\dots,\alpha_k)}, b_2 \in OSym_{(\alpha'_1,\dots,\alpha'_{k'},n')}$. It is easy to check that it is a right $OSym_\ell$-supermodule homomorphism, so it induces a surjective R_ℓ-supermodule homomorphism $\tilde{c}_{\alpha,\alpha'} : OSym_\gamma \otimes_{OSym_\ell} R_\ell \to \widetilde{V}^\ell_{n;\alpha} \otimes_{OH^\ell_{n+d}} \widetilde{V}^\ell_{n+d;\alpha'}$. This is the map (10.9).

The domain and range of $\tilde{c}_{\alpha,\alpha'}$ are free graded R_ℓ-supermodules, so to see that $\tilde{c}_{\alpha,\alpha'}$ is an isomorphism it suffices to check that they have the same graded ranks. By Lemma 10.1(1a) and (3.8), $\widetilde{V}^\ell_{n;\alpha}$ is a free graded right OH^ℓ_{n+d}-supermodule of graded rank $q^{N(\alpha)+nd}\begin{bmatrix} n+d \\ (n,\alpha_1,\dots,\alpha_k)\end{bmatrix}_{q,\pi}$. By Lemma 10.1(1b) and (3.8), $\widetilde{V}^\ell_{n+d;\alpha'}$ is a free graded left OH^ℓ_{n+d}-supermodule of graded rank $q^{N(\alpha')+n'd'}\begin{bmatrix} n'+d' \\ (\alpha'_1,\dots,\alpha'_{k'},n')\end{bmatrix}_{q,\pi}$. By Corollary 8.6, OH^ℓ_{n+d} is a free graded R_ℓ-supermodule of graded rank $q^{(n+d)(n'+d')}\begin{bmatrix} \ell \\ n+d\end{bmatrix}_{q,\pi}$. Multiplying these together and using the identity

$$N(\gamma) = N(\alpha) + N(\alpha') + nd + n'd' + (n+d)(n'+d')$$

gives that $\widetilde{V}^\ell_{n;\alpha} \otimes_{OH^\ell_{n+d}} \widetilde{V}^\ell_{n+d;\alpha'}$ is a free graded R_ℓ-supermodule of graded rank $q^{N(\gamma)}\begin{bmatrix} \ell \\ \gamma\end{bmatrix}$. This is also the graded rank of $\widetilde{V}^\ell_{n;\alpha\sqcup\alpha'}$ as a free graded R_ℓ-supermodule, as follows from Theorem 6.8 and (6.23).

We still need to show that $\tilde{c}_{\alpha,\alpha'}$ is a graded $(OH^\ell_n, OH^\ell_{n+d+d'})$-supermodule homomorphism. We just go through the details for the right action whose definition is slightly more complicated than the left action. We restrict to considering just to vectors $b_1\, \mathrm{sh}_{n+d}(b_2) \otimes 1 \in \widetilde{V}^\ell_{n;\alpha\sqcup\alpha'}$ for $b_1 \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$ and $b_2 \in OSym_{(\alpha'_1,\dots,\alpha'_{k'})}$. We can do this because these vectors generate $\widetilde{V}^\ell_{n;\alpha\sqcup\alpha'}$ as an R_ℓ-supermodule. Then we take $a \in OSym_{n+d+d'}$, write it as $a = \sum_{i=1}^p a'_i\, \mathrm{sh}_{n+d}(a''_i)$ for $a'_i \in OSym_{(n,\alpha_1,\dots,\alpha_k)}$ and $a''_i \in OSym_{(\alpha'_1,\dots,\alpha'_{k'})}$, and calculate:

$$\begin{aligned}
\tilde{c}_{\alpha,\alpha'}\left((b_1\, \mathrm{sh}_{n+d}(b_2) \otimes 1)\bar{a}\right) &= \sum_{i=1}^p \tilde{c}_{\alpha,\alpha'}\left(b_1\, \mathrm{sh}_{n+d}(b_2) a'_i\, \mathrm{sh}_{n+d}(a''_i) \otimes 1\right) \\
&= \sum_{i=1}^p (-1)^{\mathrm{par}(b_2)\,\mathrm{par}(a'_i)} \tilde{c}_{\alpha,\alpha'}\left(b_1 a'_i\, \mathrm{sh}_{n+d}(b_2 a''_i)) \otimes 1\right) \\
&= \sum_{i=1}^p (-1)^{\mathrm{par}(b_2)\,\mathrm{par}(a'_i)} (b_1 a'_i \otimes 1) \otimes (\mathrm{sh}_{n+d}(b_2)\, \mathrm{sh}_{n+d}(a''_i) \otimes 1),
\end{aligned}$$

$$\begin{aligned}
\tilde{c}_{\alpha,\alpha'}\left(b_1\, \mathrm{sh}_{n+d}(b_2) \otimes 1\right)\bar{a} = ((b_1 \otimes 1) \otimes (\mathrm{sh}_{n+d}(b_2) \otimes 1))\bar{a} &= \sum_{i=1}^p (b_1 \otimes 1) \otimes (\mathrm{sh}_{n+d}(b_2) a'_i\, \mathrm{sh}_{n+d}(a''_i) \otimes 1) \\
&= \sum_{i=1}^p (-1)^{\mathrm{par}(b_2)\,\mathrm{par}(a'_i)} (b_1 \otimes 1) \otimes \bar{a}'_i (\mathrm{sh}_{n+d}(b_2)\, \mathrm{sh}_{n+d}(a''_i) \otimes 1) \\
&= \sum_{i=1}^p (-1)^{\mathrm{par}(b_2)\,\mathrm{par}(a'_i)} (b_1 \otimes 1)\bar{a}'_i \otimes (\mathrm{sh}_{n+d}(b_2)\, \mathrm{sh}_{n+d}(a''_i) \otimes 1) \\
&= \sum_{i=1}^p (-1)^{\mathrm{par}(b_2)\,\mathrm{par}(a'_i)} (b_1 a'_i \otimes 1) \otimes (\mathrm{sh}_{n+d}(b_2)\, \mathrm{sh}_{n+d}(a''_i) \otimes 1).
\end{aligned}$$

These are equal so $\tilde{c}_{\alpha,\alpha'}$ is a right $OH^\ell_{n+d+d'}$-supermodule homomorphism. We have now established the existence of (10.9).

To obtain (10.10), we define a superbimodule isomorphism $c_{\alpha,\alpha'}$ so that the following diagram commutes:

$$\begin{array}{ccc} V^\ell_{n;\alpha\sqcup\alpha'} & \xrightarrow{c_{\alpha,\alpha'}} & V^\ell_{n;\alpha} \otimes_{OH^\ell_{n+d}} V^\ell_{n+d;\alpha'} \\ \text{id}\downarrow & & \uparrow \text{id}\otimes\text{id} \\ \widetilde{V}^\ell_{n;\alpha\sqcup\alpha'} & \xrightarrow[\tilde{c}_{\alpha,\alpha'}]{} & \widetilde{V}^\ell_{n;\alpha} \otimes_{OH^\ell_{n+d}} \widetilde{V}^\ell_{n+d;\alpha'} \end{array}$$

The vertical maps here arise from identity maps on the underlying vector spaces, which are graded superbimodule isomorphisms but they are not even of degree 0. It remains to compute $c_{\alpha,\alpha'}(v_{n;\alpha\sqcup\alpha'}(f\,\mathrm{sh}_d(f')))$ explicitly by tracing it around the other three sides of the square to see that it is exactly the map $c_{\alpha,\alpha'}$ from (10.10). The complicated sign arises because the right hand map takes $\tilde{v}_{n;\alpha}(f) \otimes \tilde{v}_{n+d;\alpha'}(f')$ to $(-1)^{((n+d)\#d')\,\mathrm{par}(f)} v_{n;\alpha}(f) \otimes v_{n+d;\alpha'}(f')$ since $\mathrm{id} : \widetilde{V}^\ell_{n+d;\alpha'} \to V^\ell_{n+d;\alpha'}$ is of parity $(n+d)\#d'$.

(2) Writing $*$ for the appropriate one of the isomorphisms from (10.3) and (10.4), we define $b_{\alpha',\alpha}$ to be the composition $(*\otimes*)\circ\tilde{c}_{\alpha',\alpha}\circ *$. The appropriate diagram is

$$\begin{array}{ccc} U^\ell_{\alpha'\sqcup\alpha;n} & \xrightarrow{b_{\alpha',\alpha}} & U^\ell_{\alpha';n+d} \otimes_{OH^\ell_{n+d}} U^\ell_{\alpha;n} \\ *\downarrow & & \uparrow *\otimes* \\ \widetilde{V}^\ell_{n';\alpha'\sqcup\alpha} & \xrightarrow[\tilde{c}_{\alpha',\alpha}]{} & \widetilde{V}^\ell_{n';\alpha'} \otimes_{OH^\ell_{n'+d'}} \widetilde{V}^\ell_{n'+d';\alpha} \end{array}$$

The resulting isomorphism $b_{\alpha',\alpha}$ is a graded $(OH^\ell_{n+d+d'}, OH^\ell_n)$-superbimodule homomorphism thanks to Lemma 10.2(1). It remains to compute $b_{\alpha',\alpha}(\mathrm{sh}_d(f')f)$ for $f \in OSym_{\alpha^\mathrm{r}}, f' \in OSym_{(\alpha')^\mathrm{r}}$. Note that $\mathrm{sh}_d(f')f = (-1)^{\mathrm{par}(f)\,\mathrm{par}(f')} f\,\mathrm{sh}_d(f') \in OSym_{(\alpha'\sqcup\alpha)^\mathrm{r}}$. Using (10.7) and (10.8), we have that

$$\begin{aligned} \tilde{c}_{\alpha',\alpha}\left(u_{\alpha'\sqcup\alpha;n}(\mathrm{sh}_d(f')f)^*\right)^{*\otimes*} &= (-1)^{n'\,\mathrm{par}(f)+n'\,\mathrm{par}(f')+\mathrm{par}(f)\,\mathrm{par}(f')}\tilde{c}_{\alpha',\alpha}\left(\tilde{v}_{n';\alpha'\sqcup\alpha}\Big(\gamma_{d+d'}(f^*\,\mathrm{sh}_d((f')^*))\Big)\right)^{*\otimes*} \\ &= (-1)^{n'\,\mathrm{par}(f)+n'\,\mathrm{par}(f')}\tilde{c}_{\alpha',\alpha}\left(\tilde{v}_{n';\alpha'\sqcup\alpha}\Big(\gamma_{d'}(f')^*\,\mathrm{sh}_{d'}(\gamma_d(f)^*)\Big)\right)^{*\otimes*} \\ &= (-1)^{n'\,\mathrm{par}(f)+n'\,\mathrm{par}(f')}\left(\tilde{v}_{n';\alpha'}(\gamma_{d'}(f')^*)\otimes\tilde{v}_{n'+d';\alpha}(\gamma_d(f)^*)\right)^{*\otimes*} \\ &= (-1)^{d'\,\mathrm{par}(f)} u_{\alpha';n+d}(f')\otimes u_{\alpha;n}(f), \end{aligned}$$

which is the formula for $b_{\alpha',\alpha}\Big(u_{\alpha'\sqcup\alpha;n}(\mathrm{sh}_d(f')f)\Big)$ from (10.11). This establishes the existence of $b_{\alpha',\alpha}$. Finally, the existence of (10.12) can now be deduced in the same way that (10.10) was deduced from (10.9) in the proof of (1). □

The next lemma gives "Schur bases" for $V^\ell_{n;(d)}$ and $U^\ell_{(d);n}$, and for various specializations in which we are viewing $\mathbb{F}$ as a graded supermodule concentrated in degree 0 and even parity in the unique possible way. Similar statements hold for $\widetilde{V}^\ell_{n;(d)}$ and $\widetilde{U}^\ell_{(d);n}$, but these will not be needed.

Lemma 10.4. *Suppose that $\ell = n + d + n'$.*

(1) The supermodule $V^\ell_{n;(d)}$ has basis $\left\{v_{n;(d)}(s^{(d)}_\lambda)\;\middle|\;\lambda\in\Lambda^+_{d\times n'}\right\}$ as a free left OH^ℓ_n-supermodule, and basis $\left\{v_{n;(d)}(\sigma^{(d)}_\lambda)\;\middle|\;\lambda\in\Lambda^+_{d\times n}\right\}$ as a free right OH^ℓ_{n+d}-supermodule. Hence, the vectors $\left\{v_{n;(d)}(\sigma^{(d)}_\lambda)\otimes 1\;\middle|\;\lambda\in\Lambda^+_{d\times n}\right\}$ give a linear basis for $V^\ell_{n;(d)}\otimes_{OH^\ell_{n+d}}\mathbb{F}$. Moreover, for $\lambda\in\Lambda^+$ and any $f\in OSym_d$, we have that

$$v_{n;(d)}(f\sigma^{(d)}_\lambda)\otimes 1 = (-1)^{\overline{NE}(\lambda)+|\lambda|(\mathrm{par}(f)+n\#d)}\,\bar{s}^{(n)}_{\lambda^\mathrm{t}}\,v_{n;(d)}(f)\otimes 1 \tag{10.13}$$

in $V^\ell_{n;(d)} \otimes_{OH^\ell_{n+d}} \mathbb{F}$. In particular, $v_{n;(d)}(\sigma^{(d)}_\lambda) \otimes 1 = 0$ unless $\lambda \in \Lambda^+_{d\times n}$.

(2) The supermodule $U^\ell_{(d);n}$ has basis $\left\{u_{(d);n}(s^{(d)}_\lambda) \;\middle|\; \lambda \in \Lambda^+_{d\times n}\right\}$ as a free left OH^ℓ_{n+d}-supermodule, and basis $\left\{u_{(d);n}(\sigma^{(d)}_\lambda) \;\middle|\; \lambda \in \Lambda^+_{d\times n'}\right\}$ as a free right OH^ℓ_n-supermodule. Hence, the vectors $\left\{1 \otimes u_{(d);n}(s^{(d)}_\lambda) \;\middle|\; \lambda \in \Lambda^+_{d\times n}\right\}$ give a linear basis for $\mathbb{F} \otimes_{OH^\ell_{n+d}} U^\ell_{(d);n}$. Moreover, for $\lambda \in \Lambda^+$ and any $f \in OSym_d$, we have that

$$1 \otimes u_{(d);n}(s^{(d)}_\lambda f) = (-1)^{\overline{NE}(\lambda)+|\lambda|(\mathrm{par}(f)+d)} 1 \otimes u_{(d);n}(f)\bar{s}^{(n)}_{\lambda^{\mathrm{t}}} \tag{10.14}$$

in $\mathbb{F} \otimes_{OH^\ell_{n+d}} U^\ell_{(d);n}$. In particular, $1 \otimes u_{(d);n}(s^{(d)}_\lambda) = 0$ unless $\lambda \in \Lambda^+_{d\times n}$.

Proof. (1) The existence of the two families of Schur bases for $V^\ell_{n;(d)}$ follow in the same way as the bases in Lemma 10.1(1) were constructed, using Corollary 6.13 in place of Corollary 6.9. To prove (10.13), we have in $OSym_{(n,d)} \otimes_{OSym_{n+d}} \mathbb{F}$ that $\mathrm{sh}_n(\sigma^{(d)}_\lambda) \otimes 1 = (-1)^{\overline{NE}(\lambda)} s^{(n)}_{\lambda^{\mathrm{t}}} \otimes 1$ thanks to Corollary 7.4. Multiplying this identity on the left by $\mathrm{sh}_n(f)$ for $f \in OSym_d$ gives that

$$\mathrm{sh}_n(f\sigma^{(d)}_\lambda) \otimes 1 = (-1)^{\overline{NE}(\lambda)+|\lambda|\,\mathrm{par}(f)} s^{(n)}_{\lambda^{\mathrm{t}}}\,\mathrm{sh}_n(f) \otimes 1.$$

This implies (10.13).

For the final assertion, it remains to observe that if $\lambda \notin \Lambda^+_{d\times n}$ then we either have that $\lambda_1 > n$, in which case $s^{(n)}_{\lambda^{\mathrm{t}}} = 0$ by Theorem 6.3, or $\mathrm{ht}(\lambda) > d$, in which case $\sigma^{(d)}_\lambda = 0$ by Corollary 6.4. Hence, $\tilde{v}_{n;(d)}(\sigma^{(d)}_\lambda) \otimes 1 = \pm\tilde{v}_{n;(d)}(1) \otimes 1 = 0$ for such λ.

(2) The existence of the two families of bases follows by applying $*$ to the bases in (1), using also Lemma 10.2(1), (10.7) and (6.7). To prove (10.14) (which is *not* what one gets by applying $*$ to (10.13)), we start from the identity $s^{(d)}_\lambda \otimes 1 = (-1)^{\overline{NE}(\lambda)}\,\mathrm{sh}_d(\sigma^{(n)}_{\lambda^{\mathrm{t}}}) \otimes 1$ in $OSym_{d,n} \otimes_{OSym_{n+d}} \mathbb{F}$ from Corollary 7.4. Applying the isomorphism $*$ that is the right hand map of (7.2) gives $1 \otimes \sigma^{(d)}_\lambda = (-1)^{\overline{NE}(\lambda)} 1 \otimes \mathrm{sh}_d(s^{(n)}_{\lambda^{\mathrm{t}}})$ in $\mathbb{F} \otimes_{OSym_{n+d}} OSym_{d,n}$. This we multiply on the right by $\gamma_d(f) \in OSym_d$ to obtain $1 \otimes \sigma^{(d)}_\lambda \gamma_d(f) = (-1)^{\overline{NE}(\lambda)+\mathrm{par}(f)|\lambda|} 1 \otimes \gamma_d(f)\,\mathrm{sh}_d(s^{(n)}_{\lambda^{\mathrm{t}}})$. This implies that

$$1 \otimes \mathrm{sh}_{n'}(\sigma^{(d)}_\lambda \gamma_d(f)) = (-1)^{\overline{NE}(\lambda)+\mathrm{par}(f)|\lambda|} 1 \otimes \mathrm{sh}_{n'}(\gamma_d(f))\,\mathrm{sh}_{n'+d}(s^{(n)}_{\lambda^{\mathrm{t}}})$$

in $\mathbb{F} \otimes_{OH^\ell_{n+d}} U^\ell_{(d);n}$. Using the definition of the right action in Lemma 10.1(2) and (10.6), this implies (10.14).

The final assertion follows as in (1). ∎

From this point onwards, we will denote $\widetilde{V}^\ell_{n;(1)}, V^\ell_{n;(1)}, U^\ell_{(1);n}$ and $\widetilde{U}^\ell_{(1);n}$ by $\widetilde{V}^\ell_n, V^\ell_n, U^\ell_n$ and $\widetilde{U}^\ell_n$, respectively. We denote the generators $\tilde{v}_{n;(1)}(f), v_{n;(1)}(f), u_{(1);n}(g)$ and $\tilde{u}_{(1);n}(g)$ from (10.5) and (10.6) by $\tilde{v}_n(f), v_n(f), u_n(g)$ and $\tilde{u}_n(g)$ for $f, g \in OSym_1$. It is also convenient to write simply x in place of x_1 when working in rank 1, i.e., we identify $OSym_1$ with the graded polynomial superalgebra $\mathbb{F}[x]$ generated by the variable x that is odd of degree 2 so that $x_1 \in OSym_1$ is identified with $x \in \mathbb{F}[x]$. So, for $f(x) \in \mathbb{F}[x]$, we have that

$$\tilde{v}_n(f(x)) = f(x_{n+1}) \otimes 1 \in \widetilde{V}^\ell_n, \qquad v_n(f(x)) = f(x_{n+1}) \otimes 1 \in V^\ell_n, \tag{10.15}$$

$$u_n(f(x)) = (-1)^{n'\,\mathrm{par}(f)} 1 \otimes f(x_{n'+1}) \in U^\ell_n, \qquad \tilde{u}_n(f(x)) = (-1)^{n'\,\mathrm{par}(f)} 1 \otimes f(x_{n'+1}) \in \widetilde{U}^\ell_n \tag{10.16}$$

for n' defined from $\ell = n+1+n'$. For further motivation for the significance of the graded (OH_n, OH_{n+1})-superbimodule V^ℓ_n and the graded (OH_{n+1}, OH_n)-superbimodule U^ℓ_n, see Corollaries 11.4 and 11.6.

Applying a sequence of the isomorphisms from (10.10) and (10.11) in any way that makes sense, we obtain even degree 0 isomorphisms

$$c_{(1)^d} : V^\ell_{n;(1^d)} \xrightarrow{\sim} V^\ell_n \otimes_{OH^\ell_{n+1}} V^\ell_{n+1} \otimes_{OH^\ell_{n+2}} \cdots \otimes_{OH^\ell_{n+d-1}} V^\ell_{n+d-1} \tag{10.17}$$

$$v_{n;(1^d)}\left(x_1^{\kappa_1}\cdots x_d^{\kappa_d}\right)\mapsto(-1)^{\sum_{i=1}^d((n+i)\#(d-i))\kappa_i}v_n\left(x^{\kappa_1}\right)\otimes\cdots\otimes v_{n+d-1}\left(x^{\kappa_d}\right)$$

of $(OH_n^\ell, OH_{n+d}^\ell)$-superbimodules, and

$$b_{(1)^d}: U_{(1^d);n}^\ell \xrightarrow{\sim} U_{n+d-1}^\ell \otimes_{OH_{n+d-1}^\ell}\cdots\otimes_{OH_{n+2}^\ell} U_{n+1}^\ell \otimes_{OH_{n+1}^\ell} U_n^\ell \tag{10.18}$$

$$u_{(1^d);n}\left(x_d^{\kappa_d}\cdots x_1^{\kappa_1}\right)\mapsto(-1)^{\sum_{i=1}^d(d-i)\kappa_i}u_{n+d-1}\left(x^{\kappa_d}\right)\otimes\cdots\otimes u_n\left(x^{\kappa_1}\right)$$

of $(OH_{n+d}^\ell, OH_n^\ell)$-superbimodules.

The parity shifts incorporated into the definitions of the actions in the next lemma have been included to ensure that the actions agree with (13.32) and (13.33) below.

Lemma 10.5. *Suppose that $\ell = n + d + n'$.*

(1) There is a left action of ONH_d on $V_{n;(1^d)}^\ell$ making it into an $(OH_n^\ell\otimes ONH_d, OH_{n+d}^\ell)$-superbimodule so that $a\cdot v_{n;(1^d)}(f) = (-1)^{(n\#(d-1))\,\mathrm{par}(a)}v_{n;(1^d)}(a\cdot f)$ for $a\in ONH_d$ and $f\in OPol_d$. Moreover, the inclusion $OSym_d\hookrightarrow OPol_d$ induces an isomorphism $V_{n;(d)}^\ell \xrightarrow{\sim} (\omega\xi)_d\cdot \widetilde{V}_{n;(1^d)}^\ell$ of $(OH_n^\ell, OH_{n+d}^\ell)$-superbimodules taking $v_{n;(d)}(f)\mapsto v_{n;(1^d)}(f)$ for $f\in OSym_d$. Hence, we have that

$$V_{n;(1^d)}^\ell \simeq \bigoplus_{w\in S_d}(\Pi Q^2)^{\ell(w)}V_{n;(d)}^\ell \tag{10.19}$$

as graded $(OH_n^\ell, OH_{n+d}^\ell)$-superbimodules.

(2) There is a right action of ONH_d on $U_{(1^d);n}^\ell$ making it an $(OH_{n+d}^\ell, OH_n^\ell\otimes ONH_d)$-superbimodule so that $u_{(1^d);n}(f)\cdot a = (-1)^{(d-1)\,\mathrm{par}(a)}u_{(1^d);n}(f\cdot a)$ for $a\in ONH_d$ and $f\in OPol_d$; the right action of ONH_d on $OPol_d$ being used here is the one from (5.31). Moreover, the inclusion $OSym_d\hookrightarrow OPol_d$ induces an isomorphism $U_{(d);n}^\ell\xrightarrow{\sim} U_{(1^d);n}^\ell\cdot(\xi\omega)_d$ taking $u_{(d);n}(f)\mapsto u_{(1^d);n}(f)$ for $f\in OSym_n$. Hence, we have that

$$U_{(1^d);n}^\ell \simeq \bigoplus_{w\in S_d}(\Pi Q^2)^{\ell(w)}U_{(d);n}^\ell \tag{10.20}$$

as graded $(OH_{n+d}^\ell, OH_n^\ell)$-superbimodules.

Proof. (1) Since $\widetilde{V}_{n;(1^d)}^\ell = OSym_{(n,1^d,n')}\otimes_{OSym_\ell} R_\ell$ and $OSym_{(n,1^d,n')} = OSym_n\otimes OPol_d\otimes OSym_{n'}$, the left action of ONH_d on $OPol_d$ from Section 5 twisted by the automorphism $\mathrm{p}^{n+d-1}: ONH_d\to ONH_d$ induces a left action of ONH_d on $\widetilde{V}_{n;(1^d)}^\ell$ such that $a\cdot v_{n;(1^d)}(f) = (-1)^{(n+d-1)\,\mathrm{par}(a)}v_{n;(1^d)}(a\cdot f)$ for $a\in ONH_d$ and $f\in OPol_d$. This supercommutes with the left action of OH_n^ℓ and commutes with the right action of OH_{n+d}^ℓ, so it makes $\widetilde{V}_{n;(1^d)}^\ell$ into an $(OH_n^\ell\otimes ONH_d, OH_{n+d}^\ell)$-superbimodule. We get the action of ONH_d on $V_{n;(1^d)}^\ell$ in the statement of the lemma on incorporating the additional sign of $(-1)^{(n\#d)\,\mathrm{par}(f)}$ into the action, which comes from the parity shift $\Pi^{n\#d}$ in the definition (10.2) of $V_{n;(1^d)}^\ell$. Since $OSym_d = (\omega\xi)_d\cdot OPol_d$, the inclusion $OSym_d\hookrightarrow OPol_d$ induce an isomorphism $V_{n;(d)}^\ell\xrightarrow{\sim}(\omega\xi)_d\cdot V_{n;(1^d)}^\ell$. The decomposition (10.19) follows from Theorem 5.4.

(2) This is similar, starting from the right action of ONH_d on $OPol_d$ discussed at the end of Section 5 composed with p^{d-1}. The last assertions in (2) follow because $OSym_d = OPol_d\cdot(\xi\omega)_d$, and (10.20) follows from (5.33). □

In view of Lemmas 10.3 and 10.5, all of the odd Grassmannian bimodules $\widetilde{V}_{n;\alpha}^\ell$, $V_{\alpha;n}^\ell$, $U_{\alpha;n}^\ell$ and $\widetilde{U}_{\alpha;n}^\ell$ are isomorphic to 1-morphisms in the 2-supercategory $\mathcal{OGBim}_\ell$ introduced in the next important definition.

Definition 10.6. The category $\mathcal{OGBim}_\ell$ of *odd Grassmannian bimodules* is the full additive graded sub-(Q,Π)-2-supercategory of the (weak) graded (Q,Π)-2-supercategory of graded superalgebras, graded superbimodules and superbimodule homomorphisms consisting of objects that are the graded superalgebras OH_n^ℓ for $0 \le n \le \ell$ plus a distinguished object that is a trivial (zero) graded superalgebra, and 1-morphisms that are generated by the odd Grassmannian bimodules $V_n^\ell \in \mathrm{Hom}_{\mathcal{OGBim}_\ell}(OH_{n+1}^\ell, OH_n^\ell)$ and $U_n^\ell \in \mathrm{Hom}_{\mathcal{OGBim}_\ell}(OH_n^\ell, OH_{n+1}^\ell)$ for $0 \le n < \ell$.

Remark 10.7. Although not a strict 2-supercategory, we often work with $\mathcal{OGBim}_\ell$ as though it was in fact strict. More formally, when we do this, we are replacing it with its strictification. The latter can be realized as a 2-subcategory of the strict graded 2-supercategory of graded supercategories, graded superfunctors and graded supernatural transformations in such a way that the graded 2-superequivalence from $\mathcal{OGBim}_\ell$ takes the graded superalgebra OH_n^ℓ to the graded supercategory OH_n^ℓ-gsMod, a graded (OH_m^ℓ, OH_n^ℓ)-superbimodule M in $\mathcal{OGBim}_\ell$ to the graded superfunctor $M \otimes_{OH_n} - : OH_n^\ell\text{-gsMod} \to OH_m^\ell\text{-gsMod}$, and a graded superbimodule endomorphism $f : M \to M'$ to the graded supernatural transformation $f \otimes \mathrm{id} : M \otimes_{OH_n} - \Rightarrow M' \otimes_{OH_n} -$.

The next lemma gives explicit presentations for the generating odd Grassmannian bimodules V_n^ℓ and U_n^ℓ. In formulating the result, we also incorporate an indeterminate t into our notation, working in the $R_\ell(\!(t^{-1})\!)$-supermodules $V_n^\ell(\!(t^{-1})\!)$ and $U_n^\ell(\!(t^{-1})\!)$, this being a natural extension of the generating function formalism developed already for odd symmetric functions.

Lemma 10.8. *Suppose that $\ell = n + 1 + n'$.*

(1) Let $V := OH_n^\ell \otimes_{R_\ell} R_\ell[x] \otimes_{R_\ell} OH_{n+1}^\ell$, which is the free graded $(OH_n^\ell, OH_{n+1}^\ell)$-superbimodule on the graded R_ℓ-supermodule $R_\ell[x]$. For $f(x) \in \mathbb{F}[x] \subset R_\ell[x]$, we denote $1 \otimes f(x) \otimes 1 \in V$ by $v(f(x))$. Let T be the sub-bimodule of V generated by either of the following equivalent relations[2]*:*

$$v(x^r)\bar{e}^{(n+1)}(t) = (-1)^{n(r+1)}\bar{e}^{(n)}((-1)^{n+r}t)v((t-x)x^r), \tag{10.21}$$

$$\bar{\varepsilon}^{(n)}(t)v(x^r) = (-1)^{n(r+1)}v(((-1)^{n+r}t - x)^{-1}x^r)\,\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t), \tag{10.22}$$

for $r \ge 0$. There is an isomorphism of graded $(OH_n^\ell, OH_{n+1}^\ell)$-superbimodules

$$V\,/\,T \xrightarrow{\sim} V_n^\ell, \qquad v(f(x)) + T \mapsto v_n(f(x)). \tag{10.23}$$

Moreover:

(a) V_n^ℓ is free as a graded left OH_n^ℓ-supermodule with basis $\{v_n(x^r) \mid 0 \le r \le n'\}$;

(b) V_n^ℓ is free as a graded right OH_{n+1}^ℓ-supermodule with basis $\{v_n(x^r) \mid 0 \le r \le n\}$;

(c) the vector $v_n(1)$ generates V_n^ℓ as a graded $(OH_n^\ell, OH_{n+1}^\ell)$-superbimodule.

(2) Let $U := OH_{n+1}^\ell \otimes_{R_\ell} R_\ell[x] \otimes_{R_\ell} OH_n^\ell$, which is the free graded $(OH_{n+1}^\ell, OH_n^\ell)$-superbimodule on the graded R_ℓ-supermodule $R_\ell[x]$. For $f \in \mathbb{F}[x] \subseteq R_\ell[x]$, we denote $1 \otimes f \otimes 1 \in U$ by $u(f)$. Let S be the sub-bimodule of U generated by either of the following equivalent relations:

$$\bar{e}^{(n+1)}(t)u(x^r) = (-1)^{n(r+1)}u((t-x)x^r)\bar{e}^{(n)}((-1)^{r+1}t), \tag{10.24}$$

$$u(x^r)\bar{\varepsilon}^{(n)}(t) = (-1)^{n(r+1)}\bar{\varepsilon}^{(n+1)}((-1)^{r+1}t)u(((-1)^{r+1}t - x)^{-1}x^r), \tag{10.25}$$

for $r \ge 0$. There is an isomorphism of graded $(OH_{n+1}^\ell, OH_n^\ell)$-superbimodules

$$U\,/\,S \xrightarrow{\sim} U_n^\ell, \qquad u(f(x)) + S \mapsto u_n(f(x)). \tag{10.26}$$

Moreover:

(a) U_n^ℓ is free as a right OH_n^ℓ-supermodule with basis $\{u_n(x^r) \mid 0 \le r \le n'\}$;

(b) U_n^ℓ is free as a graded left OH_{n+1}^ℓ-supermodule with basis $\{u_n(x^r) \mid 0 \le r \le n\}$;

[2] We mean the relations obtained by equating coefficients of powers of t on both sides.

(c) the vector $u_n(1)$ generates U_n^ℓ as a graded $(OH_{n+1}^\ell, OH_n^\ell)$-superbimodule.

Proof. Note to start with that (1a)–(1b) and (2a)–(2b) follow immediately from Lemma 10.4.

(1) In this paragraph, we prove the equivalence of the relations (10.21) and (10.22) by some formal manipulation with power series. Replacing t by $(-1)^{n+r}t$, the relations (10.21) are equivalent to

$$\bar{e}^{(n)}(t)v(((-1)^{n+r}t-x)x^r) = (-1)^{n(r+1)}v(x^r)\bar{e}^{(n+1)}((-1)^{n+r}t) \tag{10.27}$$

for all $r \geq 0$. We first show that the relations (10.27) are equivalent to the relations

$$\bar{e}^{(n)}(t)v(x^r) = (-1)^{(n+1)r}v(t(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r}t) - (-1)^{nr}v(x(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r+1}t) \tag{10.28}$$

for all $r \geq 0$. To deduce (10.27) from (10.28), we take the left hand side of (10.27), which equals $(-1)^{n+r}\bar{e}^{(n)}(t)v(tx^r) - \bar{e}^{(n)}(t)v(x^{r+1})$. Then we use (10.28) to commute $\bar{e}^{(n)}(t)$ to the right to obtain

$$\begin{aligned}&(-1)^{n(r+1)}v(t^2(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r}t) + (-1)^{(n+1)(r+1)}v(tx(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r+1}t)\\ &\quad - (-1)^{(n+1)(r+1)}v(tx(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r+1}t) + (-1)^{n(r+1)}v(x^2(t^2+x^2)^{-1}x^r)\bar{e}^{(n+1)}((-1)^{n+r}t).\end{aligned}$$

After making obvious cancellations, this is equal to the right hand side of (10.27). The reverse implication, that is, the deduction of (10.28) from (10.27) is similar: one starts with the right hand side of (10.28) then uses (10.27) to commute the $\bar{e}^{(n+1)}(\pm t)$ term to the left. So the relations (10.21) and (10.28) are equivalent. Then we prove that (10.28) and (10.22) are equivalent using the identities

$$\varepsilon^{(n)}(t) = \frac{\mathbf{i}^{1-n}e^{(n)}(\mathbf{i}t) + \mathbf{i}^n e^{(n)}(-\mathbf{i}t)}{1+\mathbf{i}}, \qquad e^{(n)}(t) = \frac{\mathbf{i}^{1-n}\varepsilon^{(n)}(\mathbf{i}t) + \mathbf{i}^n \varepsilon^{(n)}(-\mathbf{i}t)}{1+\mathbf{i}} \tag{10.29}$$

where $\mathbf{i} \in \mathbb{F}$ denotes a square root of -1, which may be verified by equating coefficients on each side. To pass from (10.28) to (10.22), we replace $\varepsilon^{(n)}(t)$ in (10.22) with this linear combination of $e^{(n)}(\pm\mathbf{i}t)$, then use (10.28) with t replaced by $\pm\mathbf{i}t$ to commute $e^{(n)}(\pm\mathbf{i}t)$ to the right. At the end, a linear factor in the denominator $t^2 - x^2 = (t-x)(t+x)$ cancels, and after that one converts back to $\varepsilon^{(n)}(\pm t)$ using (10.29) again. This calculation is quite lengthy but elementary. The argument can be reversed to obtain (10.28) from (10.22), hence, the equivalence.

Next, we first check that the images of the relations (10.21) and (10.22) hold[3] for the actions of OH_n^ℓ and OH_{n+1}^ℓ on V_n^ℓ. For (10.21), it suffices to check that $v_n(1)\bar{e}^{(n+1)}(t) = (-1)^n\bar{e}^{(n)}((-1)^n t)v_n(t-x)$, i.e., the $r = 0$ case, for then we can act on the left with $x_1^r \in ONH_1$ using Lemma 10.5 to deduce the more general formulae. This follows because

$$\tilde{v}_n(1)\bar{e}^{(n+1)}(t) = \bar{e}^{(n)}(t)\tilde{v}_n(t-x) \tag{10.30}$$

in $\widetilde{V}_n^\ell[t] = OSym_{(n,1,n')} \otimes_{OSym_\ell} R_\ell[t]$ since, by the definition of the actions and (4.47), both sides are equal to $(t-x_1)\cdots(t-x_n)(t-x_{n+1}) \otimes 1$. Similarly, for (10.22), it suffices to check that $\bar{\varepsilon}^{(n)}(t)v_n(1) = (-1)^n v_n(((-1)^n t - x)^{-1})\bar{\varepsilon}^{(n+1)}((-1)^n t)$ or, equivalently, $(-1)^n\bar{\varepsilon}^{(n)}((-1)^n t)v_n(1) = v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t)$. This follows because

$$\bar{\varepsilon}^{(n)}\tilde{v}_n(1) = \tilde{v}_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t) \tag{10.31}$$

in $\widetilde{V}_n^\ell[t] = OSym_{(n,1,n')} \otimes_{OSym_\ell} R_\ell[t]$ since both sides are equal to $(t-x_n)\cdots(t-x_1) \otimes 1$ thanks to (4.47).

The relations check made in the previous paragraph implies that there is a well-defined graded superbimodule homomorphism $V/T \to V_n^\ell$ taking $v(f(x)) + T$ to $v_n(f(x))$ for all $f(x) \in \mathbb{F}[x]$. Moreover, this map is surjective by (1a)–(1b). To show that it is an isomorphism, it suffices to show that V/T is generated as a right OH_{n+1}^ℓ-module by the vectors $v(1) + T, v(x) + T, \ldots, v(x^n) + T$. It is generated as a superbimodule by all $v(x^r) + T$ $(r \geq 0)$. The relation (10.22) implies that any $\bar{a}v(f(x))$ for $a \in OSym_n, f(x) \in \mathbb{F}[x]$ can be expanded as a linear combination of vectors of the form $v(g(x))\bar{b}$ for

[3] Since (10.21) and (10.22) are equivalent, we really only need to check one of these. We check both because it is easy and explains how we discovered the relations in the first place.

$b \in OSym_{n+1}, g(x) \in \mathbb{F}[x]$. Hence, V/T is generated just as a right OH^ℓ_{n+1}-supermodule by the vectors $v(x^r) + T$ $(r \geq 0)$. Now we let V' be the right OH^ℓ_{n+1}-submodule of V generated by T and the vectors $v(1), v(x), \dots, v(x^n)$, and complete the argument by showing that $v(x^{n+r}) \in V'$ by induction on $r = 0, 1, 2, \dots$. The base $r = 0$ is vacuous. For the induction step, take $r > 0$. Consider the relation arising from the t^{-1}-coefficients in (10.22). The left hand side is a polynomial, so this coefficient is zero on the left hand side. Hence, the t^{-1}-coefficient of the right hand side belongs to $T \subseteq V'$. Working out this coefficient explicitly reveals that it equals $\pm v(x^{n+1+r})$ plus a linear combination of terms of the form $v(x^s)\bar{a}$ for $0 \leq s \leq n + r$ and $a \in OSym_{n+1}$ of positive degree. All of these "lower terms" are in V' by induction, hence, $v(x^{n+1+r}) \in V'$.

Finally, to establish (1c), looking at the t^n-coefficients of (10.21) shows that $v(x^{r+1}) + T$ lies in the sub-bimodule generated by $v(x^r) + T$ for any $r \geq 0$. Hence, by another induction on r, the sub-bimodule of V/T generated by $v(1) + T$ contains all $v(x^r) + T$.

(2) This follows by a similar argument to the proof of (1). We just explain how to see that the relations (10.24) and (10.25) both hold for the actions of OH^ℓ_n and OH^ℓ_{n+1} on U^ℓ_n. For (10.24), it suffices to check it in the case that $r = 0$, then one can act on the right with x^r using Lemma 10.5 to get the general result. To prove it when $r = 0$, it suffices to show that $\bar{e}^{(n+1)}(t)u_n(1) = (-1)^n u_n(t-x)\bar{e}^{(n)}(-t)$. This follows because, in view of the signs in the definition of the left and right actions in Lemma 10.1(2) and also (10.6), both sides are equal to $(t-(-1)^{n'}x_{n'+1})(t-(-1)^{n'}x_{n'+2})\cdots(t-(-1)^{n'}x_\ell)$. Similarly, to prove (10.25), it suffices to check the case $r = 0$, which amounts to showing that $u_n(1)\bar{\varepsilon}^{(n)}(t) = (-1)^{n+1}\bar{\varepsilon}^{(n+1)}(-t)u_n((t + x)^{-1})$. This follows because both sides are equal to $(t + (-1)^{n'}x_\ell)\cdots(t + (-1)^{n'}x_{n'+2})$. □

The final lemma in this section is an application of the presentation for V^ℓ_n derived in Lemma 10.8(1). Recall the graded R_ℓ-superalgebra automorphism $\delta^\ell_n : OH^\ell_n \to OH^\ell_n$ from (8.7). In terms of generating functions, we have that

$$\delta^\ell_n(\bar{e}^{(n)}(t)) = (-1)^{\ell n}\bar{e}^{(n)}((-1)^\ell t) - (-1)^{\ell n}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^\ell t) + (-1)^{\ell n}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+1}t). \tag{10.32}$$

This follows from (8.9) on equating coefficients of t.

Lemma 10.9. *There is a unique even degree 0 graded R_ℓ-supermodule automorphism $\phi^\ell_n : V^\ell_n \xrightarrow{\sim} V^\ell_n$ such that $\phi^\ell_n(v_n(1)) = v_n(1)$ and $\phi^\ell_n(avb) = \delta^\ell_n(a)\phi^\ell_n(v)\delta^\ell_{n+1}(b)$ for all $a \in OH^\ell_n, b \in OH^\ell_{n+1}$ and $v \in V^\ell_n$. Moreover, we have that*

$$\phi^\ell_n(v_n(x^r)) = (-1)^{\ell r}v_n(x^r) + (-1)^{(\ell+1)(r-1)}(1 - (-1)^r)\dot{o}^{(\ell)}v_n(x^{r-1}) \tag{10.33}$$

for all $r \geq 0$.

Proof. The uniqueness is clear since V^ℓ_n is a cyclic superbimodule thanks to Lemma 10.8(1c). To prove existence, consider the $(OH^\ell_n, OH^\ell_{n+1})$-superbimodule V that is V^ℓ_n with the left and right actions of OH^ℓ_n and OH^ℓ_{n+1} defined by twisting the usual actions with the automorphisms δ^ℓ_n and δ^ℓ_{n+1}, respectively. We are trying to show that there is a superbimodule homomorphism $\phi^\ell_n : V^\ell_n \to V$ satisfying (10.33). We can check this using the presentation for the superbimodule V^ℓ_n from Lemma 10.8(1) with the defining relations (10.21). This reduces the proof to checking that the identity

$$\phi^\ell_n(v_n(x^r))\delta^\ell_{n+1}(\bar{e}^{(n+1)}(t)) = (-1)^{n(r+1)}\delta^\ell_n\Big(\bar{e}^{(n)}((-1)^{n+r}t)\Big)\phi^\ell_n\big(v_n((t-x)x^r)\big) \tag{10.34}$$

is satisfied in the superbimodule V^ℓ_n for any $r \geq 0$. Substituting the formulae from (10.32) and (10.33) into (10.34) and cancelling $(-1)^{\ell r+\ell(n+1)}$ from both sides, this expands to the equation

$$\begin{aligned}
&\Big(v_n(x^r) - (-1)^{\ell+r}(1-(-1)^r)\dot{o}^{(\ell)}v_n(x^{r-1})\Big)\Big(\bar{e}^{(n+1)}((-1)^\ell t) - t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n+1)}((-1)^\ell t) + t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n+1)}((-1)^{\ell+1}t)\Big)\\
&= \Big((-1)^{n(r+1)}\bar{e}^{(n)}((-1)^{\ell+n+r}t) - (-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r}t) + (-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)\Big)\\
&\qquad \times \Big(v_n(((-1)^\ell t - x)x^r) - (-1)^r(1-(-1)^r)t\dot{o}^{(\ell)}v_n(x^{r-1}) - (-1)^{\ell+r}(1+(-1)^r)\dot{o}^{(\ell)}v_n(x^r)\Big).
\end{aligned}$$

Now we expand both sides using that $(\dot{o}^{(\ell)})^2 = 0$. Commuting $\dot{o}^{(\ell)}$ then $\bar{e}^{(n+1)}(\pm t)$ to the left with (10.21), the left hand side contributes the sum of the following four terms:

$$\begin{aligned}
v_n(x^r)\bar{e}^{(n+1)}((-1)^\ell t) &= (-1)^{n(r+1)}\bar{e}^{(n)}((-1)^{\ell+n+r}t)v_n(((-1)^\ell t - x)x^r),\\
-(-1)^{\ell+r}(1-(-1)^r)\dot{o}^{(\ell)}v_n(x^{r-1})\bar{e}^{(n+1)}((-1)^\ell t) &= -(-1)^{(n+1)r+\ell}(1-(-1)^r)\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)\\
&\qquad\times v_n(((-1)^\ell t - x)x^{r-1}),\\
-(-1)^{n+r}t^{-1}\dot{o}^{(\ell)}v_n(x^r)\bar{e}^{(n+1)}((-1)^\ell t) &= -(-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r}t)v_n(((-1)^\ell t - x)x^r),\\
(-1)^{n+r}t^{-1}\dot{o}^{(\ell)}v_n(x^r)\bar{e}^{(n+1)}((-1)^{\ell+1} t) &= -(-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)v_n(((-1)^\ell t + x)x^r).
\end{aligned}$$

Commuting $\dot{o}^{(\ell)}$ to the left, the right hand side contributes the sum of the following five terms:

$$\begin{aligned}
&(-1)^{n(r+1)}\bar{e}^{(n)}((-1)^{\ell+n+r}t)v_n(((-1)^\ell t - x)x^r),\\
-&(-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r}t)v_n(((-1)^\ell t - x)x^r),\\
&(-1)^{(n+1)r}t^{-1}\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)v_n(((-1)^\ell t - x)x^r),\\
-&(-1)^{(n+1)r}(1-(-1)^r)t\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)v_n(x^{r-1}),\\
-&(-1)^{(n+1)r+\ell}(1+(-1)^r)\dot{o}^{(\ell)}\bar{e}^{(n)}((-1)^{\ell+n+r+1}t)v_n(x^r).
\end{aligned}$$

From this, without making any further commutations, it follows that the two sides are equal. □

11. Rigidity of $\mathcal{OGBim}_\ell$

In this section, we prove that the 2-supercategory $\mathcal{OGBim}_\ell$ from Definition 10.6 is *rigid* in the sense that all of its 1-morphisms have left and right duals. Given a formal Laurent series $f(t)$, we use the notation $[f(t)]_{t^r}$ to denote its t^r-coefficient. Similarly, we use $[f(t)]_{t^{\leq r}}$, $[f(t)]_{t^{\geq r}}$, etc. for the formal Laurent series obtained by keeping only the terms with the specified powers of t. Note also the following elementary identity: we have that

$$f(x) = \left[(t-x)^{-1}f(t)\right]_{t^{-1}} \tag{11.1}$$

for any polynomial $f(x) \in \mathbb{F}[x]$. For the first lemma, recall the notation (10.15) and (10.16).

Lemma 11.1. *Suppose that $\ell = n + 1 + n'$ and $0 \leq r, s \leq n$.*

(1) $v_n((t-x)^{-1}x^r) = \sum_{p=r}^{n} v_n(x^p)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{>t^p}\bar{\eta}^{(n+1)}(t)t^{r-p-1} - \sum_{p=0}^{r-1} v_n(x^p)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^p}\bar{\eta}^{(n+1)}(t)t^{r-p-1}.$

(2) $u_n((t-x)^{-1}x^s) = \sum_{q=s}^{n} \bar{\eta}^{(n+1)}(t)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{>t^q}u_n(x^q)t^{s-q-1} - \sum_{q=0}^{s-1} \bar{\eta}^{(n+1)}(t)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^q}u_n(x^q)t^{s-q-1}.$

Proof. (1) Remembering (10.15) and the definition of the right action of OH^ℓ_{n+1} from Lemma 10.1(1), the identity obtained by applying γ_{n+1} to Lemma 6.11(2) implies for any $m \geq 0$ that

$$v_n(x^{m+n+1}) = -\sum_{p=0}^{n} v_n(x^p)\sum_{s=0}^{m}(-1)^{m+n+1-p-s}\bar{\varepsilon}^{(n+1)}_{m+n+1-p-s}\bar{\eta}^{(n+1)}_s.$$

We multiply this by $t^{r-m-n-2}$ and sum over $m \geq 0$ to obtain

$$\sum_{m\geq 0} v_n(x^{m+n+1})t^{r-m-n-2} = -\sum_{p=0}^{n} v_n(x^p)\left(\sum_{m\geq 0}\sum_{s=0}^{m}(-1)^{m+n+1-p-s}\bar{\varepsilon}^{(n+1)}_{m+n+1-p-s}\bar{\eta}^{(n+1)}_s t^{p-m-n-1}\right)t^{r-p-1}.$$

The expression in brackets is equal to $\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^p}\bar{\eta}^{(n+1)}(t)$, as may be checked by comparing $t^{p-m-n-1}$-coefficients for all $m\in\mathbb{Z}$. Using this and (4.49), we obtain

$$\begin{aligned} v_n((t-x)^{-1}x^r) &= \sum_{p=r}^{n} v_n(x^p)t^{r-p-1} + \sum_{m\geq 0} v_n(x^{m+n+1})t^{r-m-n-2} \\ &= \sum_{p=r}^{n} v_n(x^p)\bar{\varepsilon}^{(n+1)}(t)\bar{\eta}^{(n+1)}(t)t^{r-p-1} - \sum_{p=0}^{n} v_n(x^p)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^p}\bar{\gamma}^{(n+1)}(t)t^{r-p-1}. \end{aligned}$$

It remains to write the first $\bar{\varepsilon}^{(n+1)}(t)$ as $\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{>t^p}+\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^p}$ and make some obvious cancellations to obtain the desired formula.

(2) Remembering the sign in (10.16) and the matching sign in the definition of the left action of OH^ℓ_{n+1} from Lemma 10.1(2), the identity obtained by applying $\mathrm{p}^{n'}\circ\mathrm{sh}_{n'}\circ\gamma_{n+1}$ to Lemma 6.11(1) gives

$$u_n(x^{m+n+1}) = -\sum_{q=0}^{n}\sum_{r=0}^{m}(-1)^{m+n+1-q-r}\bar{\eta}^{(n+1)}_r\bar{\varepsilon}^{(n+1)}_{m+n+1-q-r}u_n(x^q)$$

for any $m\geq 0$. We multiply this by $t^{s-m-n-2}$ and sum over $m\geq 0$ to obtain

$$\sum_{m\geq 0}u_n(x^{m+n+1})t^{s-m-n-2} = -\sum_{q=0}^{n}\left(\sum_{m\geq 0}\sum_{r=0}^{m}(-1)^{m+n+1-q-r}\bar{\eta}^{(n+1)}_r\bar{\varepsilon}^{(n+1)}_{m+n+1-q-r}t^{q-m-n-1}\right)u_n(x^q)t^{s-q-1}.$$

The expression in brackets is equal to $\bar{\gamma}^{(n+1)}(t)\left[\bar{\varepsilon}^{(n+1)}(t)\right]_{\leq t^q}$. Now the proof is completed as in (1). □

Lemma 11.2. *Suppose that $\ell = n+1+n'$. The element*

$$z := \sum_{\substack{r,s\geq 0\\ r+s\leq n}}(-1)^{r+s}v_n(x^r)\bar{\varepsilon}^{(n+1)}_{n-r-s}\otimes u_n(x^s)\in V^\ell_n\otimes_{OH^\ell_{n+1}}U^\ell_n$$

is central in the sense that $az = za$ for all $a\in OH^\ell_n$.

Proof. By Lemma 10.8(b), any element of $V^\ell_n\otimes_{OH^\ell_{n+1}}U^\ell_n$ can be written as $\sum_{p,q=0}^n v_n(x^p)f_{p,q}\otimes u_n(x^q)$ for unique $f_{p,q}\in OH^\ell_{n+1}$. So there are unique $L_{p,q}(t)=\sum_{k=0}^n L_{p,q;k}t^k, R_{p,q}(t)=\sum_{k=0}^n R_{p,q;k}t^k\in OH^\ell_{n+1}[t]$ such that

$$\bar{\varepsilon}^{(n)}(t)z = \sum_{p,q=0}^{n}(-1)^{p+q}v_n(x^p)L_{p,q}(t)\otimes u_n(x^q),\qquad z\bar{\varepsilon}^{(n)}(t) = \sum_{p,q=0}^{n}(-1)^{p+q}v_n(x^p)R_{p,q}(t)\otimes u_n(x^q).$$

To prove the lemma, it suffices to show that $\bar{\varepsilon}^{(n)}(t)z = z\bar{\varepsilon}^{(n)}(t)$, which we do by computing $L_{p,q}(t)$ and $R_{p,q}(t)$ explicitly then checking that $L_{p,q;k}=R_{p,q;k}$ for all $0\leq p,q,k\leq n$. For brevity, we adopt the convention that $\varepsilon^{(n+1)}_r = 0$ for $r<0$; this allows the restriction $r+s\leq n$ on the summation in the definition of z to be omitted.

To compute $L_{p,q}(t)$, we expand $\bar{\varepsilon}^{(n)}(t)z$ using (10.22), Lemma 11.1(1) and (4.49):

$$\begin{aligned} \bar{\varepsilon}^{(n)}(t)z &= \sum_{r,q\geq 0}(-1)^{r+q}\bar{\varepsilon}^{(n)}(t)v_n(x^r)\bar{\varepsilon}^{(n+1)}_{n-r-q}\otimes u_n(x^q) \\ &= \sum_{r,q\geq 0}(-1)^{r+q+n(r+1)}v_n(((-1)^{n+r}t-x)^{-1}x^r)\,\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\bar{\varepsilon}^{(n+1)}_{n-r-q}\otimes u_n(x^q) \\ &= \sum_{r,q\geq 0}\sum_{p=r}^{n}v_n(x^p)\left((-1)^{r+q+n(r+1)+(n+r)(r-p-1)}\left[\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\right]_{>t^p}\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{r-p-1}\right)\otimes u_n(x^q) \end{aligned}$$

$$-\sum_{r,q\geq 0}\sum_{p=0}^{r-1} v_n(x^p)\left((-1)^{r+q+n(r+1)+(n+r)(r-p-1)}\left[\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\right]_{\leq t^p}\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{r-p-1}\right)\otimes u_n(x^q).$$

Hence, using $r+q+n(r+1)+(n+r)(r-p-1)+p+q \equiv pr+pn+p+r \pmod 2$ to simplify the sign, we have that

$$L_{p,q}(t) = \sum_{r=0}^{p}(-1)^{pr+pn+p+r}\left[\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\right]_{>t^p}\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{r-p-1} - \sum_{r\geq p+1}(-1)^{pr+pn+p+r}\left[\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\right]_{\leq t^p}\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{r-p-1} \tag{11.2}$$

for $0\leq p,q\leq n$. Similarly, we compute $R_{p,q}(t)$ using (10.25), Lemma 11.1(2) and (4.49):

$$\begin{aligned} z\bar{\varepsilon}^{(n)}(t) &= \sum_{p,s\geq 0}(-1)^{p+s}v_n(x^p)\bar{\varepsilon}^{(n+1)}_{n-p-s}\otimes u_n(x^s)\bar{\varepsilon}^{(n)}(t)\\ &= \sum_{p,s\geq 0}(-1)^{p+s+n(s+1)}v_n(x^p)\bar{\varepsilon}^{(n+1)}_{n-p-s}\bar{\varepsilon}^{(n+1)}((-1)^{s+1}t)\otimes u_n(((-1)^{s+1}t-x)^{-1}x^s)\\ &= \sum_{p,s\geq 0}\sum_{q=s}^{n} v_n(x^p)\left((-1)^{p+s+n(s+1)+(s+1)(s-q-1)}\bar{\varepsilon}^{(n+1)}_{n-p-s}\left[\bar{\varepsilon}^{(n+1)}((-1)^{s+1}t)\right]_{>t^q}t^{s-q-1}\right)\otimes u_n(x^q)\\ &\quad -\sum_{p,s\geq 0}\sum_{q=0}^{s-1} v_n(x^p)\left((-1)^{p+s+n(s+1)+(s+1)(s-q-1)}\bar{\varepsilon}^{(n+1)}_{n-p-s}\left[\bar{\varepsilon}^{(n+1)}((-1)^{s+1}t)\right]_{\leq t^q}t^{s-q-1}\right)\otimes u_n(x^q).\end{aligned}$$

From this, we get that

$$R_{p,q}(t) = \sum_{s=0}^{q}(-1)^{ns+qs+n+1}\bar{\varepsilon}^{(n+1)}_{n-p-s}\left[\bar{\varepsilon}^{(n+1)}((-1)^{s+1}t)\right]_{>t^q}t^{s-q-1} - \sum_{s\geq q+1}(-1)^{ns+qs+n+1}\bar{\varepsilon}^{(n+1)}_{n-p-s}\left[\bar{\varepsilon}^{(n+1)}((-1)^{s+1}t)\right]_{\leq t^q}t^{s-q-1} \tag{11.3}$$

for $0\leq p,q\leq n$.

Now we use (11.2) and (11.3) to compute the t^k-coefficients $L_{p,q;k}$ and $R_{p,q;k}$ for $0\leq k\leq n$:

$$L_{p,q;k} = \sum_{r=0}^{k}(-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q} - \sum_{r\geq p+1}(-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q}, \tag{11.4}$$

$$R_{p,q;k} = \sum_{s=0}^{k}(-1)^{(n+k)s}\bar{\varepsilon}^{(n+1)}_{n-p-s}\bar{\varepsilon}^{(n+1)}_{n+s-q-k} - \sum_{s\geq q+1}(-1)^{(n+k)s}\bar{\varepsilon}^{(n+1)}_{n-p-s}\bar{\varepsilon}^{(n+1)}_{n+s-q-k}. \tag{11.5}$$

The details of these two computations are very similar, so we just elaborate on the first one. Note that

$$(-1)^{pr+pn+p+r}\bar{\varepsilon}^{(n+1)}((-1)^{n+r}t)\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{r-p-1} = \sum_{j\in\mathbb{Z}}(-1)^{pr+pn+p+r+(n+r)j+n+1+j}\bar{\varepsilon}^{(n+1)}_{n+1-j}\bar{\varepsilon}^{(n+1)}_{n-r-q}t^{j+r-p-1}.$$

To get a contribution of t^k from this, we must have that $j = k+p-r+1$, in which case

$$(-1)^{pr+pn+p+r+(n+r)j+n+1+j}\bar{\varepsilon}^{(n+1)}_{n+1-j}\bar{\varepsilon}^{(n+1)}_{n-r-q} = (-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q}.$$

Using these observations, it follows that the t^k-coefficients from the first summation in (11.2) contribute $\sum(-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q}$ summing over r with $0\leq r\leq p$ such that $j := k+p-r+1$ satisfies $j>p$, i.e., $0\leq r\leq \min(k,p)$. Similarly, the t^k-coefficients from the second summation in (11.2) contribute

$-\sum(-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q}$ summing over r with $r \geq p+1$ such that $j := k+p-r+1$ satisfies $j \leq p$, i.e., $r \geq \max(k,p)+1$. Thus, we have shown that

$$L_{p,q;k} = \sum_{r=0}^{\min(k,p)} (-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q} - \sum_{r\geq\max(k,p)+1} (-1)^{(n+k)(r+k)}\bar{\varepsilon}^{(n+1)}_{n+r-p-k}\bar{\varepsilon}^{(n+1)}_{n-r-q}.$$

This is exactly as in (11.4) when $k \leq p$. To see that it is also equal to (11.4) when $k > p$, one just has to cancel the overlapping terms when $p+1 \leq r \leq k$ in the first and second summations from (11.4).

It remains to see that $L_{p,q;k} = R_{p,q;k}$ for every $0 \leq p,q,k \leq n$. In fact, the first summation in (11.4) is equal to the first summation in (11.5). This is easily seen on making the substitution $s = k-r$ in one of them. To see that the second summation in (11.4) is equal to the second summation in (11.5), we substitute $m = r-p$ in (11.4) and $m = s-q$ in (11.5), and the problem reduces to showing that

$$\sum_{m\geq1}(-1)^{(n+k)(m+p+k)}\bar{\varepsilon}^{(n+1)}_{n-k+m}\bar{\varepsilon}^{(n+1)}_{n-p-q-m} = \sum_{m\geq1}(-1)^{(n+k)(m+q)}\bar{\varepsilon}^{(n+1)}_{n-p-q-m}\bar{\varepsilon}^{(n+1)}_{n-k+m}$$

for all $0 \leq p,q,k \leq n$. In fact this equality already holds in $OSym_{n+1}$ thanks to Lemma 4.9 or, rather, the identity obtained from that by applying the involution γ_{n+1}. □

Theorem 11.3. *Suppose that $\ell = n+1+n'$. There are unique even degree 0 superbimodule homomorphisms*

$$\begin{aligned}\operatorname{coev}_n : OH^\ell_n &\to V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n\\ 1 &\mapsto \sum_{\substack{r,s\geq0\\ r+s\leq n}} (-1)^{n-r-s} v_n(x^r)\bar{\varepsilon}^{(n+1)}_{n-r-s}\otimes u_n(x^s)\end{aligned} \tag{11.6}$$

and

$$\begin{aligned}\operatorname{ev}_n : U^\ell_n \otimes_{OH^\ell_n} V^\ell_n &\to OH^\ell_{n+1}\\ u_n(x^r)\otimes v_n(x^s) &\mapsto \begin{cases}\bar{\eta}^{(n+1)}_{r+s-n} & \text{if } r+s\geq n\\ 0 & \text{otherwise,}\end{cases}\end{aligned} \tag{11.7}$$

the latter being true for all $r,s \geq 0$. Moreover, the following compositions are identities:

$$U^\ell_n \xrightarrow{\text{can}} U^\ell_n \otimes_{OH^\ell_n} OH^\ell_n \xrightarrow{\text{id}\otimes\text{coev}_n} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \xrightarrow{\text{ev}_n\otimes\text{id}} OH^\ell_{n+1} \otimes_{EOH^\ell_{n+1}} U^\ell_n \xrightarrow{\text{can}} U^\ell_n, \tag{11.8}$$

$$V^\ell_n \xrightarrow{\text{can}} OH^\ell_n \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\text{coev}_n\otimes\text{id}} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\text{id}\otimes\text{ev}_n} V^\ell_n \otimes_{OH^\ell_{n+1}} OH^\ell_{n+1} \xrightarrow{\text{can}} V^\ell_n. \tag{11.9}$$

Hence, coev_n *and* ev_n *are the unit and counit of an adjunction making $\left(U^\ell_n, V^\ell_n\right)$ into a dual pair of 1-morphisms in $\mathcal{OGBim}_\ell$.*

Before proving the theorem, we write down several equivalent formulations of the definitions of coev_n and ev_n, assuming that such superbimodule homomorphisms do indeed exist. For the unit of adjunction, the element $\operatorname{coev}_n(1)$ in the statement of Theorem 11.3 is equal to $(-1)^n z$ where z is the central tensor from Lemma 11.2. In terms of generating functions, we have that

$$\operatorname{coev}_n(1) = \left[v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t)\otimes u_n((t-x)^{-1})\right]_{t^{-1}}. \tag{11.10}$$

This is easily checked by computing coefficients of t. Using (10.22) and (10.25) with $r = 0$, we have that

$$v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t) = \bar{\varepsilon}^{(n)}((-1)^n t)v_n((-1)^n), \tag{11.11}$$

$$\bar{\varepsilon}^{(n+1)}(t)u_n((t-x)^{-1}) = u_n((-1)^n)\bar{\varepsilon}^{(n)}(-t). \tag{11.12}$$

Hence, we can rewrite the right hand side of (11.10) to obtain

$$\mathrm{coev}_n(1) = \left[v_n((t-x)^{-1}) \otimes u_n((-1)^n)\bar{\varepsilon}^{(n)}(-t)\right]_{t^{-1}} = \left[\bar{\varepsilon}^{(n)}((-1)^n t)v_n((-1)^n) \otimes u_n((t-x)^{-1})\right]_{t^{-1}}. \quad (11.13)$$

Equating coefficients in (11.13) gives two more formulae:

$$\mathrm{coev}_n(1) = \sum_{r=0}^{n} v_n(x^r) \otimes u_n(1)\bar{\varepsilon}^{(n)}_{n-r} = \sum_{s=0}^{n} (-1)^{(n+1)s}\bar{\varepsilon}^{(n)}_{n-s} v_n(1) \otimes u_n(x^s). \quad (11.14)$$

For the counit ev_n, we have the following two equivalent formulations:

$$\mathrm{ev}_n\left(u_n((t-x)^{-1}f(x)) \otimes v_n(g(x))\right) = \left[\bar{\eta}^{(n+1)}(t)f(t)g(t)\right]_{<t^0} \quad (11.15)$$

$$\mathrm{ev}_n\left(u_n(f(x)) \otimes v_n(g(x)(t-x)^{-1})\right) = \left[\bar{\eta}^{(n+1)}(t)f(t)g(t)\right]_{<t^0} \quad (11.16)$$

for any $f(x), g(x) \in \mathbb{F}[x]$. This can be checked by assuming that $f(x) = x^r, g(x) = x^s$ then equating coefficients of t.

Proof of Theorem 11.3. We split the proof up into six steps.

Step one. We first construct the maps coev_n and ev_n. For coev_n, we define $\mathrm{coev}_n(a) := (-1)^n az$ for any $a \in OH^\ell_n$, where z is as in Lemma 11.2. This is an even degree 0 homomorphism of graded left OH^ℓ_n-supermodules. Since $(-1)^n az = (-1)^n za$ according to Lemma 11.2, it is also a right OH^ℓ_n-supermodule homomorphism, so it is a superbimodule homomorphism. Thus, we have defined the superbimodule homomorphism coev_n, and (11.6) holds. For ev_n, $U^\ell_n \otimes_{OH^\ell_n} V^\ell_n$ is a free graded right OH^ℓ_{n+1}-supermodule with basis $u_n(x^r) \otimes v_n(x^s)$ $(0 \leq r \leq n', 0 \leq s \leq n)$ by Lemma 10.8(1b) and (2a). So there is a unique even degree 0 graded right OH^ℓ_{n+1}-supermodule homomorphism ev_n such that (11.7) holds for $0 \leq r \leq n'$ and $0 \leq s \leq n$. It is not yet clear that (11.7) holds for other values of r and s, or that ev_n is a graded left OH^ℓ_{n+1}-supermodule homomorphism.

Step two. Next we use induction to show that (11.7) also holds for $0 \leq r \leq n'$ and all $s > n$. Fix a choice of r with $0 \leq r \leq n'$. We know by our definition that (11.7) holds for $0 \leq s \leq n$. For the induction step, we take some $s \geq n$, assume that (11.7) holds for this and all smaller values of s, and show that it also holds when s is replaced by $s+1$. The $m = 0$ case of Lemma 6.11(2) shows that $x_1^{n+1} = \sum_{p=0}^n (-1)^{n-p} x_1^p e^{(n+1)}_{n+1-p}$. Multliplying on the left by x_1^{s-n} then applying γ_{n+1}, we deduce that $x_{n+1}^{s+1} = \sum_{p=0}^n (-1)^{n-p} x_{n+1}^{p+s-n} \varepsilon^{(n+1)}_{n+1-p}$. So

$$u_n(x^r) \otimes v_n(x^{s+1}) = \sum_{p=0}^{n} (-1)^{n-p} u_n(x^r) \otimes v_n(x^{p+s-n})\bar{\varepsilon}^{(n+1)}_{n+1-p}.$$

Now we apply the right supermodule homomorphism ev_n using the induction hypothesis to see that

$$\mathrm{ev}_n\left(u_n(x^r) \otimes v_n(x^{s+1})\right) = \sum_{p=\max(0,2n-r-s)}^{n} (-1)^{n-p}\bar{\eta}^{(n+1)}_{p+r+s-2n}\bar{\varepsilon}^{(n+1)}_{n+1-p}.$$

This is equal to $\bar{\eta}^{(n+1)}_{r+s+1-n}$ by (4.49), which is what we wanted.

Step three. Since ev_n is a right supermodule homomorphism by definition, the composition of maps in (11.8) makes sense and is a right OH^ℓ_n-supermodule homomorphism[4]. To show that the composition is equal to the identity map, it suffices to show that it takes $u_n(f(x))$ to $u_n(f(x))$ for any $f(x) \in \mathbb{F}[x]$ of degree $\leq n'$, since these elements generate U^ℓ_n as a right OH^ℓ_n-supermodule by Lemma 10.8(2a). Using (11.10), we apply the even map $\mathrm{id} \otimes \mathrm{coev}_n$ to $u_n(f(x)) \otimes 1$ to obtain

$$\left[u_n(f(x)) \otimes v_n((t-x)^{-1}) \otimes \bar{\varepsilon}^{(n+1)}(t)u_n((t-x)^{-1})\right]_{t^{-1}}.$$

[4]However, (11.9) does not make sense at this point since $\mathrm{id} \otimes \mathrm{ev}_n$ is not defined until we can shown that ev_n is a left supermodule homomorphism.

By step two, (11.16) holds for $f(x)$ of degree $\leq n'$. We use it to apply $\mathrm{ev}_n \otimes \mathrm{id}$, then multiply out the tensor, to obtain

$$\left[[\bar{\eta}^{(n+1)}(t)f(t)]_{<t^0}\bar{\varepsilon}^{(n+1)}(t)u_n((t-x)^{-1})\right]_{t^{-1}}.$$

Now, (11.12) shows that $\bar{\varepsilon}^{(n+1)}(t)u_n((t-x)^{-1})$ is a polynomial in t, so we can omit the inside square brackets. Then the η and ε cancel by (4.49), leaving us with

$$\left[f(t)u_n((t-x)^{-1})\right]_{t^{-1}} = u_n(f(x)),$$

where we applied (11.1) for the final equality. Hence, the composition (11.8) is the identity.

Step four. We prove that ev_n is a left supermodule homomorphism. Since $v_n(1)$ generates V_n^ℓ as a superbimodule (Lemma 10.8(1c)) and ev_n is a right supermodule homomorphism, it suffices to show that $a\,\mathrm{ev}_n(u \otimes v_n(1)) = \mathrm{ev}_n(au \otimes v_n(1))$ for all $a \in OH_{n+1}^\ell$ and $u \in U_n^\ell$. By step three, we have that

$$a\,((\mathrm{ev}_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_n)(u \otimes 1)) = a(1 \otimes u) = (1 \otimes au) = (\mathrm{ev}_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}_n)(au \otimes 1)$$

in $OH_{n+1}^\ell \otimes_{OH_{n+1}^\ell} U_n^\ell$. Using the formula for $\mathrm{coev}_n(1)$ from (11.14), this shows that

$$\sum_{s=0}^{n}(-1)^{(n+1)s}a\,\mathrm{ev}_n\left(u \otimes \bar{\varepsilon}_{n-s}^{(n)}v_n(1)\right) \otimes u_n(x^s) = \sum_{s=0}^{n}(-1)^{(n+1)s}\,\mathrm{ev}_n\left(au \otimes \bar{\varepsilon}_{n-s}^{(n)}v_n(1)\right) \otimes u_n(x^s).$$

By Lemma 10.8(2b), U_n^ℓ is a free left OH_{n+1}^ℓ-supermodule with basis $u_n(x^s)$ $(0 \leq s \leq n)$, so we can project to $- \otimes u_n(x^n)$-components in the identity just proved to obtain the desired equality $a\,\mathrm{ev}_n\,(u \otimes v_n(1)) = \mathrm{ev}_n\,(au \otimes v_n(1))$.

Step five. We know already that (11.7) holds for $0 \leq r \leq n'$ and $s \geq 0$. We now show that it holds for all remaining $r > n'$ and $s \geq 0$. Take $r \geq n'$ and assume by induction that (11.7) holds for this value of r and all $s \geq 0$. Equating t^n-coefficients in (10.24) gives that $u_n(x^{r+1}) = \bar{o}^{(n+1)}u_n(x^r) + (-1)^r u_n(x^r)\bar{o}^{(n)}$. Equating t^n-coefficients in (10.21) (with r replaced by s) gives that $v_n(x^{s+1}) = v_n(x^s)\bar{o}^{(n+1)} - (-1)^{n+s}\bar{o}^{(n)}v_n(x^s)$. Making these substitutions, it is then easy to check that

$$u_n(x^{r+1}) \otimes v_n(x^s) = (-1)^{n+r+s+1}u_n(x^r) \otimes v_n(x^{s+1}) + (-1)^{n+r+s}u_n(x^r) \otimes v_n(x^s)\bar{o}^{(n+1)} + \bar{o}^{(n+1)}u_n(x^r) \otimes v_n(x^s).$$

Now we compute $\mathrm{ev}_n\left(u_n(x^{r+1}) \otimes v_n(x^s)\right)$ by applying the superbimodule homomorphism ev_n to the right hand side of the equation just derived and using the induction hypothesis. If $r + s + 1 < n$ the right hand side evaluates to 0 so $\mathrm{ev}_n\left(u_n(x^{r+1}) \otimes v_n(x^s)\right) = 0$ as required. If $r+s+1 = n$ the right hand side evaluates to $\bar{\eta}_{r+1+s-n}^{(n+1)}$ as required. If $r + s + 1 > n$ the right hand side evaluates to

$$(-1)^{n+r+s+1}\bar{\eta}_{r+1+s-n}^{(n+1)} + (-1)^{n+r+s}\bar{\eta}_{r+s-n}^{(n+1)}\bar{o}^{(n+1)} + \bar{o}^{(n+1)}\bar{\eta}_{r+s-n}^{(n+1)}.$$

If $n + r + s$ is odd then $\bar{\eta}_{r+s-n}^{(n+1)}$ and $\bar{o}^{(n+1)}$ commute by the image of the defining relation (4.1) under γ, so this simplifies to the desired $\bar{\eta}_{r+1+s-n}^{(n+1)}$. If $n + r + s$ is even then $\bar{\eta}_{r+s-n}^{(n+1)}\bar{o}^{(n+1)} + \bar{o}^{(n+1)}\bar{\eta}_{r+s-n}^{(n+1)} = 2\bar{\eta}_{r+1+s-n}^{(n+1)}$ by the image of the second relation from (4.52) under γ, so again the expression simplifies to $\bar{\eta}_{r+1+s-n}^{(n+1)}$.

Step six. It remains to check that the composition (11.9), which makes sense as ev_n is a left supermodule homomorphism, is the identity. We do this by showing that it takes $v_n(f(x))$ to $v_n(f(x))$ for any $f(x) \in \mathbb{F}[x]$. The argument is similar to step three. By (11.10), the map $\mathrm{coev}_n \otimes \mathrm{id}$ takes $1 \otimes v_n(f(x))$ to

$$\left[v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t) \otimes u_n((t-x)^{-1}) \otimes v_n(f(x))\right]_{t^{-1}}.$$

Then we apply the even map $\mathrm{id} \otimes \mathrm{ev}_n$ using (11.15) (whose validity relies on the conclusion of step five) to get

$$\left[v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t)[\bar{\eta}^{(n+1)}(t)f(t)]_{<t^0}\right]_{t^{-1}}.$$

Since $v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t)$ is a polynomial in t by (11.11), we can omit the inside square brackets. After doing that, ε and η cancel using (4.49), so we obtain

$$\left[v_n\left((t-x)^{-1}\right)f(t)\right]_{t^{-1}} = v_n(f(x)),$$

where we used (11.1) for the final equality. □

For the following corollary, recall from Corollary 9.6 that ONH_n^ℓ is a subalgebra of ONH_{n+1}^ℓ in a natural way for $0 \leq n < \ell$.

Corollary 11.4. *For $0 \leq n < \ell$, the following diagrams of graded superfunctors commute up to even degree 0 isomorphisms:*

$$\begin{array}{ccc} ONH_{n+1}^\ell\text{-gsMod} & \xrightarrow{(\Pi Q^{-2})^n \operatorname{Res}^{ONH_{n+1}^\ell}_{ONH_n^\ell}} & ONH_n^\ell\text{-gsMod} \\ {\scriptstyle \overline{(\omega\xi)}_{n+1}-}\big\downarrow & & \big\downarrow{\scriptstyle \overline{(\omega\xi)}_n-} \\ OH_{n+1}^\ell\text{-gsMod} & \xrightarrow[V_n^\ell \otimes_{OH_{n+1}^\ell} -]{} & OH_n^\ell\text{-gsMod} \end{array} \qquad \begin{array}{ccc} ONH_n^\ell\text{-gsMod} & \xrightarrow{(\Pi Q^{2})^n \operatorname{Ind}^{ONH_{n+1}^\ell}_{ONH_n^\ell}} & ONH_{n+1}^\ell\text{-gsMod} \\ {\scriptstyle \overline{(\omega\xi)}_n-}\big\downarrow & & \big\downarrow{\scriptstyle \overline{(\omega\xi)}_{n+1}-} \\ OH_n^\ell\text{-gsMod} & \xrightarrow[U_n^\ell \otimes_{OH_n^\ell} -]{} & OH_{n+1}^\ell\text{-gsMod} \end{array}$$

(The vertical arrows are the equivalences of graded supercategories from Corollary 9.3.)

Proof. Note that $(\Pi Q^2)^n \operatorname{Ind}^{ONH_{n+1}^\ell}_{ONH_n^\ell}$ is left adjoint to $(\Pi Q^{-2})^n \operatorname{Res}^{ONH_{n+1}^\ell}_{ONH_n^\ell}$. Also $U_n^\ell \otimes_{OH_n^\ell} -$ is left adjoint to $V_n^\ell \otimes_{OH_{n+1}^\ell} -$ by Theorem 11.3. Hence, using the uniqueness of left adjoints, it suffices to prove that the first square commutes. We show equivalently that the following commutes up to even degree 0 isomorphism:

$$\begin{array}{ccc} ONH_{n+1}^\ell\text{-gsMod} & \xrightarrow{\operatorname{Res}^{ONH_{n+1}^\ell}_{ONH_n^\ell}} & ONH_n^\ell\text{-gsMod} \\ {\scriptstyle OPol_{n+1}\otimes_{OSym_{n+1}} OH_{n+1}^\ell \otimes_{OH_{n+1}^\ell} -}\big\uparrow & & \big\downarrow{\scriptstyle \overline{(\omega\xi)}_n-} \\ OH_{n+1}^\ell\text{-gsMod} & \xrightarrow[\widetilde{V}_n^\ell \otimes_{OH_{n+1}^\ell} -]{} & OH_n^\ell\text{-gsMod} \end{array}$$

Here, we have removed the degree and parity shifts on the horizontal arrows and we have replaced the equivalence $\overline{(\omega\xi)}_{n+1}- \simeq \operatorname{Hom}_{ONH_{n+1}^\ell}(ONH_{n+1}^\ell\overline{(\omega\xi)}_{n+1}, -)$ on the left hand vertical arrow with the quasi-inverse equivalence $ONH_{n+1}^\ell\overline{(\omega\xi)}_{n+1} \otimes_{OH_{n+1}^\ell} - \simeq OPol_{n+1} \otimes_{OSym_{n+1}} OH_{n+1}^\ell \otimes_{OH_{n+1}^\ell} -$. To prove this new diagram commutes, it suffices to show that

$$(\omega\xi)_n OPol_{n+1} \otimes_{OSym_{n+1}} OH_{n+1}^\ell \simeq \widetilde{V}_n^\ell$$

as $(OH_n^\ell, OH_{n+1}^\ell)$-superbimodules. As the proof of Lemma 10.1(1) in the case $d = 0$ shows, the isomorphism $OH_{n+1}^\ell \xrightarrow{\sim} OSym_{(n+1,n')} \otimes_{OSym_\ell} R_\ell$ of Theorem 8.5(1) is an isomorphism of graded $(OH_{n+1}^\ell, OH_{n+1}^\ell)$-superbimodules. So $(\omega\xi)_n OPol_{n+1} \otimes_{OSym_{n+1}} OH_{n+1}^\ell \simeq (\omega\xi)_n OPol_{n+1} \otimes_{OSym_{n+1}} OSym_{(n+1,n')} \otimes_{OSym_\ell} R_\ell$. By (5.30), we have that $(\omega\xi)_n OPol_{n+1} \simeq OSym_{(n,1)}$, so this is $\simeq OSym_{(n,1,n')} \otimes_{OSym_\ell} R_\ell$, which is exactly the definition of $\widetilde{V}_n^\ell$. □

The next theorem, which we prove by twisting Theorem 11.3 with some automorphisms, gives the second adjunction. The explicit formulae for this are not as nice as for the first adjunction.

Theorem 11.5. *Suppose that $\ell = n + 1 + n'$. There are unique even degree 0 superbimodule homomorphisms*

$$\widetilde{\operatorname{coev}}_n : OH_{n+1}^\ell \to \widetilde{U}_n^\ell \otimes_{OH_n^\ell} \widetilde{V}_n^\ell$$

$$1 \mapsto \sum_{s=0}^{n'} (-1)^{\ell s+\binom{s}{2}} (\psi^\ell_{n+1})^{-1}\left(\bar{\varepsilon}^{(n')}_{n'-s}\right)\tilde{u}_n(1) \otimes \tilde{v}_n(x^s)$$

and

$$\widetilde{\mathrm{ev}}_n : \widetilde{V}^\ell_n \otimes_{OH^\ell_{n+1}} \widetilde{U}^\ell_n \to OH^\ell_n$$

$$\tilde{v}_n(x^r) \otimes \tilde{u}_n(x^s) \mapsto \begin{cases} (-1)^{n(r+s)+\binom{r}{2}+\binom{s+1}{2}}\Big[(\psi^\ell_n)^{-1}\Big(\bar{\eta}^{(n'+1)}_{r+s-n'} + (-1)^{n+1}(1-(-1)^s)\bar{\eta}^{(n'+1)}_{r+s-n'-1}\dot{o}^{(\ell)}\Big)\Big] & \textit{if } r+s>n' \\ (-1)^{n(r+s)+\binom{r}{2}+\binom{s+1}{2}} & \textit{if } r+s=n' \\ 0 & \textit{otherwise} \end{cases}$$

giving the unit and counit of an adjunction making $\left(\widetilde{V}^\ell_n, \widetilde{U}^\ell_n\right)$ *into another dual pair of 1-morphisms in* $\mathcal{OGBim}_\ell$.

Proof. Let $\dagger := \phi^\ell_{n'} \circ * : \widetilde{U}^\ell_n \to V^\ell_{n'}$, where $\phi^\ell_{n'}$ is the map from Lemma 10.9. Since $\psi^\ell_{n+1} = \delta^\ell_{n'} \circ (\psi^\ell_{n'})^{-1}$ and $\psi^\ell_n = \delta^\ell_{n'+1} \circ (\psi^\ell_{n'+1})^{-1}$ according to the definition (8.7), Lemma 10.9 and Lemma 10.2(2) imply that

$$\left(\bar{b}_1 u \bar{b}_2\right)^\dagger = \psi^\ell_{n+1}(\bar{b}_1)u^\dagger \psi^\ell_n(\bar{b}_2) \tag{11.17}$$

for $u \in \widetilde{U}^\ell_n$, $b_1 \in OSym_{n+1}$ and $b_2 \in OSym_n$. Now we define even degree 0 graded R_ℓ-supermodule homomorphisms $\widetilde{\mathrm{coev}}_n$ and $\widetilde{\mathrm{ev}}_n$ so that the following diagrams commute:

$$\begin{array}{ccc} OH^\ell_{n+1} & \xrightarrow{\widetilde{\mathrm{coev}}_n} & \widetilde{U}^\ell_n \otimes_{OH^\ell_n} \widetilde{V}^\ell_n \\ {\scriptstyle \psi^\ell_{n+1}}\big\downarrow & & \big\downarrow{\scriptstyle \dagger\otimes *} \\ OH^\ell_{n'} & \xrightarrow[\mathrm{coev}_{n'}]{} & V^\ell_{n'} \otimes_{OH^\ell_{n'+1}} U^\ell_{n'} \end{array} \qquad \begin{array}{ccc} \widetilde{V}^\ell_n \otimes_{OH^\ell_{n+1}} \widetilde{U}^\ell_n & \xrightarrow{\widetilde{\mathrm{ev}}_n} & OH^\ell_n \\ {\scriptstyle *\otimes\dagger}\big\downarrow & & \big\downarrow{\scriptstyle \psi^\ell_n} \\ U^\ell_{n'} \otimes_{OH^\ell_{n'}} V^\ell_{n'} & \xrightarrow[\mathrm{ev}_{n'}]{} & OH^\ell_{n'+1} \end{array}$$

To see that the vertical maps $\dagger \otimes *$ and $* \otimes \dagger$ in these diagrams make sense, one needs to check that they are balanced, which follows using Lemma 10.2(1) and (11.17). In fact, $\widetilde{\mathrm{coev}}_n$ and $\widetilde{\mathrm{ev}}_n$ defined in this way are superbimodule homomorphisms. This again follows using Lemma 10.2(1) and (11.17) since $\mathrm{coev}_{n'}$ and $\mathrm{ev}_{n'}$ are superbimodule homomorphisms.

The zig-zag identities for $\widetilde{\mathrm{coev}}_n$ and $\widetilde{\mathrm{ev}}_n$ follow from their definitions using the zig-zag identities (11.8) and (11.9) for $\mathrm{coev}_{n'}$ and $\mathrm{ev}_{n'}$. Hence they give the unit and counit of an adjunction.

It just remains to compute the explicit formulae for the maps given in the statement of the corollary. To see that $\widetilde{\mathrm{coev}}_n(1) = \sum_{s=0}^{n'}(-1)^{\ell s+\binom{s}{2}}\Big[(\psi^\ell_{n+1})^{-1}\Big](\bar{\varepsilon}^{(n')}_{n'-s})\tilde{u}_n(1)\otimes\tilde{v}_n(x^s)$, the image of 1 under the southwest pair of maps in the diagram defining $\widetilde{\mathrm{coev}}_n$ is $\sum_{s=0}^{n'}(-1)^{(n'+1)s}\bar{\varepsilon}^{(n')}_{n'-s}v_{n'}(1)\otimes u_{n'}(x^s)$ thanks to (11.14). This is also the image of $\sum_{s=0}^{n'}(-1)^{\ell s+\binom{s}{2}}\Big[(\psi^\ell_{n+1})^{-1}(\bar{\varepsilon}^{(n')}_{n'-s})\Big]\tilde{u}_n(1) \otimes \tilde{v}_n(x^s)$ under the right hand map $\dagger \otimes *$ by (10.7), (10.8), (10.33) and (11.17).

To compute $\widetilde{\mathrm{ev}}_n(\tilde{v}_n(x^r) \otimes \tilde{u}_n(x^s))$, we use the diagram defining $\widetilde{\mathrm{ev}}_n$. From (10.7), (10.8) and (10.33), one sees that

$$(* \otimes \dagger)(\tilde{v}_n(x^r) \otimes \tilde{u}_n(x^s)) = (-1)^{\binom{r}{2}+\binom{s+1}{2}+n(r+s)}u_{n'}(x^r) \otimes \Big[v_{n'}(x^s) + (-1)^{n+1}(1-(-1)^s)v_{n'}(x^{s-1})\dot{o}^{(\ell)}\Big].$$

Using (11.7), the image of this under the homomorphism $(\psi^\ell_n)^{-1} \circ \mathrm{ev}_{n'}$ is easily seen to be equal to the formula for $\widetilde{\mathrm{ev}}_n(\tilde{v}_n(x^r) \otimes \tilde{u}_n(x^s))$ given in the statement of the lemma. $\square$

Corollary 11.6. *For $0 \leq n < \ell$, the following diagrams of graded superfunctors commute up to even degree 0 isomorphisms:*

$$
\begin{array}{ccc}
ONH^\ell_{n+1}\text{-gsMod} & \xrightarrow{\operatorname{Res}^{ONH^\ell_{n+1}}_{ONH^\ell_n}} & ONH^\ell_n\text{-gsMod} \\
{\scriptstyle (\overline{\omega\xi})_{n+1}-}\big\downarrow & & \big\downarrow{\scriptstyle (\overline{\omega\xi})_n-} \\
OH^\ell_{n+1}\text{-gsMod} & \xrightarrow[\widetilde{V}^\ell_n \otimes_{OH^\ell_{n+1}} -]{} & OH^\ell_n\text{-gsMod}
\end{array}
\qquad
\begin{array}{ccc}
ONH^\ell_n\text{-gsMod} & \xrightarrow{\operatorname{Coind}^{ONH^\ell_{n+1}}_{ONH^\ell_n}} & ONH^\ell_{n+1}\text{-gsMod} \\
{\scriptstyle (\overline{\omega\xi})_n-}\big\downarrow & & \big\downarrow{\scriptstyle (\overline{\omega\xi})_{n+1}-} \\
OH^\ell_n\text{-gsMod} & \xrightarrow[\widetilde{U}^\ell_n \otimes_{OH^\ell_n} -]{} & OH^\ell_{n+1}\text{-gsMod}
\end{array}
$$

(The vertical arrows are the equivalences of graded supercategories from Corollary 9.3.)

Proof. The commutativity of the first diagram follows from Corollary 11.4, then the second follows using Theorem 11.5 and the uniqueness of right adjoints. □

The following generalizes Corollary 9.7.

Corollary 11.7. *For $0 \leq n < \ell$, we have that $\operatorname{Ind}^{ONH^\ell_{n+1}}_{ONH^\ell_n} \simeq (\Pi Q^2)^{\ell-2n-1} \operatorname{Coind}^{ONH^\ell_{n+1}}_{ONH^\ell_n}$. Hence, ONH^ℓ_{n+1} is a graded Frobenius extension of ONH^ℓ_n of degree $2(\ell-2n-1)$ and parity $\ell - 2n - 1 \pmod 2$.*

Proof. The second statement follows from the first statement by the general theory of Frobenius extensions explained after (2.11). To prove the first statement, Corollaries 11.4 and 11.6 show that under the Morita equivalence $\operatorname{Ind}^{ONH^\ell_{n+1}}_{ONH^\ell_n}$ corresponds to $(\Pi Q^2)^{-n} U_n \otimes_{OH^\ell_n} -$ and $\operatorname{Coind}^{ONH^\ell_{n+1}}_{ONH^\ell_n}$ corresponds to $\widetilde{U}_n \otimes_{OH^\ell_n} -$. Now the result follows because $(\Pi Q^2)^{-n} U_n \simeq (\Pi Q^2)^{\ell-2n-1} \widetilde{U}_n$ according to (10.2). □

The final task in this section is to compute various mates of the endomorphisms of $U^\ell_{(1^d);n}$ defined by the action of ONH_d from Lemma 10.5(2). Specifically, we need to work out the endomorphisms in $\mathcal{OGBim}_\ell$ that correspond to the diagrams (13.4) and (13.8) in the graphical calculus to be introduced later in the article. Suppose that $\ell = n+d+n'$ for $d \geq 0$. We define graded superalgebra homomorphisms

$$\rho_{(1^d);n} : ONH_d \to \operatorname{End}_{OH^\ell_{n+d}\text{-}OH^\ell_n}(U^\ell_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n)^{\mathrm{sop}} \tag{11.18}$$

$$\lambda_{n;(1^d)} : ONH_d \to \operatorname{End}_{OH^\ell_n\text{-}OH^\ell_{n+d}}(V^\ell_n \otimes_{OH^\ell_{n+1}} \cdots \otimes_{OH^\ell_{n+d-1}} (V^\ell_{n+d-1})) \tag{11.19}$$

as follows. For $a \in ONH_d$, $\rho_{(1^d);n}(a)$ is defined to be the top map the following diagram commute:

$$
\begin{array}{ccc}
U^\ell_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n & \xrightarrow{\rho_{(1^d);n}(a)} & U^\ell_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n \\
{\scriptstyle (b_{(1)^d})^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle b_{(1)^d}} \\
U^\ell_{(1^d);n} & \xrightarrow[u_{(1^d);n}(f) \mapsto (-1)^{\mathrm{par}(a)\,\mathrm{par}(f)} u_{(1^d);n}(f)\cdot a]{} & U^\ell_{(1^d);n}
\end{array}
\tag{11.20}
$$

Also $\lambda_{n;(1^d)}(a)$ is the top map making the following commute:

$$
\begin{array}{ccc}
V^\ell_n \otimes_{OH^\ell_{n+1}} \cdots \otimes_{OH^\ell_{n+d-1}} V^\ell_{n+d-1} & \xrightarrow{\lambda_{n;(1^d)}(a)} & V^\ell_n \otimes_{OH^\ell_{n+1}} \cdots \otimes_{OH^\ell_{n+d-1}} V^\ell_{n+d-1} \\
{\scriptstyle (c_{(1)^d})^{-1}}\big\downarrow & & \big\uparrow{\scriptstyle c_{(1)^d}} \\
V^\ell_{n;(1^d)} & \xrightarrow[v_{n;(1^d)}(f) \mapsto a\cdot v_{n;(1^d)}(f)]{} & V^\ell_{n;(1^d)}
\end{array}
\tag{11.21}
$$

The vertical maps in (11.20) and (11.21) come from (10.17) and (10.18). We also remind the reader that $u_{(1^d);n}(f) \cdot a = (-1)^{(d-1)\,\mathrm{par}(a)} u_{(1^d);n}(f \cdot a)$ and $a \cdot v_{n;(1^d)}(f) = (-1)^{(n\#(d-1))\,\mathrm{par}(a)} v_{n;(1^d)}(a \cdot f)$.

Lemma 11.8. *For $0 \leq n < \ell$, the mate of the $(OH^\ell_{n+1}, OH^\ell_n)$-superbimodule endomorphism $\rho_{(1);n}(x_1)$ under the adjunction from Theorem 11.3, that is, the composition*

$$V^\ell_n \xrightarrow{\text{can}} OH^\ell_n \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\text{coev}_n \otimes \text{id}} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\text{id} \otimes \rho_{(1);n}(x_1) \otimes \text{id}}$$

$$V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\text{id} \otimes \text{ev}_n} V^\ell_n \otimes_{OH^\ell_{n+1}} OH^\ell_{n+1} \xrightarrow{\text{can}} V^\ell_n,$$

is equal to $\lambda_{n;(1)}(x_1)$.

Proof. Since V^ℓ_n is generated as a superbimodule by $v_n(1)$ by Lemma 10.8(1c), it suffices to show that

$$(\text{id} \otimes \text{ev}_n) \circ (\text{id} \otimes \rho_{(1);n}(x_1) \otimes \text{id}) \circ (\text{coev}_n \otimes \text{id})(1 \otimes v_n(1)) = \lambda_{n;(1)}(x_1)(v_n(1)) \otimes 1.$$

This is a calculation from the definitions. The right hand side is $v_n(x) \otimes 1$ by the definition (11.21). For the left hand side, we first apply $\text{coev}_n \otimes \text{id}$ using (11.13) to get

$$\left[\bar{\varepsilon}^{(n)}((-1)^n t) v_n((-1)^n) \otimes u_n((t-x)^{-1}) \otimes v_n(1)\right]_{t^{-1}}.$$

Then we apply the odd endomorphism $\text{id} \otimes \rho_{(1);n}(x_1) \otimes \text{id}$ defined by (11.20) to get

$$\left[\bar{\varepsilon}^{(n)}((-1)^{n+1} t) v_n((-1)^n) \otimes u_n((t+x)^{-1} x) \otimes v_n(1)\right]_{t^{-1}}.$$

Finally we apply $\text{id} \otimes \text{ev}_n$ using (11.15) with t replaced by $-t$ to get

$$\left[\bar{\varepsilon}^{(n)}((-1)^{n+1} t) v_n((-1)^n) [\bar{\eta}^{(n+1)}(-t) t]_{<t^0}\right]_{t^{-1}} \otimes 1.$$

Computing the coefficients explicitly, this is equal to $v_n(1)\bar{\eta}^{(n+1)}_1 - (-1)^n \bar{\varepsilon}^{(n)}_1 v_n(1)$. Since $\eta^{(n+1)}_1 = x_1 + \cdots + x_{n+1}$ and $\varepsilon^{(n)}_1 = x_1 + \cdots + x_n$ (and $(-1)^n$ cancels when $\varepsilon^{(n)}_1$ acts on $v_n(1)$ due to the parity shift) this is $v_n(x) \otimes 1$. ◻

Lemma 11.9. *For $0 < n < \ell$, the (OH^ℓ_n, OH^ℓ_n)-superbimodule endomorphism σ_n that is defined by the composition*

$$U^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \xrightarrow{\text{can}} OH^\ell_n \otimes_{OH^\ell_n} U^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \xrightarrow{\text{coev}_n \otimes \text{id} \otimes \text{id}}$$

$$V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} U^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \xrightarrow{\text{id} \otimes \rho_{(1^2);n-1}(\tau_1) \otimes \text{id}} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} U^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1}$$

$$\xrightarrow{\text{id} \otimes \text{id} \otimes \text{ev}_{n-1}} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} OH^\ell_n \xrightarrow{\text{can}} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n$$

maps $u_{n-1}(x^r) \otimes v_{n-1}(x^s)$ to

$$(-1)^{nr+rs+r+s+n+1} v_n(x^s) \otimes u_n(x^r) + \sum_{p=0}^{n} \sum_{q=0}^{r+s-n} (-1)^{nq+pq+rq+r+q} v_n(x^p) \otimes u_n(x^q) \bar{\varepsilon}^{(n)}_{n-p} \bar{\eta}^{(n)}_{r+s-n-q} \tag{11.22}$$

for any $r \geq 0$ and $0 \leq s \leq n-1$.

Proof. First, we show that $\rho_{(1^2);n-1}(\tau_1)\Big(u_n((t-x)^{-1}) \otimes u_{n-1}(x^r)\Big)$ equals

$$-u_n((t+x)^{-1} x^r) \otimes u_{n-1}\Big((t-(-1)^r x)^{-1}\Big) + \sum_{q=0}^{r-1} (-1)^{q+r+rq} u_n((t+x)^{-1} x^q) \otimes u_{n-1}(x^{r-q-1}). \tag{11.23}$$

To do this, according to the definition (11.20), we first use the inverse of the map $b_{(1)^2}$ from (10.18) to pass to $U^\ell_{(1^2);n-1}$. This maps $u_n((t-x)^{-1}) \otimes u_{n-1}(x^r)$ to $(-1)^r u_{(1^2);n-1}((t-x_2)^{-1}x_1^r)$. The application of $\rho_{(1^2);n-1}(\tau_1)$ takes this to $-u_{(1^2);n-1}((t+x_2)^{-1}x_1^r \cdot \tau_1)$. This we can compute with Lemma 5.11 to get

$$-u_{(1^2);n-1}\Big((t+x_2)^{-1}x_2^r(t+(-1)^r x_1)^{-1}\Big) - \sum_{q=0}^{r-1}(-1)^{rq}u_{(1)^2;n-1}\Big((t+x_2)^{-1}x_2^q x_1^{r-q-1}\Big).$$

After that we apply $b_{(1)^2}$ to obtain the vector in $U^\ell_n \otimes_{OH^\ell_n} U^\ell_{n-1}$ displayed in (11.23).

Now to prove the lemma, we again calculate with generating functions. Start with the vector $u_{n-1}(x^r) \otimes v_{n-1}(x^s)$ for $0 \leq s \leq n-1$ (this assumption on s will be crucial shortly). By (11.10), the map $\mathrm{coev}_n \otimes \mathrm{id} \otimes \mathrm{id}$ takes it to

$$\Big[v_n((t-x)^{-1})\bar{\varepsilon}^{(n+1)}(t) \otimes u_n((t-x)^{-1}) \otimes u_{n-1}(x^r) \otimes v_{n-1}(x^s)\Big]_{t^{-1}}. \tag{11.24}$$

Then we apply the odd homomorphism $\mathrm{id} \otimes \rho_{(1^2);n-1}(\tau_1) \otimes \mathrm{id}$ using (11.23) to obtain

$$\Big[v_n((t+x)^{-1})\bar{\varepsilon}^{(n+1)}(-t) \otimes u_n((t+x)^{-1}x^r) \otimes u_{n-1}\Big((t-(-1)^r x)^{-1}\Big) \otimes v_{n-1}(x^s)\Big]_{t^{-1}}$$
$$+ \sum_{q=0}^{r-1}(-1)^{q+r+rq+1}\Big[v_n((t+x)^{-1})\bar{\varepsilon}^{(n+1)}(-t) \otimes u_n((t+x)^{-1}x^q) \otimes u_{n-1}(x^{r-q-1}) \otimes v_{n-1}(x^s)\Big]_{t^{-1}}. \tag{11.25}$$

It just remains to apply $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}_{n-1}$. We treat the two terms in (11.25) separately. For the first term, we have that $\mathrm{ev}_{n-1}\Big(u_{n-1}\Big((t-(-1)^r x)^{-1}\Big) \otimes v_{n-1}(x^s)\Big) = (-1)^{rs+r}\Big[\bar{\eta}^{(n)}((-1)^r t)t^s\Big]_{<t^0}$ by (11.15) (with t replaced by $(-1)^r t$). The assumption that $s \leq n-1$ means that we can omit the truncation to $< t^0$ here. So the first term contributes

$$(-1)^{rs+r}\Big[v_n((t+x)^{-1}) \otimes \bar{\varepsilon}^{(n+1)}(-t)u_n((t+x)^{-1}x^r) \otimes \bar{\eta}^{(n)}((-1)^r t)t^s\Big]_{t^{-1}}.$$

Now we use (10.25) to rewrite this as

$$(-1)^{nr+rs+r+n+1}\Big[v_n((t+x)^{-1}) \otimes u_n(x^r)\bar{\varepsilon}^{(n)}((-1)^r t)\bar{\eta}^{(n)}((-1)^r t)t^s \otimes 1\Big]_{t^{-1}}.$$

The ε and η cancel by the infinite Grassmannian relation to leave

$$(-1)^{nr+rs+r+n+1}\Big[v_n((t+x)^{-1}t^s) \otimes u_n(x^r) \otimes 1\Big]_{t^{-1}} \overset{(11.1)}{=} (-1)^{nr+rs+r+s+n+1}v_n(x^s) \otimes u_n(x^r) \otimes 1.$$

It remains to consider the term obtained by applying $\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{ev}_{n-1}$ to the second term from (11.25). Using (10.25) and (11.7), this contributes

$$\sum_{q=0}^{r+s-n}(-1)^{q+r+rq+1}\Big[v_n((t+x)^{-1}) \otimes \bar{\varepsilon}^{(n+1)}(-t)u_n((t+x)^{-1}x^q) \otimes \bar{\eta}^{(n)}_{r+s-q-n}\Big]_{t^{-1}} =$$
$$\sum_{q=0}^{r+s-n}(-1)^{q+r+rq+nq+n}\Big[v_n((t+x)^{-1}) \otimes u_n(x^q)\bar{\varepsilon}^{(n)}((-1)^q t)\bar{\eta}^{(n)}_{r+s-q-n} \otimes 1\Big]_{t^{-1}}.$$

It remains to work out the t^{-1}-coefficient explicitly to complete the proof. □

Lemma 11.10. *For $0 < n < \ell$, the mate of $\rho_{(1^2);n-1}(\tau_1)$ under the adjunction from Theorem 11.3, that is, the composition*

$$V^\ell_{n-1} \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\mathrm{can}} OH^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\mathrm{coev}_{n-1} \otimes \mathrm{id} \otimes \mathrm{id}}$$
$$V^\ell_{n-1} \otimes_{OH^\ell_n} U^\ell_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \otimes_{OH^\ell_n} V^\ell_n \xrightarrow{\mathrm{id} \otimes \sigma_n \otimes \mathrm{id}} V^\ell_{n-1} \otimes_{OH^\ell_n} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n$$

$$\xrightarrow{\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_n} V^\ell_{n-1}\otimes_{OH^\ell_n} V^\ell_n \otimes_{OH^\ell_{n+1}} OH^\ell_{n+1} \xrightarrow{\mathrm{can}} V^\ell_{n-1}\otimes_{OH^\ell_n} V^\ell_n$$

where σ_n is the superbimodule homomorphism[5] *defined in Lemma 11.9, is equal to $\lambda_{n-1;(1^2)}(\tau_1)$.*

Proof. By Lemma 10.8(1b)–(1c), $V^\ell_{n-1}\otimes_{OH^\ell_n} V^\ell_n$ is generated as an $(OH^\ell_{n-1}, OH^\ell_{n+1})$-superbimodule by the vectors $v_{n-1}(1)\otimes v_n(x^s)$ $(0\le s\le n)$. Therefore it suffices to show that

$$(\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_n)\circ(\mathrm{id}\otimes\sigma_n\otimes\mathrm{id})\circ(\mathrm{coev}_{n-1}\otimes\mathrm{id}\otimes\mathrm{id})(1\otimes v_{n-1}(1)\otimes v_n(x^s)) = \lambda_{n-1;(1^2)}(\tau_1)(v_{n-1}(1)\otimes v_n(x^s))\otimes 1$$

for $0\le s\le n$. The right hand side may be computed directly from (10.17) and (11.21). It equals

$$(-1)^n c_{(1)^2}\Big(v_{n-1;(1^2)}(\tau_1\cdot x_2^s)\Big)\otimes 1 \overset{(5.10)}{=} (-1)^{n+1}c_{(1)^2}\Big(v_{n-1;(1^2)}(\eta^{(2)}_{s-1})\Big)\otimes 1.$$

So to complete the proof we must show that

$$\Big(c_{(1)^2}\otimes\mathrm{id}\Big)^{-1}\Big((\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_n)\circ(\mathrm{id}\otimes\sigma_n\otimes\mathrm{id})\circ(\mathrm{coev}_{n-1}\otimes\mathrm{id}\otimes\mathrm{id})(1\otimes v_{n-1}(1)\otimes v_n(x^s))\Big)$$
$$= (-1)^{n+1}v_{n-1;(1^2)}(\eta^{(2)}_{s-1})\otimes 1. \quad (11.26)$$

To compute the left hand side, we first use (11.14) to get

$$(\mathrm{coev}_{n-1}\otimes\mathrm{id}\otimes\mathrm{id})(1\otimes v_{n-1}(1)\otimes v_n(x^s)) = \sum_{r=0}^{n-1}(-1)^{nr}\bar\varepsilon^{(n-1)}_{n-1-r}v_{n-1}(1)\otimes u_{n-1}(x^r)\otimes v_{n-1}(1)\otimes v_n(x^s).$$

Then we apply $\mathrm{id}\otimes\sigma_n\otimes\mathrm{id}$ using (11.22), noting also that σ_n is *odd*. Since $r\le n-1$ in this expression, the summation over q on the right hand side of (11.22) is actually an empty sum, so zero, and we get simply

$$(-1)^{n+1}\sum_{r=0}^{n-1}\bar\varepsilon^{(n-1)}_{n-1-r}v_{n-1}(1)\otimes v_n(1)\otimes u_n(x^r)\otimes v_n(x^s).$$

Next we apply $\mathrm{id}\otimes\mathrm{id}\otimes\mathrm{ev}_n$. We must have that $r+s\ge n$ so $r\ge n-s$, and the final expression is

$$(-1)^{n+1}\sum_{r=n-s}^{n-1}\bar\varepsilon^{(n-1)}_{n-1-r}v_{n-1}(1)\otimes v_n(1)\bar\eta^{(n+1)}_{r+s-n} = (-1)^{n+1}\sum_{r=0}^{s-1}\bar\varepsilon^{(n-1)}_{r}v_{n-1}(1)\otimes v_n(1)\bar\eta^{(n+1)}_{s-1-r}.$$

Applying $(c_{(1)^2}\otimes\mathrm{id})^{-1}$ as is required for (11.26), we get

$$(-1)^{n+1}\sum_{r=0}^{s-1}\bar\varepsilon^{(n-1)}_{r}v_{n-1;(1^2)}(1)\bar\eta^{(n+1)}_{s-1-r}.$$

There is a sign change of $(-1)^r$ due to the parity shift $(n-1)\#2\equiv 1 \pmod 2$. Also we have that $\sum_{r=0}^{s-1}(-1)^r\varepsilon^{(n-1)}_r\eta^{(n+1)}_{s-1-r} = \mathrm{sh}_{n-1}(\eta^{(2)}_{s-1})$ by a similar argument to the proof of Lemma 6.10. So this is $(-1)^{n+1}v_{n-1;(1^2)}(\eta^{(2)}_{s-1})$ exactly as in (11.26). □

12. Singular Rouquier complex

Throughout the section, we fix $\ell\in\mathbb{N}$. The graded (Q,Π)-2-supercategory $\mathcal{OGBim}_\ell$ categorifies the locally unital $\mathbb{Z}[q,q^{-1}]^\pi$-algebra that is the image of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ in its representation on $\mathbf{V}(-\ell)$, notation as at the end of Section 3. To make this statement precise, let $K_0(OH^\ell_n)$ be the Grothendieck group of OH^ℓ_n; recall this means the split Grothendieck group of the category OH^ℓ_n-pg$\underline{\text{smod}}$. Since OH^ℓ_n is positively

[5] Diagrammatically, we have rotated through 180° by rotating by 90° twice, see (13.4) and (13.8).

graded with degree 0 component that is the ground field $\mathbb{F}$, this is nothing more than the free $\mathbb{Z}[q,q^{-1}]^\pi$-module generated by the isomorphism class $[OH_n^\ell]$ of the regular module, with the actions of π and q induced by the parity and degree shift functors Π and Q, respectively. So we can identify

$$\mathbf{V}(-\ell) \equiv \bigoplus_{n=0}^{\ell} K_0(OH_n^\ell), \qquad\qquad b_n^\ell \equiv [OH_n^\ell]. \tag{12.1}$$

The following matches up the action of generators of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ on $\mathbf{V}(-\ell)$ with endomorphisms of the Grothendieck group induced by tensoring with odd Grassmannian bimodules.

Theorem 12.1. *Under the identification (12.1) of $\bigoplus_{n=0}^{\ell} K_0(OH_n^\ell)$ with $\mathbf{V}(-\ell)$, the $\mathbb{Z}[q,q^{-1}]^\pi$-module endomorphisms induced by tensoring with odd Grassmannian bimodules correspond to endomorphisms defined by actions of elements of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ according to the following dictionary:*

(1) $[U_n^\ell \otimes_{OH_n^\ell} -] \equiv q^n E 1_{2n-\ell}$ and, more generally, $[U_{(d);n}^\ell \otimes_{OH_n^\ell} -] \equiv q^{nd} E^{(d)} 1_{2n-\ell}$;

(2) $[V_n^\ell \otimes_{OH_{n+1}^\ell} -] \equiv q^{\ell-3n-1} 1_{2n-\ell} F$ and $[V_{n;(d)}^\ell \otimes_{OH_{n+d}^\ell} -] \equiv q^{d(\ell-3n-2d+1)} 1_{2n-\ell} F^{(d)}$.

Also, for $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod 2$, the map $T : 1_{-k}\mathbf{V}(-\ell) \to 1_k\mathbf{V}(-\ell)$ from Theorem 3.6 corresponds to the $\mathbb{Z}[q,q^{-1}]^\pi$-module homomorphism

$$T : K_0(OH_n^\ell) \to K_0(OH_{n'}^\ell), \qquad\qquad [OH_n^\ell] \mapsto (-1)^n (\pi q^2)^{\binom{n+1}{2}+nk} q^{-nk} [OH_{n'}^\ell] \tag{12.2}$$

where $n := \frac{\ell-k}{2}$ and $n' := \frac{\ell+k}{2}$.

Proof. (1) Note here we are assuming implicitly that $0 \leq n \leq \ell-1$ and $0 \leq n \leq \ell-d$, respectively so that U_n^ℓ and $U_{(d);n}^\ell$ are defined. By Lemma 10.8(2b), we have that $[U_n^\ell \otimes_{OH_n^\ell} OH_n^\ell] = q^n [n+1]_{q,\pi} [OH_{n+1}^\ell]$. Also $E b_n^\ell = [n+1]_{q,\pi} b_n^\ell$. It follows that $[U_n^\ell \otimes_{OH_n^\ell} -]$ and $q^n E 1_{2n-\ell}$ define the same endomorphisms of $\mathbf{V}(-\ell)$. For the more general assertion, take $d \geq 1$. By Lemma 10.5(2) and (3.3), we have that $[U_{(1^d);n}^\ell \otimes_{OH_n^\ell} -] = q^{\binom{d}{2}} [d]^!_{q,\pi} [U_{(d);n}^\ell \otimes_{OH_n^\ell} -]$. Also $U_{(1^d);n}^\ell \simeq U_{n+d-1}^\ell \otimes_{OH_{n+d-1}^\ell} \cdots \otimes_{OH_{n+1}^\ell} U_n^\ell$ by (10.18), so we deduce using the special case already treated that

$$q^{\binom{d}{2}} [d]^!_{q,\pi} [U_{(d);n}^\ell \otimes_{OH_n^\ell} -] = q^{nd+\binom{d}{2}} E^d 1_{2n-\ell} = q^{nd+\binom{d}{2}} [d]^!_{q,\pi} E^{(d)} 1_{2n-\ell}.$$

Cancelling $q^{\binom{d}{2}} [d]^!_{q,\pi}$ gives the required conclusion.

(2) Again we are assuming that $0 \leq n \leq \ell-1$ and $0 \leq n \leq \ell-d$, respectively. The first step is to show that $[\widetilde{V}_n^\ell \otimes_{OH_{n+1}^\ell} -] = q^{\ell-n-1}\pi^n 1_{2n-\ell} F$, which follows from Lemma 10.8(1a) like in the proof of (1). Hence, since $V_n^\ell = (\Pi Q^{-2})^n \widetilde{V}_n^\ell$, we get that $[V_n^\ell \otimes_{OH_{n+1}^\ell} -] = q^{\ell-3n-1} 1_{2n-\ell} F$. The passage from this to the more general result about $V_{n;(d)}^\ell$ follows in a similar way to the argument given in (1).

Now consider the final statement about T. Take k and $n = \frac{\ell-k}{2}, n' = \frac{\ell+k}{2}$ as in the statement of the theorem. We saw in (3.20) that $T(b_n^\ell) = (-1)^n \pi^{\binom{n}{2}+nn'} q^{n+nn'} b_{n'}^\ell$. Using the identification (12.1), it follows that $T\big([OH_n^\ell]\big) = (-1)^n \pi^{\binom{n}{2}+nn'} q^{n+nn'} [OH_{n'}^\ell]$. On replacing n' by $k+n$, this becomes the formula in the statement of the theorem. □

The goal in the remainder section is to categorify $T : 1_{-k}\mathbf{V}(-\ell) \to 1_k\mathbf{V}(-\ell)$ for all $-\ell \leq k \leq \ell$ with $k \equiv \ell \pmod 2$. Throughout, we let $n := \frac{\ell-k}{2}$ and $n' := \frac{\ell+k}{2} = n+k$ so that $2n-\ell = -k$ and $2n'-\ell = k$. Since $n + n' = \ell$, Theorem 8.5(4) shows that the graded superalgebras OH_n^ℓ and $OH_{n'}^\ell$ are isomorphic.

Definition 12.2. For $0 \leq d \leq n$, let

$$C_d := \begin{cases} U_{(k+d);n-d}^\ell \otimes_{OH_{n-d}^\ell} V_{n-d;(d)}^\ell & \text{if } d \geq -k, \\ 0 & \text{otherwise.} \end{cases} \tag{12.3}$$

The *singular Rouquier complex* for odd Grassmannian bimodules is the following sequence of graded $(OH^\ell_{n'}, OH^\ell_n)$-superbimodules and even degree 0 superbimodule homomorphisms in $\mathcal{OGBim}_\ell$:

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_{d+1}} C_d \xrightarrow{\partial_d} C_{d-1} \xrightarrow{\partial_{d-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{12.4}$$

where $\partial_d = 0$ unless $\max(0, -k) < d \leq n$, in which case $\partial_d : C_d \to C_{d-1}$ is the even degree 0 superbimodule homomorphism defined by the composition

$$U^\ell_{(k+d);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(d)} \xrightarrow{\text{inc}} U^\ell_{(k+d-1,1);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(1,d-1)} \xrightarrow{c'_{(k+d-1),(1)} \otimes c_{(1),(d-1)}}$$

$$U^\ell_{(k+d-1);n-d+1} \otimes_{OH^\ell_{n-d+1}} U^\ell_{n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1;(d-1)} \xrightarrow{\text{id} \otimes \text{ev}_{n-d} \otimes \text{id}}$$

$$U^\ell_{(k+d-1);n-d+1} \otimes_{OH^\ell_{n-d+1}} OH^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1;(d-1)} \xrightarrow{\text{can}} U^\ell_{(k+d-1);n-d+1} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1;(d-1)}.$$

Theorem 12.3. *The singular Rouquier complex* (12.4) *is a chain complex with homology that is zero in all except for the top (nth) homological degree. Moreover, as a graded $(OH^\ell_{n'}, OH^\ell_n)$-superbimodule the top homology is $\simeq (\Pi Q^2)^{\binom{n+1}{2}+nk} OH^\ell_{n'}$ viewed as a graded left $OH^\ell_{n'}$-supermodule by the natural action and as a graded right OH^ℓ_n-supermodule by restricting the natural right action of $OH^\ell_{n'}$ along some graded superalgebra isomorphism $OH^\ell_n \xrightarrow{\sim} OH^\ell_{n'}$.*

To formulate a corollary, let $K^b(OH^\ell_n\text{-pgsmod})$ be the bounded homotopy supercategory of the graded supercategory of finitely generated projective graded left OH^ℓ_n-supermodules; in the definition of this we require that differentials and chain homotopies are even of degree 0 but chain maps between cochain complexes can be constructed using arbitrary morphisms in OH^ℓ_n-pgsmod. By Euler characteristic (e.g., see [R]), the triangulated Grothendieck group of the underlying ordinary category is identified with $K_0(OH^\ell_n)$, hence, via (12.1), with $1_{-k}\mathbf{V}(-\ell)$.

Corollary 12.4. *The graded superfunctor $K^b(OH^\ell_n\text{-pgsmod}) \to K^b(OH^\ell_{n'}\text{-pgsmod})$ defined by tensoring with the singular Rouquier complex* (12.4) *(viewed now as a cochain complex) then taking the total complex is an equivalence of triangulated graded supercategories. The induced $\mathbb{Z}[q, q^{-1}]^\pi$-module isomorphism $1_{-k}\mathbf{V}(-\ell) \xrightarrow{\sim} 1_k\mathbf{V}(-\ell)$ at the level of Grothendieck groups is equal to $q^{nk}T$ for T as in* (12.2)*.*

Proof. The theorem shows that the singular Rouquier complex is quasi-isomorphic to the cochain complex which is the graded superbimodule $(\Pi Q^2)^{\binom{n+1}{2}+nk} OH^\ell_{n'}$ in cohomological degree $-n$ and zero elsewhere. So it defines an equivalence of triangulated graded supercategories $D^-(OH^\ell_n\text{-gsMod}) \to D^-(OH^\ell_{n'}\text{-gsMod})$ between the bounded-above derived categories. Since $D^-(OH^\ell_n\text{-gsMod})$ is equivalent to $K^-(OH^\ell_n\text{-pgsMod})$ and similarly for $OH^\ell_{n'}$, we deduce that the functor arising from tensoring with the singular Rouquier complex defines an equivalence of triangulated graded supercategories $K^-(OH^\ell_n\text{-pgsMod}) \to K^-(OH^\ell_{n'}\text{-pgsMod})$. The first part of the corollary follows on restricting this equivalence to $K^b(OH^\ell_n\text{-pgsmod})$. The second part follows using also (12.2) because the functor takes the cochain complex that is OH^ℓ_n concentrated in cohomological degree 0 to a cochain complex with the same Euler characteristic as $(\Pi Q^2)^{\binom{n+1}{2}+nk} OH^\ell_{n'}$ concentrated in cohomological degree $-n$. □

The remainder of the section is devoted to the proof of Theorem 12.3, which will be carried out with a series of lemmas. We assume for simplicity of notation that $k \geq 0$, although with obvious modifications the arguments work for negative k too.

Lemma 12.5. *We have that $\partial_{d-1} \circ \partial_d = 0$ for $d = 1, \ldots, n+1$, hence,* (12.4) *is a chain complex.*

Proof. By the super interchange law, $\partial_{d-1} \circ \partial_d$ factorizes as the composition first of the embedding

$$C_d = U^\ell_{(k+d);n-d} \otimes_{OH^\ell_{n-d}} V_{n-d;(d)} \xrightarrow{\text{inc}} U^\ell_{(k+d-2,2);n-d} \otimes_{OH^\ell_{n-d}} V_{n-d;(2,d-2)} \xrightarrow{b_{(k+d-2),(2)} \otimes c_{(2),(d-2)}} U^\ell_{(k+d-2);n-d+2} \otimes_{OH^\ell_{n-d+2}} U^\ell_{(2);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(2)} \otimes_{OH^\ell_{n-d+2}} V^\ell_{n-d+2;(d-2)}$$

then the map $\mathrm{id} \otimes (\partial \circ \iota) \otimes \mathrm{id}$ from there to $C_{d-2} = U^\ell_{(k+d-2);n-d+2} \otimes_{OH^\ell_{n-d+2}} V^\ell_{n-d+2;(d-2)}$, where

$$\iota : U^\ell_{(2);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(2)} \xrightarrow{\text{inc}} U^\ell_{(1,1);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(1,1)} \xrightarrow{b_{(1),(1)} \otimes c_{(1),(1)}} U^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} U^\ell_{n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1},$$

$$\partial : U^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} U^\ell_{n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1} \xrightarrow{\mathrm{id} \otimes \mathrm{ev}_{n-d} \otimes \mathrm{id}} U^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} OH^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1} \cong U^\ell_{n-d+1} \otimes_{OH^\ell_{n-d+1}} V^\ell_{n-d+1} \xrightarrow{\mathrm{ev}_{n-d+1}} OH^\ell_{n-d+2}.$$

Thus, we are reduced to proving that ∂ takes vectors in the image of ι to zero. By Lemma 10.5, the image of ι is equal to the image of the projection $\rho_{(1^2);n-d}((\xi\omega)_2) \otimes \lambda_{n-d;(1^2)}((\omega\xi)_2)$. This projection equals

$$\rho_{(1^2);n-d}(x_1\tau_1) \otimes \lambda_{n-d;(1^2)}(\tau_1 x_1) = (\rho_{(1^2);n-d}(\tau_1) \otimes \lambda_{n-d;(1^2)}(\tau_1)) \circ (\rho_{(1^2);n-d}(x_1) \otimes \lambda_{n-d;(1^2)}(x_1)).$$

Finally, to complete the proof, we observe that $\partial \circ (\rho_{(1^2);n-d}(\tau_1) \otimes \lambda_{n-d;(1^2)}(\tau_1)) = 0$ because

$$\partial \circ (\rho_{(1^2);n-d}(\tau_1) \otimes \mathrm{id} \otimes \mathrm{id}) = \partial \circ (\mathrm{id} \otimes \mathrm{id} \otimes \lambda_{n-d;(1^2)}(\tau_1))$$

thanks to Lemma 11.10, and $\lambda_{n-d;(1^2)}(\tau_1) \circ \lambda_{n-d;(1^2)}(\tau_1) = \lambda_{n-d;(1^2)}(\tau_1^2) = 0$. □

Now we need to understand the "numerology" of (12.4). In fact, the combinatorial Lemma 3.3 derived long ago is just what we need for this. Recall the definitions of $b_{m,n}(r), c_{m,n}(r) \in \mathbb{Z}[q,q^{-1}]^\pi$ made in the statement of that lemma. The following shows that $c_{n+k,n}(d)$ is the graded superrank of C_d either as a free graded right OH^d_n-supermodule or a free graded left $OH^d_{n'}$-supermodule. We will also see in a bit that $b_{n+k,n}(d)$ is the graded rank of $\mathrm{im}\,\partial_d$ for $d = 0, 1, \dots, n$.

Lemma 12.6. *The vectors*

$$\left\{ u_{(k+d);n-d}(\bar{\sigma}^{(k+d)}_\lambda) \otimes v_{n-d;(d)}(\bar{\sigma}^{(d)}_\mu) \,\middle|\, (\lambda, \mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)} \right\} \tag{12.5}$$

give a basis for C_d as a free right OH^ℓ_n-supermodule. Hence, as a graded right OH^ℓ_n-supermodule, C_d is free of graded superrank $c^{(d)}_{n+k,n}$. It is also free as a graded left $OH^\ell_{n'}$-superbimodule with the same graded superrank.

Proof. Lemma 10.4 implies that it is free as a graded right OH^ℓ_n-supermodule with basis (12.5). The formula for its graded superrank then follows using Corollary 3.2. It is also free as a graded left $OH^\ell_{n'}$-supermodule thanks to Lemma 10.4 again. Since $OH^\ell_{n'} \cong OH^\ell_n$, its graded superrank for $OH^\ell_{n'}$ is the same as for OH^ℓ_n. □

Recall that OH^ℓ_n is positively graded with degree 0 component isomorphic the ground field $\mathbb{F}$. We apply the functor $- \otimes_{OH^\ell_n} \mathbb{F}$ to (12.4) to obtain the chain complex

$$0 \xrightarrow{\bar\partial_{n+1}} \overline{C}_n \xrightarrow{\bar\partial_n} \cdots \xrightarrow{\bar\partial_{d+1}} \overline{C}_d \xrightarrow{\bar\partial_d} \overline{C}_{d-1} \xrightarrow{\bar\partial_{d-1}} \cdots \xrightarrow{\bar\partial_2} \overline{C}_1 \xrightarrow{\bar\partial_1} \overline{C}_0 \xrightarrow{\bar\partial_0} 0 \tag{12.6}$$

of graded left $\overline{OH}^\ell_{n'}$-supermodules. Lemma 12.6 implies that $\overline{C}_d$ is of graded superdimension $c_{n+k,n}(d)$.

Lemma 12.7. *Suppose we are given that* $\dim \operatorname{im} \bar\partial_d \geq |b_{n+k,n}(d)|$ *for* $d = 1, \dots, n$*, where* $|b_{n+k,n}(d)|$ *denotes the natural number obtained by applying the evaluation map* $\mathbb{Z}[q, q^{-1}]^\pi \to \mathbb{Z}, q \mapsto 1, \pi \mapsto 1$ *to* $b_{n+k,n}(d)$*. Then Theorem 12.3 is true.*

Proof. We first consider the specialized complex (12.6), showing that $\operatorname{im} \bar\partial_d = \ker \bar\partial_{d-1}$ and that it is of graded superdimension $b_{n+k,n}(d)$ for each $d = 1, \dots, n$. This follows by induction on d, defining $\bar\partial_{-1}$ to be the zero map so that we can start the induction at $d = 0$. The induction base holds because $b_{n+k,n}(0) = 0$. For the induction step, take $0 \leq d < n$ and assume that $\operatorname{im} \bar\partial_d = \ker \bar\partial_{d-1}$ is of graded superdimension $b_{n+k,n}(d)$. We have that $\overline{C}_d = \overline{B}'_d \oplus \overline{Z}_d$ where $\overline{Z}_d := \ker \bar\partial_d$ and $\overline{B}'_d \simeq \operatorname{im} \bar\partial_d$ is a complementary graded superspace. By induction, $\dim_{q,\pi} \overline{B}'_d = b_{n+k,n}(d)$. We have that $\dim \overline{C}_d = \dim \operatorname{im} \bar\partial_d + \dim \ker \bar\partial_d$ so, using Lemma 3.3 for the final equality, get that

$$|c_{n+k,n}(d)| = |b_{n+k,n}(d)| + \dim \ker \bar\partial_d \geq |b_{n+k,n}(d)| + \dim \operatorname{im} \bar\partial_{d+1} \geq |b_{n+k,n}(d)| + |b_{n+k,n}(d+1)| = |c_{n+k,n}(d)|.$$

This means that equality holds throughout, thereby proving that $\operatorname{im} \bar\partial_{d+1} = \ker \bar\partial_d$. The same sequence of equalities without evaluating at $q = \pi = 1$ now gives that $\dim_{q,\pi} \operatorname{im} \bar\partial_{d+1} = b_{n+k,k}(d+1)$, and the argument is complete.

Next we show that $\operatorname{im} \partial_d = \ker \partial_{d-1}$ and that it is free as a graded right OH^ℓ_n-supermodule of graded superrank $b_{n+k,n}(d)$ for each $d = 1, \dots, n$. This is a similar induction to the one in the previous paragraph. For the induction step, we take $0 \leq d < n$ and assume that we have shown already that $\operatorname{im} \partial_d$ is free of graded superrank $b_{n+k,n}(d)$. Consider the short exact sequence

$$0 \longrightarrow Z_d \longrightarrow C_d \longrightarrow \operatorname{im} \partial_d \longrightarrow 0$$

where $Z_d := \ker \partial_d$. Since $\operatorname{im} \partial_d$ is free, this short exact sequence splits, so we have that $C_d = B'_d \oplus Z_d$ where $B'_d \simeq \operatorname{im} \partial_d$ is a complement to Z_d in C_d as a graded right OH^ℓ_n-supermodule. Moreover, $\overline{Z}_d$ from the previous paragraph is $Z_d \otimes 1$. As it is a summand of C_d, which is free, we deduce that Z_d is a free graded right OH^ℓ_n-supermodule with $\operatorname{rk}_{q,\pi} Z_d = \dim_{q,\pi} \overline{Z}_d = b_{n+k,k}(d+1)$. The map $\partial_{d+1} : C_{d+1} \to Z_d$ is surjective because $\operatorname{id} \otimes \partial_{d+1} : \overline{C}_{d+1} \to \overline{Z}_d$ is surjective according to the previous paragraph. We deduce that $\operatorname{im} \partial_{d+1} = \ker \partial_d$ is free of graded superrank $b_{n+k,k}(d+1)$, and the argument is complete.

So now we have shown that $\operatorname{im} \partial_d = \ker \partial_{d-1}$ is free as a graded right OH^ℓ_n-supermodule of graded superrank $b_{n+k,n}(d)$ for $d = 1, \dots, n$. The same is true as a graded left $OH^\ell_{n'}$-supermodule since $OH^\ell_{n'} \cong OH^\ell_n$ and all of the numerology is the same.

To complete the proof of Theorem 12.3, it just remains to prove the assertion about the top degree homology. As $OH^\ell_{n'} \cong OH^\ell_n$, it suffices to show that it is free of graded superrank $(\pi q^2)^{\binom{n+1}{2}+nk}$ both as a graded right OH^ℓ_n- and as a graded left $OH^\ell_{n'}$-supermodule. We have already shown that the image of ∂_n is free of graded superrank $b_{n+k,k}(n)$. Hence, since C_n is free of graded superrank $c_{n+k,n}(n)$ by Lemma 12.6, we deduce that $\ker \partial_n$ is free of graded superrank $c_{n+k,n}(n) - b_{n+k,k}(n)$. Thus, we are reduced to showing that $c_{n+k,n}(n) - b_{n+k,k}(n) = (\pi q^2)^{\binom{n+1}{2}+nk}$. This follows from the following calculation:

$$\begin{aligned} c_{n+k,n}(n) - b_{n+k,k}(n) &= (\pi q^{-2})^{\binom{n}{2}} q^{(n+k)n} \begin{bmatrix} 2n+k \\ n \end{bmatrix}_{q,\pi} \\ &\qquad - (\pi q^{-2})^{\binom{n}{2}} \sum_{s=0}^{n-1} (\pi q^2)^{(n+k)(n-s-1)} q^{(n+k-1)(s+1)} \begin{bmatrix} n+k+s \\ s+1 \end{bmatrix}_{q,\pi} \\ &= (\pi q^{-2})^{\binom{n}{2}} \sum_{s=0}^{n} (\pi q^2)^{(n+k)(n-s)} q^{(n+k-1)s} \begin{bmatrix} n+k+s-1 \\ s \end{bmatrix}_{q,\pi} \\ &\qquad - (\pi q^{-2})^{\binom{n}{2}} \sum_{s=1}^{n} (\pi q^2)^{(n+k)(n-s)} q^{(n+k-1)s} \begin{bmatrix} n+k+s-1 \\ s \end{bmatrix}_{q,\pi} \end{aligned}$$

$$= (\pi q^2)^{(n+k)n-\binom{n}{2}} = (\pi q^2)^{\binom{n+1}{2}+nk}.$$

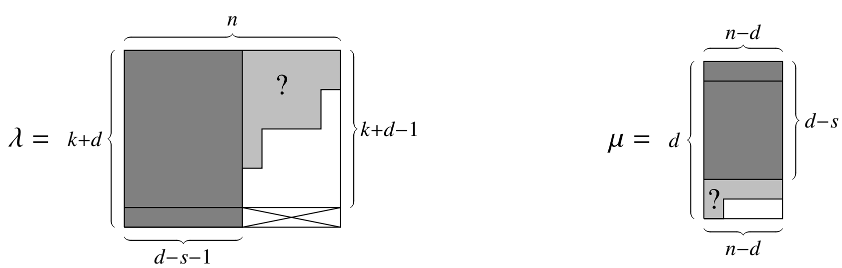
Here, we have used the definitions in Lemma 3.3 for the first equality and Corollary 3.4 for the second. ◻

Remark 12.8. From Theorem 12.1(1)–(2), tensoring with the dth term $U^{\ell}_{(k+d);n-d} \otimes_{OH^{\ell}_{n-d}} V^{\ell}_{n-d;(d)}$ in the singular Rouquier complex corresponds at the level of Grothendieck groups to the $\mathbb{Z}[q,q^{-1}]^{\pi}$-module homomorphism $1_{-k}\mathbf{V}(-\ell) \to 1_k\mathbf{V}(-\ell)$ defined by the action of

$$q^{(n-d)(k+d)+d(\ell-3(n-d)-2d+1)}E^{(k+d)}F^{(d)}1_{2n-\ell} = q^{nk}\left(q^d E^{(k+d)}F^{(d)}1_{2n-\ell}\right).$$

From the original definition of the map T in Theorem 3.6, it follows that multiplication by the Euler characteristic of (12.4) corresponds to $q^{nk}T : 1_{-k}\mathbf{V}(-\ell) \to 1_k\mathbf{V}(-\ell)$. Given that the homology of the complex vanishes in all but the top degree, it then follows by (12.2) that the top homology is $(\pi q^2)^{\binom{n+1}{2}+nk}\left[OH^{\ell}_{n'}\right]$ which, reassuringly, agrees with the final assertion of Theorem 12.3.

Proof of Theorem 12.3. Fix d with $1 \leq d \leq n$. Define an *initial pair* to be $(\lambda,\mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)}$ such that $\lambda_{k+d} = d-1-s$ and $\mu_{d-s} = n-d$ for some $0 \leq s \leq d-1$. This condition is illustrated by the following picture:

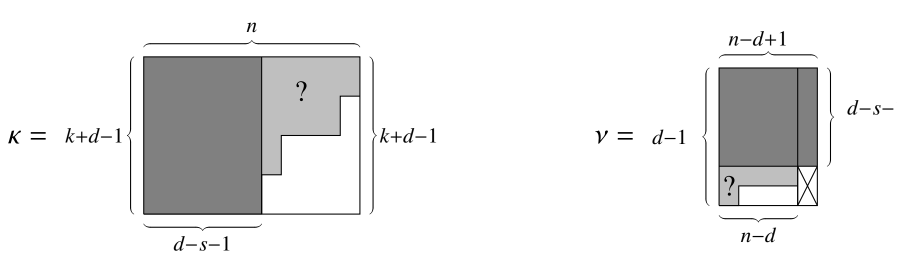


Let I be the set of all initial pairs. Note that

$$|I| = |b_{n+k,n}(d)|. \tag{12.7}$$

To see this, the number of $(\lambda,\mu) \in I$ with $\lambda_{k+d} = d-1-s$ and $\mu_{d-s} = n-d$ is $|\Lambda^+_{(k+d-1)\times(n-d+s+1)} \times \Lambda^+_{s\times(n-d)}|$, which is $\binom{n+k+s}{n-d+s+1}\binom{n-d+s}{s}$. Summing[6] over $s = 0,1,\dots,d-1$ gives $|b_{n+k,n}(d)|$ by the original definition of this natural number.

Also define a *terminal pair* to be $(\kappa,\nu) \in \Lambda^+_{(k+d-1)\times n} \times \Lambda^+_{(d-1)\times(n-d+1)}$ such that $\kappa_{k+d-1} \geq d-1-s$ and $\nu_{d-s} < \nu_{d-s-1} = n-d+1$ for some $0 \leq s \leq d-1$:



Let T be the set of all terminal pairs. Our final combinatorial observation is that there is a bijection

$$f : I \xrightarrow{\sim} T \tag{12.8}$$

[6]This is all we need here, but with a little more care using also Corollary 3.2, this argument can be used to show that the coefficient of $q^{2r}\pi^r$ in $b_{n+k,n}(d)$ is equal to the number of $(\lambda,\mu) \in I$ with $|\lambda| + |\mu| = 2r$, explaining the definition of $b_{n+k,n}(d)$ itself rather than merely its evaluation at $q = \pi = 1$.

taking $(\lambda,\mu) \in I$ to $(\lambda^-,\mu^+) \in T$ where λ^- is obtained from λ by removing the bottom row of its Young diagram, and μ^+ is obtained from μ by adding one box to the end of the first $(d-s)$ rows then removing completely the top row.

Now we are going to make an explicit computation of the differential $\bar{\partial}_d$ in terms of the basis for $\overline{C}_d = C_d \otimes_{OH_n^\ell} \mathbb{F}$ consisting of the vectors

$$w(\lambda,\mu) := u_{n-d;(k+d)}(\bar{\sigma}_\lambda^{(k+d)}) \otimes v_{n-d;(d)}(\bar{\sigma}_\mu^{(d)}) \otimes 1 \in U^\ell_{(k+d);n-d} \otimes_{OH^\ell_{n-d}} V^\ell_{n-d;(d)} \otimes_{OH_n^\ell} \mathbb{F} \tag{12.9}$$

for $(\lambda,\mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)}$; cf. Lemma 12.6. Order pairs $(\kappa,\nu) \in \Lambda^+_{(k+d-1)\times n} \times \Lambda^+_{(d-1)\times(n-d+1)}$ so that $(\kappa',\nu') < (\kappa,\nu)$ if either $|\kappa'| < |\kappa|$, or $|\kappa'| = |\kappa|$ and $\nu' <_{\text{lex}} \nu$. We claim for $(\lambda,\mu) \in I$ that

$$\bar{\partial}_d(w(\lambda,\mu)) = \pm w(\lambda^-,\mu^+) + \text{ (a linear combination of } w(\kappa,\nu) \text{ for } (\kappa,\nu) < (\lambda^-,\mu^+)). \tag{12.10}$$

Given the claim, it follows by (12.7) and (12.8) that $|\operatorname{im}\bar{\partial}_d| \geq |b_{n+k,n}(d)|$, so that the theorem follows by Lemma 12.7.

It remains to prove (12.10). Take $(\lambda,\mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)}$ and consider $\bar{\partial}_d(w(\lambda,\mu))$. According to the definition of $\bar{\partial}_d$, we have to apply three different maps to $w(\lambda,\mu)$ arising from $b_{(d+k-1),(1)}$, $c_{(1),(d-1)}$ and ev_{n-d}. We apply these maps one by one.

- First, the map $b_{(d+k-1),(1)}$ comes from the embedding

$$OSym_{k+d} \hookrightarrow OSym_{(k+d-1,1)} \xrightarrow{\sim} OSym_{k+d-1} \otimes \mathbb{F}[x].$$

  Lemma 7.7 shows that this embedding takes $s_\lambda^{(k+d)}$ to $\sum_\kappa \pm s_\kappa^{(k+d-1)} \otimes x^{|\lambda|-|\kappa|}$ summing over all $\kappa \in \Lambda^+_{(k+d-1)\times n}$ whose Young diagram is obtained by removing boxes from the bottoms of different columns of the Young diagram of λ, necessarily including all λ_{k+d} boxes on its $(k+d)$th row. We say simply "κ obtained by removing a row strip from λ" for this from now on.
- Next, we apply the map $c_{(1),(d-1)}$, which comes from the embedding

$$OSym_d \hookrightarrow OSym_{(1,d-1)} \xrightarrow{\sim} \mathbb{F}[x] \otimes OSym_{d-1},$$

  plus some extra signs due to the parity shift. The version of Pieri obtained by applying γ_d to Lemma 7.7 (using also (6.8)) shows that this takes $\sigma_\mu^{(d)}$ to $\sum_\delta \pm x^{|\mu|-|\delta|} \otimes \sigma_\delta^{(d-1)}$ summing over $\delta \in \Lambda^+_{(d-1)\times(n-d)}$ whose Young diagram is obtained by removing boxes from the bottoms of different columns of the Young diagram of μ, necessarily including all μ_d boxes on its dth row, to obtain the Young diagram of partition δ. We say simply "δ obtained by removing a row strip from μ" for this from now on.
- So far, remembering (10.6), we have shown that $(b_{(k+d-1),(1)} \otimes\ c_{(1),(d-1)} \otimes \mathrm{id}) \circ (\mathrm{inc} \otimes \mathrm{id})$ takes $w(\lambda,\mu)$ to

$$\sum_{(\kappa,\delta)} \pm u_{(k+d-1);n-d+1}(\sigma_\kappa^{(k+d-1)}) \otimes u_{n-d}(x^{|\lambda|-|\kappa|}) \otimes v_{n-d}(x^{|\mu|-|\delta|}) \otimes v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}) \otimes 1$$

  summing over $(\kappa,\delta) \in \Lambda^+_{(k+d-1)\times n} \times \Lambda^+_{(d-1)\times(n-d)}$ such that κ is obtained by removing a row strip from λ and δ is obtained by removing a row strip from μ. Then we use the definition in Theorem 11.3 to apply $(\mathrm{can} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{ev}_{n-d} \otimes \mathrm{id} \otimes \mathrm{id})$, giving

$$\bar{\partial}_d(w(\lambda,\mu)) = \sum_{(\kappa,\delta)} \pm u_{(k+d-1);n-d+1}(\sigma_\kappa^{(k+d-1)}) \otimes \bar{\eta}^{(n-d+1)}_{(|\lambda|-|\kappa|)+(|\mu|-|\delta|)-(n-d)} v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}) \otimes 1.$$

  summing over all $(\kappa,\delta) \in \Lambda^+_{(k+d-1)\times n} \times \Lambda^+_{(d-1)\times(n-d)}$ obtained by removing a row strip from $(\lambda,\mu) \in \Lambda^+_{(k+d)\times n} \times \Lambda^+_{d\times(n-d)}$.

It remains to commute the elements $\bar{\eta}^{(n-d+1)}_{(|\lambda|-|\kappa|)+(|\mu|-|\delta|)-(n-d)}$ to the right hand side in this expression. In view of Lemma 10.4(1) and degree considerations, this will produce some linear combination of basis vectors of the form $w(\kappa,\nu)$ for $\nu \in \Lambda^+_{(d-1)\times(n-d+1)}$ with $|\nu| = |\mu| + (|\lambda| - |\kappa|) - (n-d)$. We just need to show that $w(\lambda^-,\mu^+)$ appears with coefficient ±1 and all other $w(\kappa,\nu)$ that arise satisfy $(\kappa,\nu) < (\lambda^-,\mu^+)$. This is clearly the case if $|\kappa| < |\lambda^-|$, so we may assume from now on that κ, like λ^-, is obtained from λ by removing the minimal number of boxes, i.e., just its bottom row. So we have that $\kappa = \lambda^-$ and $|\lambda|-|\kappa| = \lambda_{k+d}$, which equals $d-s-1$ for a unique $0 \le s < d$. Also let $p := (d-s+1)+(|\mu|-|\delta|)-(n-d)$ for short and consider

$$\bar{\eta}_p^{(n-d+1)} v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}) \otimes 1.$$

By Theorem 6.3, we have that $\eta_p^{(n-d+1)} = s_{(p)}^{(n-d+1)}$ plus a linear combination of other $s_\tau^{(n-d+1)}$ for partitions τ with $|\tau| = p$ and $\mathrm{ht}(\tau) > 1$. Also $\sigma_{(1^p)}^{(d-1)} = \varepsilon_p^{(d-1)}$. Using (10.13), we deduce that

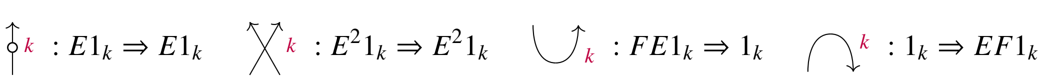
$$\bar{\eta}_p^{(n-d+1)} v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}) \otimes 1 = \pm v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}\varepsilon_p^{(d-1)}) \otimes 1 + (*) \tag{12.11}$$

where $(*)$ is a linear combination of terms of the form $v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}\sigma_\tau^{(d-1)})\otimes 1$ for partitions τ with $|\tau| = p$ and $\tau_1 > 1$. We can compute all of these products of dual Schur polynomials by conjugating with γ_{d-1} and using the odd Littlewood-Richardson rule; see Remark 7.8. Remembering also that $v_{n-d+1;(d-1)}(\sigma_\nu^{(d-1)}) \otimes 1 = 0$ unless $\nu \in \Lambda^+_{(d-1)\times(n-d+1)}$ by Lemma 10.4(1) again, we obtain a linear combination of basis vectors $v_{n-d+1;(d-1)}(\sigma_\nu^{(d-1)}) \otimes 1$ for $\nu \in \Lambda^+_{(d-1)\times(n-d+1)}$ obtained from δ by adding p boxes in particular ways. If $p < d-s-1$ then we cannot have $\nu = \mu^+$ since that has $d-s+1$ boxes in the rightmost column, whereas that column is empty in δ. Now suppose that $p = d-s-1$; then δ is μ with all $(n-d)$ boxes in its first row removed. In this case, the leading term $\pm v_{n-d+1;(d-1)}\big(\sigma_\delta^{(d-1)}\varepsilon_p^{(d-1)}\big) \otimes 1$ computed via the odd Littlewood-Richardson rule does produce $\pm v_{n-d+1;(d-1)}(\sigma_{\mu^+}^{(d-1)})\otimes 1$ when a column strip of p boxes is added at the top right of the Young diagram of δ. All other basis vectors coming from this leading term are of the form $v_{n-d+1;(d-1)}(\sigma_\nu^{(d-1)}) \otimes 1$ for $\nu <_{\mathrm{lex}} \mu^+$. The basis vectors coming from the lower terms $v_{n-d+1;(d-1)}(\sigma_\delta^{(d-1)}\sigma_\tau^{(d-1)}) \otimes 1$ for τ with $|\tau| = p$ and $\tau_1 > 1$ must also all be of the form $v_{n-d+1;(d-1)}(s_\nu^{(d-1)}) \otimes 1$ for $\nu <_{\mathrm{lex}} \mu^+$ since when $\tau_1 > 1$ the odd Littlewood-Richardson rule does not allow all p boxes to be added to the same column of the Young diagram of δ. □

13. Non-degeneracy of the odd 2-category $\mathfrak{U}(\mathfrak{sl}_2)$

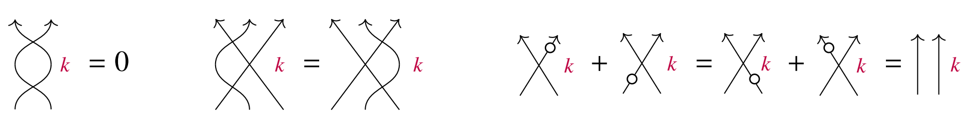
In this section, we will use the string calculus for strict graded monoidal supercategories and 2-supercategories adopting all of the conventions from [BE1]. In particular, $f \circ g$ is vertical composition (f on top of g) and $f \otimes g$ or simply fg is horizontal composition (f to the left of g). The following definition originated in [EL] and was reformulated in the present terms in [BE2].

Definition 13.1. The *odd* $\mathfrak{sl}_2$ *2-category* is the strict graded 2-supercategory $\mathfrak{U}(\mathfrak{sl}_2)$ with object set $\mathbb{Z}$, generating 1-morphisms $E1_k = 1_{k+2}E : k \to k+2$ and $1_kF = F1_{k+2} : k+2 \to k$ for each $k \in \mathbb{Z}$ whose identity 2-morphisms are represented graphically by $\uparrow k = k{+}2 \uparrow$ and $k \downarrow = \downarrow k{+}2$, respectively, and generating 2-morphisms



(13.1)

which are odd of degree 2, odd of degee -2, even of degree $k+1$, and even of degree $1-k$, respectively. Then there are three families of relations. First we have the odd nil-Hecke relations (in the standard formulation rather than the modified version from (5.1) to (5.6)):

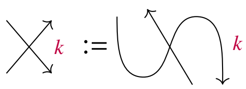


(13.2)

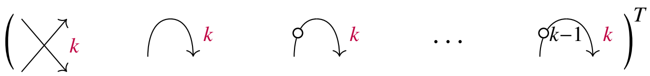
Next we have the *right adjunction relations* asserting that $Q^{-k-1}1_kF$ is right dual to $E1_k$ in the (Q,Π)-envelope of $\mathfrak{U}(\mathfrak{sl}_2)$ (cf. Lemma 2.1(1)):

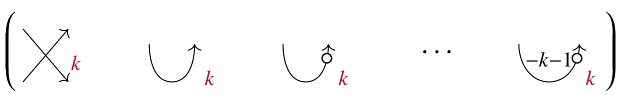
 (13.3)

Finally there are some *inversion relations*. To formulate these, we first introduce new 2-morphisms

 $: EF1_k \to FE1_k.$ (13.4)

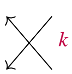
Then, denoting powers of the dot generator by labelling it with a natural number, we require that the following (not necessarily homogeneous) 2-morphisms are isomorphisms:

 $: EF1_k \xrightarrow{\sim} FE1_k \oplus 1_k^{\oplus k}$ for $k \geq 0$ (13.5)

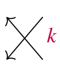
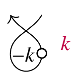
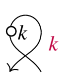
 $: EF1_k \oplus 1_k^{\oplus(-k)} \xrightarrow{\sim} FE1_k$ for $k \leq 0.$ (13.6)

The morphisms depicted in (13.5) and (13.6) represent a $(k+1) \times 1$ matrix and a $1 \times (1-k)$ matrix of 2-morphisms in $\mathfrak{U}(\mathfrak{sl}_2)$, respectively, i.e., they are 2-morphisms in the additive envelope of $\mathfrak{U}(\mathfrak{sl}_2)$. Saying that they are isomorphisms means that there are some further generating 2-morphisms in $\mathfrak{U}(\mathfrak{sl}_2)$ which provide the matrix entries of two-sided inverses to these morphisms.

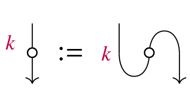
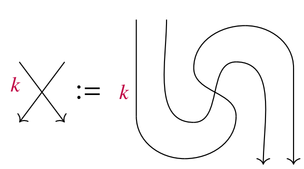
The defining relations (13.1) to (13.3), (13.5) and (13.6) look quite innocent but they imply many further relations. In order to record some of these, we first need to introduce some further shorthand for generating 2-morphisms whose existence is provided by the inversion relation. First, we have the leftward crossing and the leftward cups and caps

 $: FE1_k \Rightarrow EF1_k$  $: 1_kEF \Rightarrow 1_k$  $: 1_k \Rightarrow FE1_k$ (13.7)

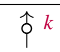
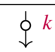
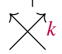
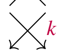
which are defined as follows.

- We let  $: FE1_k \Rightarrow EF1_k$ be the *negation* of the leftmost entry of the $1 \times (k+1)$ matrix that is the two-sided inverse of (13.5) if $k \geq 0$, or the *negation* of the top entry of the $(1-k) \times 1$ matrix that is the two-sided inverse of (13.6) if $k \leq 0$; cf. [BE2, (2.8)–(2.9)].
- We let  be the rightmost entry of the $1 \times (k+1)$ matrix that is the two-sided inverse of (13.5) if $k > 0$, or $(-1)^{k+1}$  if $k \leq 0$; cf. [BE2, (2.10)].
- We let  be the bottom entry of the $(1-k) \times 1$ matrix that is the two-sided inverse of (13.6) if $k < 0$, or $(-1)^{k+1}$  if $k \geq 0$; cf. [BE2, (2.11)]

Finally, we have the downward dot and the downward crossing, which are the right mates of the upward dot and the upward crossing:

 $: 1_kF \Rightarrow 1_kF$  $: 1_kF^2 \Rightarrow 1_kF^2.$ (13.8)

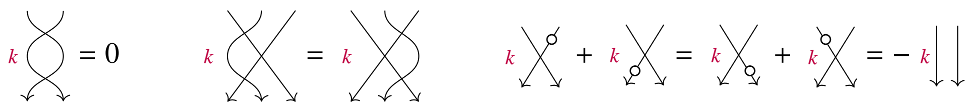
The following table summarizes the parity and degree information about all of the 2-morphisms defined thus far.

Generator	Degree	Parity	Generator	Degree	Parity
	2	$\bar{1}$		2	$\bar{1}$
	-2	$\bar{1}$		0	$\bar{1}$
	-2	$\bar{1}$		0	$\bar{1}$
	$k+1$	$\bar{0}$		$1-k$	$\bar{k}+\bar{1}$
	$1-k$	$\bar{0}$		$k+1$	$\bar{k}+\bar{1}$

(13.9)

The following relations are derived from the defining relations in [BE2].

- *Downward odd nil-Hecke relations*; cf. [BE2, (3.7),(3.9),(3.5)–(3.6)].



(13.10)

- *Left adjunction relations*; cf. [BE2, (6.6)].

(13.11)

Recalling Lemma 2.1(1), these imply that $Q^{k+1}\Pi^{k+1}E1_k$ is right dual to 1_kF in the (Q,Π)-envelope of $\mathfrak{U}(\mathfrak{sl}_2)$.

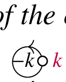
- *Infinite Grassmannian relation*; cf. [BE2, (5.3)–(5.7)]. Recall that R is the largest supercommutative quotient of *OSym* described explicitly in Corollary 4.12. For each $k \in \mathbb{Z}$, there is a graded superalgebra homomorphism[7]

$$\beta_k : R \to \operatorname{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k), \tag{13.12}$$

$\dot{\varepsilon}_r \mapsto (-1)^{(k+1)r}{}_{r+k-1}$  if $r \geq 1-k$,

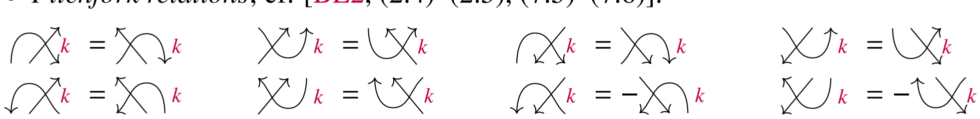
$\dot{\eta}_r \mapsto (-1)^{(k+1)r}{}_k$  ${}_{r-k-1}$ if $r \geq k+1$.

Following Lauda's convention from [L1, L2], we introduce new shorthands for endomorphisms of 1_k called "fake bubbles": we have clockwise bubbles decorated by $r+k-1$ dots on their left boundary for all $r \leq -k$ which denote $(-1)^{(k+1)r}\beta_k(\dot{\varepsilon}_r)$ if $r \geq 0$ or 0 if $r < 0$, and we have counterclockwise bubbles decorated by $r-k-1$ dots on their right boundary for all $r \leq k$ which denote $(-1)^{(k+1)r}\beta_k(\dot{\eta}_r)$ if $r \geq 0$ or 0 if $r < 0$.

- *Centrality of the odd bubble*; cf. [BE2, (7.15)]. The "odd bubble"  is shorthand for  if $k \geq 0$ or  if $k \leq 0$. So it is $(-1)^{k+1}\beta_k(\dot{o}) \in \operatorname{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k)$. These are odd 2-morphisms whose square is zero. Moreover, they are strictly central:

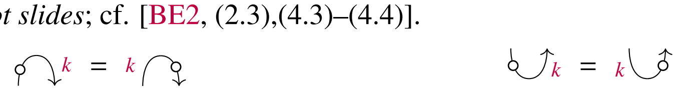
(13.13)

- *Pitchfork relations*; cf. [BE2, (2.4)–(2.5), (7.5)–(7.6)].



(13.14)
(13.15)

- *Dot slides*; cf. [BE2, (2.3),(4.3)–(4.4)].



(13.16)

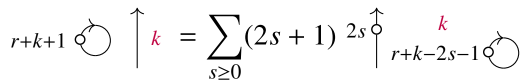
[7]There is some freedom in defining β_k—it is unique only up to an automorphism of R. The specific choice here has been made to facilitate Corollary 13.4.


(13.17)

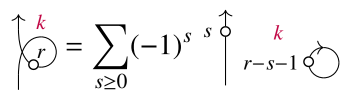
- *Almost pivotal structure*[8]; cf. [BE2, (1.27)–(1.28)].


(13.18)

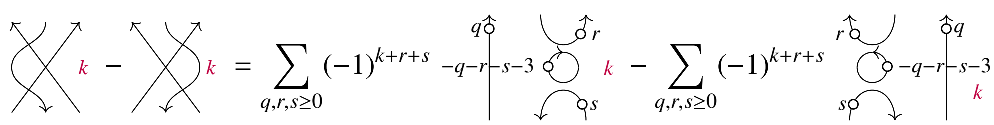
- *Bubble slides*; cf. [BE2, (7.10)].


(13.19)

- *Curl relations*; cf. [BE2, (5.21)].


(13.20)

- *Alternating braid relation*; cf. [BE2, (7.20)].


(13.21)

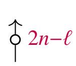
The following theorem is an odd analog of [L2, Th. 4.12]. Our proof is shorter since we are using the more efficient presentation of Definition 13.1 (although afterwards there is still work to do to determine the images of the leftward cups and caps).

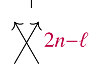
Theorem 13.2. *Fix $\ell \geq 0$. There is a graded 2-superfunctor $\Psi_\ell : \mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{OGBim}_\ell$ with the following properties.*

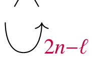
(1) On objects, Ψ_ℓ takes $2n-\ell$ to the graded superalgebra OH^ℓ_n for $0 \leq n \leq \ell$. All other objects of $\mathfrak{U}(\mathfrak{sl}_2)$ go to the trivial graded superalgebra.

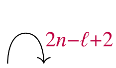
(2) On generating 1-morphisms, Ψ_ℓ takes $E1_{2n-\ell}$ to the graded superbimodule $Q^{-n}U^\ell_n$ and $1_{2n-\ell}F$ to the graded superbimodule $Q^{3n-\ell+1}V^\ell_n$, respectively, both for $0 \leq n < \ell$. All other generating 1-morphisms necessarily go to trivial graded superbimodules.

(3) On generating 2-morphisms, Ψ_ℓ takes

 $\mapsto \left(\rho_{(1);n}(x_1) : Q^{-n}U^\ell_n \to Q^{-n}U^\ell_n\right)$ $(0 \leq n \leq \ell-1)$

 $\mapsto \left(-\rho_{(1^2);n}(\tau_1) : Q^{-2n-1}U^\ell_{n+1} \otimes_{OH^\ell_{n+1}} U^\ell_n \to Q^{-2n-1}U^\ell_{n+1} \otimes_{OH^\ell_{n+1}} U^\ell_n\right)$ $(0 \leq n \leq \ell-2)$

 $\mapsto \left(\mathrm{coev}_n : OH^\ell_n \to Q^{2n-\ell+1}V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n\right)$ $(0 \leq n \leq \ell-1)$

 $\mapsto \left(\mathrm{ev}_n : Q^{2n-\ell+1}U^\ell_n \otimes_{OH^\ell_n} V^\ell_n \to OH^\ell_{n+1}\right)$ $(0 \leq n \leq \ell-1)$

Here, $\rho_{(1^d);n}(a)$ is the superbimodule endomorphism from (11.20), and ev_n and coev_n are as in Theorem 11.3[9]*. All other generating 2-morphisms are taken to zero.*

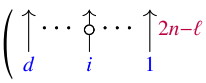
[8] Here, we have corrected a sign error in [BE2, (1.28)], and another one is corrected in (13.21) below. These mistakes were uncovered in [DEL].

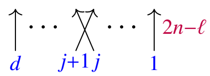
[9] We mean the same underlying linear maps—in some places we have applied some degree shifts (but no parity shifts) so that they are not now being viewed as homomorphisms between exactly the same *graded* superbimodules as before.

Proof. Note to start with that the assignments in (3) are superbimodule homomorphisms of the correct degrees and parities; cf. (13.9). Viewing $\mathcal{OGBim}_\ell$ as a strict graded 2-supercategory as explained in Remark 10.7, we will simply construct Ψ_ℓ as a strict graded 2-superfunctor by checking that the defining relations from Definition 13.1 are all satisfied. There are three sets of relations, (13.2), (13.3) and (13.5)–(13.6).

The right adjunction relations (13.3) follow immediately from Theorem 11.3.

Let us check the odd nil-Hecke relations from (13.2). The formulation of these relations in (13.2) differs by signs from the formulation in (5.1) to (5.6). This discrepancy is explained by the signs in formula (10.18). To be clear about this, for $a \in ONH_d$, let $\rho_{(1^d);n}(a)$ be the $(OH^\ell_{n+d}, OH^\ell_n)$-superbimodule endomorphism from (11.20) viewed now as an endomorphism of the degree-shifted $Q^{-nd-\binom{d}{2}} U^\ell_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n$, for $0 \le d \le n$ and $0 \le n \le \ell - d$. The definition from (3) implies more generally that

$\Psi_\ell \Bigg($  $\Bigg) = (-1)^{i-1} \rho_{(1^d);n}(x_i), \qquad (13.22)$

$\Psi_\ell \Bigg($  $\Bigg) = -(-1)^{j-1} \rho_{(1^d);n}(\tau_j). \qquad (13.23)$

We check (13.23), leaving the easier (13.22) to the reader. We must show that

$$\mathrm{id} \otimes \cdots \otimes \rho_{(1^2);n+j-1}(\tau_1) \otimes \cdots \otimes \mathrm{id} = (-1)^{j-1} \rho_{(1^d);n}(\tau_j).$$

We do this by checking that both sides take the same value on $u_{n+d-1}(x^{\kappa_d}) \otimes \cdots \otimes u_n(x^{\kappa_1})$ for any $\kappa \in \mathbb{N}^d$. Let $\sum_{\kappa'}$ denote summation over $\kappa' \in \mathbb{N}^d$ with $\kappa'_i = \kappa_i$ for $i \neq j, j+1$. Suppose that $x_{j+1}^{\kappa_{j+1}} x_j^{\kappa_j} \cdot \tau_j = \sum_{\kappa'} c_{\kappa'} x_{j+1}^{\kappa'_{j+1}} x_j^{\kappa'_j}$. Then, using (10.18), the left hand side gives

$$\sum_{\kappa'} (-1)^{1+(\kappa_j - \kappa'_j)+\kappa_j+\cdots+\kappa_d} c_{\kappa'} u_{n+d-1}(x^{\kappa'_d}) \otimes \cdots \otimes u_n(x^{\kappa'_1})$$

and the right hand side gives

$$(-1)^{j-1} \sum_{\kappa'} (-1)^{d-1+(d-j)(\kappa_j-\kappa'_j)+(d-j-1)(\kappa_{j+1}-\kappa'_{j+1})+\kappa_j+\cdots+\kappa_d} c_{\kappa'} u_{n+d-1}(x^{\kappa'_d}) \otimes \cdots \otimes u_n(x^{\kappa'_1}).$$

These are equal because $\kappa_j - \kappa'_j + \kappa_{j+1} - \kappa'_{j+1} = 1$ whenever $c_{\kappa'} \neq 0$.

With (13.22) and (13.23) in hand, the relations (13.2) are easily checked. For example, to check the length three braid relation, we must show that

$$\rho_{(1^3);n}(\tau_2) \circ \big(-\rho_{(1^3);n}(\tau_1) \big) \circ \rho_{(1^3);n}(\tau_2) = \big(-\rho_{(1^3);n}(\tau_1) \big) \circ \rho_{(1^3);n}(\tau_2) \circ \big(-\rho_{(1^3);n}(\tau_1) \big).$$

Translating to $U^\ell_{(1^3);n}$ using the isomorphism $b_{(1)^3}$, the left hand side becomes the map $u_{(1^3);n}(f) \mapsto (-1)^{\mathrm{par}(f)} u_{(1^3);n}(f \cdot \tau_2 \tau_1 \tau_2)$ and the right hand side becomes $u_{(1^3);n}(f) \mapsto -(-1)^{\mathrm{par}(f)} u_{(1^3);n}(f \cdot \tau_1 \tau_2 \tau_1)$. These are equal due to the sign in the relation (5.5). To check the third relation in (13.2), we must show that

$$\rho_{(1^2);n}(x_1) \circ \big(-\rho_{(1^2);n}(\tau_1) \big) + \rho_{(1^2);n}(\tau_1) \circ \rho_{(1^2);n}(x_2) = \rho_{(1^2);n}(\mathrm{id}).$$

The left hand side corresponds to the map $u_{(1^2);n}(f) \mapsto u_{(1^2);n}(f \cdot \tau_1 x_1 - f \cdot x_2 \tau_1)$, which equals $u_{(1^2);n}(f)$ by (5.6).

Next we check (13.5) and (13.6). There is nothing to do if $\ell = 0$ (the zero map is an isomorphism between zero superbimodules!) so assume that $\ell > 0$. Take a weight $k = 2n - \ell$ of $\mathbf{V}(-\ell)$ for some $0 \le n \le \ell$. Setting $n' := \ell - n - 1$, we have that $k = n - n' - 1$. For brevity, we will ignore grading shifts since they play no role in this place, i.e., we work with ordinary rather than graded superbimodules.

We first check (13.5), so $k \geq 0$ or, equivalently, $n \geq n' + 1$. We need to show that the superbimodule homomorphism f defined by the $(n-n') \times 1$ matrix

$$\left(\sigma_n \quad \mathrm{ev}_{n-1} \quad \cdots \quad \mathrm{ev}_{n-1} \circ (\rho_{(1);n-1}(x)^{n-n'-2} \otimes \mathrm{id})\right)^T : U_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \to V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \oplus (OH^\ell_n)^{\oplus(n-n'-1)}$$

is an isomorphism where σ_n, the image of the rightward crossing, is the superbimodule homomorphism described explicitly in Lemma 11.9, or the zero map in the extremal case $n = \ell, n' = -1$. By Lemma 10.8, the domain of f is free as a right OH^ℓ_n-supermodule with basis $\{u_{n-1}(x^r) \otimes v_{n-1}(x^s) \mid 0 \leq r \leq n'+1, 0 \leq s \leq n-1\}$, and the codomain of f is free as a right OH^ℓ_n-supermodule with basis $\{v_n(x^s) \otimes u_n(x^r) \mid 0 \leq r \leq n', 0 \leq s \leq n\} \cup \{b_1, \dots, b_{n-n'-1}\}$ where b_i is the identity element in the ith copy of OH^ℓ_n. Both of these sets are of size $nn' + 2n$, so it suffices to show that f is *surjective*. To prove this, since OH^ℓ_n is graded local, it is enough to show that the homomorphism $\bar{f} := f \otimes 1$ obtained by applying the functor $- \otimes_{OH^\ell_n} \mathbb{F}$ is surjective. Let $uv(r,s)$ denote $u_{n-1}(x^r) \otimes v_{n-1}(x^s) \otimes 1$ and $vu(s,r)$ denote $v_n(x^s) \otimes u_n(x^r) \otimes 1$. Thus, the domain of $\bar{f}$ has linear basis $\{uv(r,s) \mid 0 \leq r \leq n'+1, 0 \leq s \leq n-1\}$ and the codomain has linear basis $\{vu(s,r) \mid 0 \leq r \leq n', 0 \leq s \leq n\} \cup \{b_1 \otimes 1, \dots, b_{n-n'-1} \otimes 1\}$. By (11.22) and Theorem 11.3, we have that

$$\bar{f}(uv(r,s)) = \pm vu(n, r+s-n) \pm b_{n-r-s} \otimes 1 \pm vu(s,r) \tag{13.24}$$

for $0 \leq r \leq n'+1$ and $0 \leq s \leq n-1$, where the first term should be interpreted as zero if $r+s-n < 0$ and the middle term should be interpreted as zero if $n-r-s < 1$ or $n-r-s > n-n'-1$. Note also that the last term $vu(s,r)$ is zero for $r > n'$ by degree considerations. The argument is completed with the following observations.

- We get the vectors $vu(n,r)$ for $0 \leq r \leq n'$ from the images of the basis vectors $uv(n'+1, s)$ for $n-n'-1 \leq s \leq n-1$. Indeed, in the formula (13.24) for $\bar{f}(uv(n'+1,s))$ for these values of s, the second and third terms on the right hand side are both zero.
- Modulo the span of vectors already obtained, we get the vectors $b_1 \otimes 1, \dots, b_{n-n'-1} \otimes 1$ from the images of the basis vectors $uv(n'+1,s)$ for $0 \leq s \leq n-n'-2$. Indeed, in the formula for $f(uv(n'+1,s))$ for these values of s the third term on the right hand side is zero.
- Modulo the span of vectors already obtained, we get the vectors $vu(s,r)$ for $0 \leq r \leq n'$ and $0 \leq s \leq n-1$ from the images of the remaining basis vectors $uv(r,s)$ for these values of r and s.

Now consider (13.6), so $k \leq 0$ and $n' \geq n-1$. We need to show that the superbimodule homomorphism f defined by the $1 \times (n'-n+2)$ matrix

$$\left(\sigma_n \quad \mathrm{coev}_n \quad \cdots \quad (\mathrm{id} \otimes \rho_{(1);n}(x)^{n'-n}) \circ \mathrm{coev}_n\right) : U_{n-1} \otimes_{OH^\ell_{n-1}} V^\ell_{n-1} \oplus (OH^\ell_n)^{\oplus(n'-n+1)} \to V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n$$

is an isomorphism, where σ_n is as in Lemma 11.9 or the zero map in the extremal case $n = 0, n' = \ell - 1$. By Lemma 10.8, the domain of f is free as a right OH^ℓ_n-supermodule with basis $\{u_{n-1}(x^r) \otimes v_{n-1}(x^s) \mid 0 \leq r \leq n'+1, 0 \leq s \leq n-1\} \cup \{b_1, \dots, b_{n'-n+1}\}$, where b_i is the identity element in the ith copy of OH^ℓ_n, and the codomain of f is free as a right OH^ℓ_n-supermodule with basis $\{v_n(x^s) \otimes u_n(x^r) \mid 0 \leq r \leq n', 0 \leq s \leq n\}$. Both of these sets are of size $nn' + n + n' + 1$, so it suffices to see that f is surjective. Again, we apply $- \otimes_{OH^\ell_n} \mathbb{F}$ and show that the resulting map $\bar{f} := f \otimes 1$ is surjective. Let $uv(r,s) := u_{n-1}(x^r) \otimes v_{n-1}(x^s) \otimes 1$ and $vu(s,r) := v_n(x^s) \otimes u_n(x^r) \otimes 1$ for short. So the domain of $\bar{f}$ has linear basis $\{uv(r,s) | 0 \leq r \leq n'+1, 0 \leq s \leq n-1\} \cup \{b_1 \otimes 1, \dots, b_{n'-n+1} \otimes 1\}$ and the codomain has linear basis $\{vu(s,r) \mid 0 \leq r \leq n', 0 \leq s \leq n\}$. By (11.22) and (11.14), we have that

$$\bar{f}(uv(r,s)) = \pm vu(n, r+s-n) \pm vu(s,r), \qquad \bar{f}(b_i \otimes 1) = vu(n, i-1), \tag{13.25}$$

for $0 \leq r \leq n'+1$, $0 \leq s \leq n-1$ and $1 \leq i \leq n'-n+1$, interpreting $vu(r+s-n)$ as zero if $r+s-n < 0$ and s. The proof is completed by the following.

- We get $vu(n,r)$ for $0 \leq r \leq n'-n$ from the images of the vectors $b_i \otimes 1$ for $i = 1, \dots, n'-n+1$.

- We get $vu(n,r)$ for $n'-n+1 \leq r \leq n'$ from the images of the vectors $uv(n'+1,r)$ for $0 \leq r \leq n-1$. This uses the observation that $vu(r,n'+1) = 0$.
- Modulo the span of vectors already obtained, we get the remaining $vu(s,r)$ for $0 \leq s \leq n-1$ and $0 \leq r \leq n'$ from the images of the vectors $uv(r,s)$ for the same values of r and s.

□

In the next theorem, we give explicit descriptions of the images of the leftward cups and caps under the graded 2-superfunctor Ψ_ℓ from Theorem 13.2, that is, the superbimodule homomorphisms

$$\mathrm{coev}'_n := \Psi_\ell\Big(\circlearrowleft\!\bigcup\nolimits_{2n-\ell+2}\Big) : OH^\ell_{n+1} \to Q^{2n-\ell+1} U^\ell_n \otimes_{OH^\ell_n} V^\ell_n, \tag{13.26}$$

$$\mathrm{ev}'_n := \Psi_\ell\Big(\swarrow\!\!\bigcap{}^{2n-\ell}\Big) : Q^{2n-\ell+1} V^\ell_n \otimes_{OH^\ell_{n+1}} U^\ell_n \to OH^\ell_n \tag{13.27}$$

for $0 \leq n \leq \ell-1$ (these maps are zero for all other n). Let n' be defined so that $\ell = n+1+n'$. Then, by (13.9), coev'_n is a superbimodule homomorphism of degree $n'-n$ and parity $n'-n \pmod 2$, and ev'_n is of degree $n-n'$ and parity $n-n' \pmod 2$. Recall also the maps $\widetilde{\mathrm{ev}}_n$ and $\widetilde{\mathrm{coev}}_n$ from Theorem 11.5.

Theorem 13.3. *For $\ell = n+1+n'$ as above, we have that*

$$\mathrm{coev}'_n = (-1)^{\binom{n}{2}+(n+1)n'}(p_{n'}^{-1} \otimes q_n) \circ \widetilde{\mathrm{coev}}_n, \qquad \mathrm{ev}'_n = (-1)^{\binom{n+1}{2}+(n+1)n'}\widetilde{\mathrm{ev}}_n \circ (q_n^{-1} \otimes p_{n'}),$$

where $p_{n'} : Q^{-n}U^\ell_n \to \widetilde{U}^\ell_n$ and $q_n : \widetilde{V}^\ell_n \to Q^{3n-\ell+1}V^\ell_n$ are the superbimodule isomorphisms that are the identity maps on the underlying vector spaces. Moreover:

$$\Psi_\ell\Big(\ {\scriptstyle n-n'+r}\circlearrowleft\!\!\bigcirc\ {\scriptstyle n-n'+1}\ \Big)(1) = \sum_{s=0}^{r}(-1)^{(n'+1)r+(n+1)s+\binom{s}{2}}\Big[(\psi^\ell_{n+1})^{-1}(\bar\varepsilon^{(n')}_{r-s})\Big]\bar\eta^{(n+1)}_s \qquad \text{if } r \geq n'-n, \tag{13.28}$$

$$\Psi_\ell\Big(\ {\scriptstyle n-n'-1}\ \bigcirc\!\!\circlearrowright{\scriptstyle n'-n+r}\ \Big)(1) = \sum_{s=0}^{r}(-1)^{(n'+s)r+(n+1)s+\binom{s}{2}}\Big[(\psi^\ell_{n})^{-1}(\bar\eta^{(n'+1)}_{r-s})\Big]\bar\varepsilon^{(n)}_s \qquad \text{if } r \geq n-n'. \tag{13.29}$$

Proof. Recalling (10.2), the map $p_{n'}$ is of degree $n-2n'$ and parity $n' \pmod 2$, and q_n is of degree $n-\ell+1$ and parity $n \pmod 2$. The inverse of the map $p_{n'}^{-1} \otimes q_n$ is $(-1)^{nn'}p_{n'} \otimes q_n^{-1}$. With this in mind, we let

$$\widehat{\mathrm{coev}}_n := (-1)^{\binom{n}{2}+n'}(p_{n'} \otimes q_n^{-1}) \circ \mathrm{coev}'_n, \qquad \widehat{\mathrm{ev}}_n := (-1)^{\binom{n+1}{2}+n'}\,\mathrm{ev}'_n \circ (q_n \otimes p_{n'}^{-1}).$$

These are both even of degree 0. To prove the first part of the lemma, we must show that $\widehat{\mathrm{coev}}_n = \widetilde{\mathrm{coev}}_n$ and $\widehat{\mathrm{ev}}_n = \widetilde{\mathrm{ev}}_n$.

We first show that $\widehat{\mathrm{coev}}_n$ and $\widehat{\mathrm{ev}}_n$ are the counit and unit of an adjunction. This follows from the left adjunction relations (13.11):

$$\begin{aligned}
(\mathrm{id} \otimes \widehat{\mathrm{ev}}_n) \circ (\widehat{\mathrm{coev}}_n \otimes \mathrm{id}) &= (-1)^{\binom{n}{2}+\binom{n+1}{2}}(\mathrm{id} \otimes \mathrm{ev}'_n) \circ (\mathrm{id} \otimes q_n \otimes p_{n'}^{-1}) \circ (p_{n'} \otimes q_n^{-1} \otimes \mathrm{id}) \circ (\mathrm{coev}'_n \otimes \mathrm{id})\\
&= (-1)^{n+n'}(\mathrm{id} \otimes \mathrm{ev}'_n) \circ (p_{n'} \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes p_{n'}^{-1}) \circ (\mathrm{coev}'_n \otimes \mathrm{id})\\
&= (-1)^{\ell-1}p_{n'} \circ (\mathrm{id} \otimes \mathrm{ev}'_n) \circ (\mathrm{coev}'_n \otimes \mathrm{id}) \circ p_{n'}^{-1} = \mathrm{id},\\
(\widehat{\mathrm{ev}}_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \widehat{\mathrm{coev}}_n) &= (-1)^{\binom{n}{2}+\binom{n+1}{2}}(\mathrm{ev}'_n \otimes \mathrm{id}) \circ (q_n \otimes p_{n'}^{-1} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes p_{n'} \otimes q_n^{-1}) \circ (\mathrm{id} \otimes \mathrm{coev}'_n)\\
&= (\mathrm{ev}'_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{id} \otimes q_n^{-1}) \circ (q_n \otimes \mathrm{id} \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}'_n)\\
&= q_n^{-1} \circ (\mathrm{ev}'_n \otimes \mathrm{id}) \circ (\mathrm{id} \otimes \mathrm{coev}'_n) \circ q_n = \mathrm{id}.
\end{aligned}$$

Here, for brevity, we are implicitly assuming that $\mathcal{OGBim}_\ell$ is strict as in Remark 10.7.

So now we have two adjunctions making $(\widetilde{V}^\ell_n, \widetilde{U}^\ell_n)$ into a dual pair, one A_1 with unit $\widehat{\mathrm{ev}}_n$ and counit $\widehat{\mathrm{coev}}_n$ just constructed, and the other A_2 with unit $\widetilde{\mathrm{ev}}_n$ and counit $\widetilde{\mathrm{coev}}_n$ coming from Theorem 11.5. Any such adjunction A induces a degree 0 even $(OH^\ell_n, OH^\ell_{n+1})$-superbimodule isomorphism $\alpha : \widetilde{U}^\ell_n \xrightarrow{\sim}$

$\mathrm{Hom}_{OH_n^\ell\text{-}}(\widetilde{V}_n^\ell, OH_n^\ell)$. So from A_1 and A_2 we get isomorphisms α_1 and α_2, hence, an even degree 0 automorphism $\alpha_2^{-1}\circ\alpha_1$ of $\widetilde{U}_n^\ell$. By Lemma 10.8(2c), $\widetilde{U}_n^\ell$ is cyclic generated by the vector $\tilde{u}_n(1)$. Moreover, this vector spans the (one-dimensional) graded component of $\widetilde{U}_n^\ell$ of lowest degree. So we must have that $\alpha_2^{-1}\circ\alpha_1 = c_n\,\mathrm{id}$ for $c_n \in \mathbb{F}^\times$.

The argument in the previous paragraph shows that $\widehat{\mathrm{coev}}_n = c_n\ \widetilde{\mathrm{coev}}_n$ and $\widehat{\mathrm{ev}}_n = c_n^{-1}\ \widetilde{\mathrm{ev}}_n$ for some $c_n \in \mathbb{F}^\times$. To complete the proof of the first part of the lemma, we must show that $c_n = 1$. To see this, we will show that

$$\Psi_\ell\Big(\ {\scriptstyle n-n'+r}\circlearrowleft{\scriptstyle n-n'+1}\ \Big)(1) = c_n\sum_{s=0}^{r}(-1)^{(n'+1)r+(n+1)s+\binom{s}{2}}\Big[(\psi_{n+1}^\ell)^{-1}(\bar{\varepsilon}_{r-s}^{(n')})\Big]\bar{\eta}_s^{(n+1)} \qquad \text{if } r\geq n'-n, \tag{13.30}$$

$$\Psi_\ell\Big(\ {\scriptstyle n-n'-1}\circlearrowright{\scriptstyle n'-n+r}\ \Big)(1) = c_n^{-1}\sum_{s=0}^{r}(-1)^{(n'+s)r+(n+1)s+\binom{s}{2}}\Big[(\psi_n^\ell)^{-1}(\bar{\eta}_{r-s}^{(n'+1)})\Big]\bar{\varepsilon}_s^{(n)} \qquad \text{if } r\geq n-n'. \tag{13.31}$$

Given this, taking $r=0$ in one of these equations and using that the "bottom bubbles" ${\scriptstyle n-n'}\circlearrowleft{\scriptstyle n-n'+1}$ and ${\scriptstyle n-n'-1}\circlearrowright{\scriptstyle n'-n}$ are identities if $n\geq n'$ or $n\leq n'$, respectively, gives that $c_n=1$, and the lemma follows.

In this paragraph, we prove (13.30). We need to apply $\mathrm{ev}_n\circ(\rho_{(1);n}(x)^{n-n'+r}\otimes\mathrm{id})\circ\mathrm{coev}'_n$ to $1\in OH_{n+1}^\ell$ using that $\mathrm{coev}'_n = (-1)^{\binom{n}{2}+(n+1)n'}c_n\ (p_{n'}^{-1}\otimes q_n)\circ\widetilde{\mathrm{coev}}_n$. Applying $\widetilde{\mathrm{coev}}_n$ to 1 using the second formula for that in Theorem 11.5 gives

$$\sum_{s=0}^{n'}(-1)^{\ell s+\binom{s}{2}}\Big[(\psi_{n+1}^\ell)^{-1}(\bar{\varepsilon}_{n'-s}^{(n')})\Big]\tilde{u}_n(1)\otimes\tilde{v}_n(x^s).$$

Then we scale by $(-1)^{\binom{n}{2}+(n+1)n'}c_n$ and apply $(p_{n'}^{-1}\otimes q_n)$ to get

$$c_n\sum_{s=0}^{n'}(-1)^{\binom{n}{2}+(n+1)n'+\ell s+\binom{s}{2}+ns+n'(n'-s)}\Big[(\psi_{n+1}^\ell)^{-1}(\bar{\varepsilon}_{n'-s}^{(n')})\Big]u_n(1)\otimes v_n(x^s).$$

This is $\mathrm{coev}'_n(1)$. Then we apply $\rho_{(1);n}(x)^{n-n'+r}\otimes\mathrm{id}$ (the dots on the left boundary of the bubble) using (11.20) to get

$$c_n\sum_{s=0}^{n'}(-1)^{\binom{n}{2}+(n+1)n'+\ell s+\binom{s}{2}+ns+n'(n'-s)+(n-n'+r)(n'-s)+\binom{n-n'+r}{2}}\Big[(\psi_{n+1}^\ell)^{-1}(\bar{\varepsilon}_{n'-s}^{(n')})\Big]\tilde{u}_n(x^{n-n'+r})\otimes\tilde{v}_n(x^s).$$

Finally we apply ev_n using the formula from (11.7) to obtain

$$c_n\sum_{s=n'-r}^{n'}(-1)^{\binom{n}{2}+(n+1)n'+\ell s+\binom{s}{2}+ns+n'(n'-s)+(n-n'+r)(n'-s)+\binom{n-n'+r}{2}}\Big[(\psi_{n+1}^\ell)^{-1}(\bar{\varepsilon}_{n'-s}^{(n')})\Big]\bar{\eta}_{r+s-n'}^{(n+1)}.$$

It remains to reindex the summation replacing s by $s+(n'-r)$ and to simplify the signs to obtain (13.30).

To prove (13.31), we first note that ${\scriptstyle n-n'-1}\circlearrowright{\scriptstyle n'-n+r} = (-1)^{\binom{n'-n+r}{2}}\ {\scriptstyle n'-n+r}\circlearrowleft{\scriptstyle n-n'-1}$ by (13.16) and the super interchange law. Also $\Psi_\ell\Big(\ {\scriptstyle n-n'-1}\ \updownarrow\ \Big) = \lambda_{n;(1)}(x_1)$ by the definition (13.8), Theorem 13.2 and Lemma 11.8. Now we calculate by applying $(-1)^{\binom{n'-n+r}{2}}\,\mathrm{ev}'_n\circ(\lambda_{n;(1)}(x)^{n'-n+r}\otimes\mathrm{id})\circ\mathrm{coev}_n$ to $1\in OH_n^\ell$ using that $\mathrm{ev}'_n = (-1)^{\binom{n+1}{2}+(n+1)n'}c_n^{-1}\ \widetilde{\mathrm{ev}}_n\circ(q_n^{-1}\otimes p_{n'})$. By (11.14), we have that

$$\mathrm{coev}_n(1) = \sum_{s=0}^{n}v_n(x^s)\otimes u_n(1)\bar{\varepsilon}_{n-s}^{(n)}.$$

Then we scale by $(-1)^{\binom{n'-n+r}{2}+\binom{n+1}{2}+(n+1)n'}c_n^{-1}$ and apply $\lambda_{n;(1)}(x)^{n'-n+r}\otimes\mathrm{id}$ to obtain

$$c_n^{-1}\sum_{s=0}^{n}(-1)^{\binom{n'-n+r}{2}+\binom{n+1}{2}+(n+1)n'}v_n(x^{n'-n+r+s})\otimes u_n(1)\bar{\varepsilon}_{n-s}^{(n)}.$$

Next $q_n^{-1}\otimes p_{n'}$ gives

$$c_n^{-1}\sum_{s=0}^{n}(-1)^{\binom{n'-n+r}{2}+\binom{n+1}{2}+(n+1)n'+n'(n'+r+s)}\tilde{v}_n(x^{n'-n+r+s})\otimes\tilde{u}_n(1)\bar{\varepsilon}_{n-s}^{(n)}.$$

Finally we apply $\widetilde{\mathrm{ev}}_n$ using the formula in Theorem 11.5 to obtain

$$c_n^{-1}\sum_{s=n-r}^{n}(-1)^{\binom{n'-n+r}{2}+\binom{n+1}{2}+(n+1)n'+n'(n'+r+s)+n(n'-n+r+s)+\binom{n'-n+r+s}{2}}\Big[(\psi_n^\ell)^{-1}\Big(\bar{\eta}_{r+s-n}^{(n'+1)}\Big)\Big]\bar{\varepsilon}_{n-s}^{(n)}.$$

It remains to replace s by $n-s$ and simplify the sign to obtain (13.31). □

The formulae for the positively dotted bubbles in (13.28) and (13.29) are rather complicated. To simplify, one can apply the homomorphism α_n^ℓ from Lemma 8.3, to obtain the following.

Corollary 13.4. *For $\ell=n+n'$ and $k=n-n'$, the graded superalgebra homomorphism defined by the composition*

$$R\xrightarrow{\beta_k}\mathrm{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k)\xrightarrow{\Psi_\ell}\mathrm{End}_{OH_n\text{-}OH_n}(OH_n)\xrightarrow{\theta\mapsto\theta(1)}OH_n\xrightarrow{\alpha_n^\ell}R_{n'}$$

is equal to the canonical quotient map $R\twoheadrightarrow R_{n'},\dot{c}\mapsto\dot{c}^{(n')}$.

Proof. We note first that this composition is indeed a graded superalgebra homomorphism. Now we use (13.28) and (13.29) to show for all $r\geq 1$ that it takes $\dot{\varepsilon}_r\mapsto\dot{\varepsilon}_r^{(n')}$ in the case $k\geq 0$ and $\dot{\eta}_r\mapsto\dot{\eta}_r^{(n')}$ in the case $k\leq 0$. The arguments are similar in the two cases, so we just give the details for $k\geq 0$, i.e., $n\geq n'$. If $\ell=0$ the result is trivial, so we may assume $\ell>0$, hence, $n\geq 1$. Remembering the definition of $\beta_k(\dot{\varepsilon}_r)$ from (13.12), we apply (13.28) with n replaced by $n-1$ to get that

$$\Psi_\ell(\beta_k(\dot{\varepsilon}_r))(1)=(-1)^{(n-n'+1)r}\sum_{s=0}^{r}(-1)^{(n'+1)r+ns+\binom{s}{2}}\Big[(\psi_n^\ell)^{-1}(\bar{\varepsilon}_{r-s}^{(n')})\Big]\bar{\eta}_s^{(n)}.$$

From (8.6), it follows that $\alpha_n^\ell\Big((\psi_n^\ell)^{-1}(\bar{e}_r^{(n')})\Big)=(-1)^{nr}\dot{e}_r^{(n')}$, hence, $\alpha_n^\ell\Big((\psi_n^\ell)^{-1}(\bar{\varepsilon}_r^{(n')})\Big)=(-1)^{nr}\dot{\varepsilon}_r^{(n')}$. So $\alpha_n^\ell(\Psi_\ell(\beta_k(\varepsilon_r))(1))=(-1)^{(n-n'+1)r+(n'+1)r+nr}\varepsilon_r^{(n')}=\dot{\varepsilon}_r^{(n')}$. □

The results so far in this section have an application to prove the *non-degeneracy* of $\mathfrak{U}(\mathfrak{sl}_2)$, which was conjectured in [EL, BE2]. This asserts that the 2-morphism spaces in $\mathfrak{U}(\mathfrak{sl}_2)$ have the expected graded dimensions. The result may be formulated as follows. For any $k,\ell\in\mathbb{Z}$ and 1-morphisms $X,Y\in\mathrm{Hom}_{\mathfrak{U}(\mathfrak{sl}_2)}(k,\ell)$ (i.e., words consisting of m letters E and n letters F such that $\ell=k+2m-2n$) we view the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{sl}_2)}(X,Y)$ as a graded right R-supermodule so that $\dot{c}\in R$ acts by horizontally composing on the right with $\beta_k(\dot{c})$.

Theorem 13.5. *For $k,\ell\in\mathbb{Z}$ and $X,Y\in\mathrm{Hom}_{\mathfrak{U}(\mathfrak{sl}_2)}(k,\ell)$, the 2-morphism space $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{sl}_2)}(X,Y)$ is free as a graded right R-supermodule with basis given by a set of representatives for equivalence classes of decorated reduced (X,Y)-matchings in the sense defined in* [BE2, Sec.8]. *In particular, $\beta_k:R\to\mathrm{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k)$ is an isomorphism for all $k\in\mathbb{Z}$.*

Proof. The "easy" step in the proof is to show that $\mathrm{Hom}_{\mathfrak{U}(\mathfrak{sl}_2)}(X,Y)$ is spanned as a right R-supermodule by the 2-morphisms that are the representatives for equivalence classes of decorated reduced (X,Y)-matchings. This is proved by exhibiting an explicit straightening algorithm going by induction on the number of crossings. See [BE2, Th. 8.1], which simply cites [KL, Prop. 3.11] as the argument is the same as in the purely even setting, or [DEL] for a more systematic treatment. Note the straightening algorithm requires all of the relations described above, including the alternating braid relation.

The "hard" step is to establish the linear independence. By a standard reduction, which is again the same as in the ordinary even setting as in [KL, Rem. 3.16], it suffices to treat the case that $X = Y = E^d$ for some $d \geq 0$. In this case, the decorated reduced (X, Y)-matchings consist of d strings oriented from bottom to top decorated with some dots close to the top boundary. We index them by pairs (κ, w) for $\kappa \in \mathbb{N}^d$ and $w \in S_d$. For such a pair, the corresponding 2-morphism $f(\kappa, w)$ has κ_i dots at the top of the ith string, with the strings below arranged so that they represent some reduced expression for w. Consider some linear relation

$$f := \sum_{\kappa\in\mathbb{N}^d, w\in S_d} f(\kappa, w)\beta_k(\dot{c}_{\kappa,w}) = 0$$

for $\dot{c}_{\kappa,w} \in R$. Each $\dot{c}_{\kappa,w}$ is an $\mathbb{F}$-linear combination of basis vectors $\dot{h}_\lambda$ of R for λ in some finite set P_κ of partitions. Pick $0 \leq n \leq \ell$ with $k = 2n - \ell$ in such a way that n and $\ell - n$ are both very large relative to $|\kappa|$ and $|\lambda|$ for all $\lambda \in P_\kappa, \kappa \in \mathbb{N}^n$ with $\dot{c}_{\kappa,w} \neq 0$ for some $w \in S_n$. Then we apply the 2-superfunctor Ψ_ℓ to f to obtain the relation

$$\Psi_\ell(f) = \sum_{\kappa\in\mathbb{N}^d, w\in S_d} \Psi_\ell(f(\kappa, w))\Psi_\ell(\beta_k(\dot{c}_{\kappa,w})) = 0$$

in $\mathrm{End}_{OH^\ell_{n+d}\text{-}OH^\ell_n}(U_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n)$. Conjugating with the isomorphism $b_{(1)^d;n}$ from (10.18), we get from $\Psi_\ell(f)$ a superbimodule endomorphism $\tilde{f} = 0$ of $U^\ell_{(1^d);n}$. Using (13.22) and (13.23), it follows that

$$\tilde{f} = \sum_{\kappa\in\mathbb{N}^d, w\in S_d} \pm x^\kappa \tau_w \otimes \Psi_\ell(\beta_k(\dot{c}_{\kappa,w}))$$

for some signs, where this is being viewed as an endomorphism of the free right OH^ℓ_n-superbimodule $U^\ell_{(1^d);n}$ using the right action of ONH_d from Lemma 10.5(2). By the large choice of n and ℓ, the endomorphisms defined by each $x^\kappa\tau_w$ are linearly independent; cf. the proof of Theorem 5.2. We deduce that $\Psi_\ell(\beta_k(\dot{c}_{\kappa,w})) = 0$ for all κ and w.

It remains to show that $\Psi_\ell(\beta_k(\dot{c}_{\kappa,w})) = 0$ implies that $\dot{c}_{\kappa,w} = 0$ for sufficiently large n and ℓ. Assume that $\Psi_\ell(\beta_k(\dot{c}_{\kappa,w})) = 0$. Remembering that $\dot{c}_{\kappa,w}$ is an $\mathbb{F}$-linear combination of $\dot{h}_\lambda$ for λ with $|\lambda|$ small, this follows on evaluating at $1 \in OH^\ell_n$ then applying the homomorphism $\alpha^\ell_n : OH^\ell_n \to R_{\ell-n}$. The point here is that by Corollary 13.4 we have that

$$\alpha^\ell_n\Big(\Psi_\ell(\beta_k(\dot{h}_\lambda))(1)\Big) = \dot{h}^{(\ell-n)}_\lambda.$$

These elements of $R_{\ell-n}$ are linearly independent for small λ, so we can conclude that the coefficients of all $\dot{h}_\lambda$ in $\dot{c}_{\kappa,w}$ are zero. □

The following corollary is well known; see also [BE2, Th. 11.7] for the explicit definition of the isomorphism. We just note a different convention for (q, π)-integers is used in [BE2, Sec. 9] compared to (3.1). This accounts for the difference in the defining relation [BE2, (9.2)] for $U_{q,\pi}(\mathfrak{sl}_2)$ compared to the relation (3.11) being used for it here.

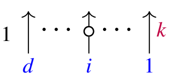
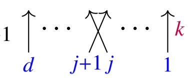
Corollary 13.6. *The split Grothendieck ring $K_0(\underline{\mathrm{gsKar}}(\mathfrak{U}(\mathfrak{sl}_2)))$ is isomorphic as a $\mathbb{Z}[q, q^{-1}]^\pi$-algebra to the integral form $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ of $U_{q,\pi}(\mathfrak{sl}_2)$ defined at the end of Section 3. Under the isomorphism, the isomorphism classes of the 1-morphisms $E1_k$ and $F1_k$ correspond to the elements of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ denoted by the same notation.*

Proof. See [BE2, Th. 12.1], which explains how to deduce this from the non-degeneracy given by Theorem 13.5. □

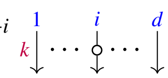
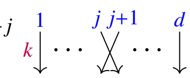
Remark 13.7. Theorem 13.5 is not new—it was already been established in [DEL] by a completely different technique. Also a version of Corollary 13.6 already appeared in [EL]. The proof of Theorem 13.5 given here is in the same spirit as the proof of non-degeneracy of the ordinary $\mathfrak{sl}_2$ 2-category given in [L1, Prop. 8.2] and the more general proof of non-degeneracy for $\mathfrak{sl}_n$ given in [KL].

For $d \geq 1$ and $k \in \mathbb{Z}$, there are graded superalgebra homomorphisms

$$\rho_d^{(k)} : ONH_d \to \mathrm{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(E^d 1_k)^{\mathrm{sop}}, \tag{13.32}$$

$x_i \mapsto (-1)^{i-1}$  , $\qquad \tau_j \mapsto -(-1)^{j-1}$ 

$$\lambda_d^{(k)} : ONH_d \to \mathrm{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(1_k F^d), \tag{13.33}$$

$x_i \mapsto (-1)^{d-i}$  , $\qquad \tau_j \mapsto -(-1)^{d-j}$ 

This follows from the relations (13.2) and (13.10), with the signs in (13.32) and (13.33) accounting for the difference between these and our preferred relations for ONH_n from (5.1) to (5.6). Another consequence of Theorem 13.5 is that both $\rho_d^{(k)}$ and $\lambda_d^{(k)}$ are *injective*.

Remark 13.8. On comparing with (13.22) and (13.23), it follows that the composition of $\rho_d^{(2n-\ell)}$ with the homomorphism $\mathrm{End}_{\mathfrak{U}(\mathfrak{sl}_2)}(E^d 1_k)^{\mathrm{sop}} \to \mathrm{End}_{OH^\ell_{n+d}\text{-}OH^\ell_n}(Q^{-nd-\binom{d}{2}} U^\ell_{n+d-1} \otimes_{OH^\ell_{n+d-1}} \cdots \otimes_{OH^\ell_{n+1}} U^\ell_n)^{\mathrm{sop}}$ induced by the 2-superfunctor Ψ_ℓ is equal to the anti-homomorphism $\rho_{(1^d);n}$ from (11.18) (up to a degree shift). One can check similarly starting from Lemma 11.8 that the composition of $\lambda_d^{(2n-\ell)}$ with the homomorphism induced by Ψ_ℓ is equal to the homomorphism $\lambda_{n;(1^d)}$ from (11.19) (up to degree shift).

To conclude the section, we explain how to define divided powers. In $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$, there are 1-morphisms

$$E^{(d)} 1_k := Q^{\binom{d}{2}}\Big(E^d 1_k, \rho_d^{(k)}((\xi\omega)_d)\Big) : k \to k + 2d, \tag{13.34}$$

$$1_k F^{(d)} := Q^{\binom{d}{2}}\Big(1_k F^d, \lambda_d^{(k)}((\omega\xi)_d)\Big) : k + 2d \to k. \tag{13.35}$$

By Lemma 5.8 plus (5.29), we have that

$$E^d 1_k \simeq \bigoplus_{w \in S_d} Q^{2\ell(w)-\binom{d}{2}} \Pi^{\ell(w)} E^{(d)} 1_k, \qquad 1_k F^d \simeq \bigoplus_{w \in S_d} Q^{2\ell(w)-\binom{d}{2}} \Pi^{\ell(w)} 1_k F^{(d)}. \tag{13.36}$$

In view of (3.3), it follows that $E^{(d)} 1_k$ and $F^{(d)} 1_k$ categorify the divided powers (3.12), i.e., the isomorphism classes of the former 1-morphisms under the isomorphism from Corollary 13.6 give the latter elements of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$. Note $E^{(d)} 1_k$ and $1_k F^{(d)}$ are obtained by upshifting the bottom degree summands of $E^d 1_k$ and $1_k F^d$. It is also useful to have available the following, which are downshifts of the top degree summands:

$$\overline{E}^{(d)} 1_k := Q^{-\binom{d}{2}}\Big(E^d 1_k, \rho_d^{(k)}((\omega\xi)_d)\Big) : k \to k + 2d, \tag{13.37}$$

$$1_k \overline{F}^{(d)} := Q^{-\binom{d}{2}}\Big(1_k F^d, \lambda_d^{(k)}((\xi\omega)_d)\Big) : k + 2d \to k. \tag{13.38}$$

These categorify the elements $\overline{E}^{(d)} 1_k$ and $1_k \overline{F}^{(d)}$ of $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$ from (3.13) since, by Lemma 5.8 plus (5.29) again, we have that

$$E^d 1_k \simeq \bigoplus_{w \in S_d} Q^{\binom{d}{2}-2\ell(w)} \Pi^{\ell(w)} \overline{E}^{(d)} 1_k, \qquad 1_k F^d \simeq \bigoplus_{w \in S_d} Q^{\binom{d}{2}-2\ell(w)} \Pi^{\ell(w)} 1_k \overline{F}^{(d)}. \tag{13.39}$$

Mirroring (3.14), we have that

$$\overline{E}^{(d)}1_k \simeq \Pi^{\binom{d}{2}}E^{(d)}1_k, \qquad 1_k\overline{F}^{(d)} \simeq \Pi^{\binom{d}{2}}1_kF^{(d)}. \tag{13.40}$$

This follows because the idempotents $(\omega\xi)_d$ and $(\xi\omega)_d$ are conjugate as discussed after (5.30)[10].

Lemma 13.9. *In* gsKar($\mathfrak{U}(\mathfrak{sl}_2)$)*, the 1-morphism* $Q^{-d(k+d)}\Pi^{\binom{d}{2}}1_kF^{(d)}$ *is right dual to* $E^{(d)}1_k$*, and the 1-morphism* $Q^{d(k+d)}\Pi^{d(k+d)+\binom{d}{2}}E^{(d)}1_{k-2d}$ *is right dual to* $F^{(d)}1_k$*.*

Proof. We first show that $Q^{-d(k+d)}1_kF^{(d)}$ is right dual to $E^{(d)}1_k$ in the (Q,Π)-envelope gsKar($\mathfrak{U}(\mathfrak{sl}_2)$). By (13.3), $Q^{-k-1}1_kF$ is right dual to $E1_k$. Hence, $Q^{-d(k+d)}Q^{-\binom{d}{2}}1_kF^d$ is right dual to $Q^{\binom{d}{2}}E^d1_k$. By definition, $E^{(d)}1_k$ is the summand of $Q^{\binom{d}{2}}E^d1_k$ defined by the idempotent $Q^{\binom{d}{2}}\rho_d^{(k)}((\xi\omega)_d)$ and $Q^{-d(k+d)}1_k\overline{F}^{(d)}$ is the summand of $Q^{-d(k+d)}Q^{-\binom{d}{2}}1_kF^d$ defined by the the idempotent $Q^{-d(k+d)}Q^{-\binom{d}{2}}\lambda_d^{(k)}((\xi\omega)_d)$. Now we observe using Lemma 2.1(2), (13.8) and (5.29) that $Q^{-d(k+d)}Q^{-\binom{d}{2}}\lambda_d^{(k)}((\xi\omega)_d)$ is the right mate of $Q^{\binom{d}{2}}\rho_d^{(k)}((\xi\omega)_d)$. Hence, we get that $Q^{-d(k+d)}1_k\overline{F}^{(d)}$ is right dual to $E^{(d)}1_k$. It remains to apply (13.40) to pass from $Q^{-d(k+d)}1_k\overline{F}^{(d)}$ to $Q^{-d(k+d)}\Pi^{\binom{d}{2}}1_kF^{(d)}$.

The proof that $Q^{d(k+d)}\Pi^{d(k+d)+\binom{d}{2}}E^{(d)}1_k$ is right dual to $1_kF^{(d)}$ is similar using (13.11) instead of (13.3). By (13.11) and Lemma 2.1(1), $Q^{k+1}\Pi^{k+1}E1_k$ is right dual to 1_kF. Hence, $Q^{d(k+dx)-\binom{d}{2}}\Pi^{d(k+d)}E^d1_k$ is right dual to $Q^{\binom{d}{2}}1_kF^d$. Since (13.18) is more complicated than (13.8), it is no longer true that the idempotent $Q^{d(k+d)-\binom{d}{2}}\Pi^{d(k+d)}\rho_d^{(k)}((\xi\omega)_d)$ is *equal* to the right mate of the idempotent $Q^{\binom{d}{2}}\lambda_d^{(k)}((\xi\omega)_d)$, but these two idempotents are conjugate via even degree 0 units. This follows by the Krull-Schmidt theorem applied to the finite-dimensional algebra that is the even degree 0 component of $ONH_d \otimes R$. Hence, $Q^{d(k+d)}\Pi^{d(k+d)}\overline{E}^{(d)}1_k$ is right dual to $1_kF^{(d)}$. It remains to appeal to (13.40) one more time. □

14. Some graded 2-representation theory

In this section, we develop some 2-representation theory of the $\mathfrak{sl}_2$ 2-category $\mathfrak{U}(\mathfrak{sl}_2)$ from Definition 13.1. We work throughout in the graded setting, but all the definitions and results here have analogs with the $\mathbb{Z}$-grading forgotten. The following is modelled on [R1, Def. 5.1.1].

Definition 14.1. By a *graded 2-representation* $\mathcal{V}$ of $\mathfrak{U}(\mathfrak{sl}_2)$, we mean a strict graded 2-superfunctor $\mathcal{V} : \mathfrak{U}(\mathfrak{sl}_2) \to \mathcal{gsCat}$. Decoding the definition, $\mathcal{V}$ consists of the following data:

- a graded supercategory $\mathcal{V}$ with a given decomposition into *weight subcategories* $\mathcal{V} = \coprod_{k\in\mathbb{Z}} \mathcal{V}_k$ (or $\mathcal{V} = \bigoplus_{k\in\mathbb{Z}} \mathcal{V}_k$ when $\mathcal{V}$ is additive);
- graded superfunctors $E : \mathcal{V} \to \mathcal{V}$ and $F : \mathcal{V} \to \mathcal{V}$ such that $E|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_{k+2}$ and $F|_{\mathcal{V}_k} : \mathcal{V}_k \to \mathcal{V}_{k-2}$ for each $k \in \mathbb{Z}$;
- graded supernatural transformations $x : E \Rightarrow E$ and $\tau : E^2 \Rightarrow E^2$ which are odd of degrees 2 and -2, respectively;
- (inhomogeneous) graded supernatural transformations $\eta : \mathrm{Id} \Rightarrow FE$ and $\varepsilon : EF \Rightarrow \mathrm{Id}$ whose restrictions $\eta : \mathrm{Id}_{\mathcal{V}_k} \Rightarrow FE|_{\mathcal{V}_k}$ and $\varepsilon : EF|_{\mathcal{V}_{k+2}} \Rightarrow \mathrm{Id}_{\mathcal{V}_{k+2}}$ are even of degrees $k+1$ and $-k-1$, respectively.

Then there are the axioms:

- the relations from (13.2) hold: $\tau \circ \tau = 0$, $(\tau E) \circ (E\tau) \circ (\tau E) = (E\tau) \circ (\tau E) \circ (E\tau)$ and $(Ex) \circ \tau + (xE) \circ \tau = (xE) \circ \tau + \tau \circ (Ex) = E^2$;

[10] It could also be deduced from (13.36) and (13.39) using Krull-Schmidt, but we prefer the argument given since it constructs the isomorphism explicitly.

- η and ε satisfy the zig-zag relations: $(F\varepsilon) \circ (\eta F) = F$ and $(\varepsilon E) \circ (E\eta) = E$ (equivalently, they define units and counits of adjunctions making $Q^{-k-1}F|_{\mathcal{V}_{k+2}}$ into a right adjoint to $E|_{\mathcal{V}_k}$ for each $k \in \mathbb{Z}$);
- letting $\sigma := (FE\varepsilon) \circ (F\tau F) \circ (\eta EF) : EF \Rightarrow FE$ be the image of the rightward crossing under $\mathcal{V}$, the following inhomogeneous matrices of supernatural transformations are isomorphisms:

$$\begin{pmatrix} \sigma & \varepsilon & \varepsilon \circ (xF) & \cdots & \varepsilon \circ (xF)^{k-1} \end{pmatrix}^T : EF|_{\mathcal{V}_k} \Rightarrow FE|_{\mathcal{V}_k} \oplus \mathrm{Id}_{\mathcal{V}_k}^{\oplus k} \qquad \text{for } k \geq 0$$
$$\begin{pmatrix} \sigma & \eta & (Fx) \circ \eta & \cdots & (Fx)^{-k-1} \circ \eta \end{pmatrix} : EF|_{\mathcal{V}_k} \oplus \mathrm{Id}_{\mathcal{V}_k}^{\oplus(-k)} \Rightarrow FE|_{\mathcal{V}_k} \qquad \text{for } k \leq 0$$

There are natural notations of (full) *sub-2-representations* (which are called "invariant ideals" in [BD, S4.2]), *quotient 2-representations*, and *morphisms* of graded 2-representations. The latter definition, which is the super analog of [R1, Def. 2.3], is equivalent to the following, which is similar to the formulation adopted in [CR, Sec. 5.2.1]; the terminology being used is the same as in [BD, Def. 4.6] (and actually goes back to Ben Webster).

Definition 14.2. Let $\mathcal{V}$ and $\mathcal{W}$ be two graded 2-representations of $\mathfrak{U}(\mathfrak{sl}_2)$. A *strongly equivariant graded superfunctor* $\Omega : \mathcal{V} \rightarrow \mathcal{W}$ is a graded superfunctor such that $\Omega|_{\mathcal{V}_k} : \mathcal{V}_k \rightarrow \mathcal{W}_k$ for each $k \in \mathbb{Z}$, plus a degree 0 even graded supernatural isomorphism $\zeta : E\Omega \stackrel{\sim}{\Rightarrow} \Omega E$, such that the following holds

- the supernatural transformation $(F\Omega\varepsilon) \circ (F\zeta F) \circ (\eta\Omega F) : \Omega F \Rightarrow F\Omega$ is invertible;
- we have that $(\Omega x) \circ \zeta = \zeta \circ (x\Omega)$;
- we have that $(\Omega\tau) \circ (\zeta E) \circ (E\zeta) = (\zeta E) \circ (E\zeta) \circ (\tau\Omega)$.

A *strongly equivariant graded superequivalence* is a strongly equivariant graded superfunctor which is also a superequivalence of supercategories.

Remark 14.3. For strongly equivariant graded superequivalences, the first axiom in Definition 14.2 actually holds automatically; see [BD, Rem. 4.8] where this is explained (in the purely even setting). Also in [BD], the diagrammatic interpretation of these definitions is discussed, which we still find helpful.

Remark 14.4. There is an obvious way to make the composition of two strongly equivariant graded superfunctors into a strongly equivariant graded superfunctor in its own right. Also the identity functor Id is strongly equivariant with $\zeta := 1_E$. So there is a category $\mathcal{Rep}(\mathfrak{U}(\mathfrak{sl}_2))$ consisting of graded 2-representations and strongly equivariant graded superfunctors.

Usually, the graded supercategories $\mathcal{V}_k$ in a graded 2-representation $\mathcal{V}$ will have some extra structure, such as being additive or (Q, Π)-complete. We are mainly interested here in what we call *graded Karoubian 2-representations*. By definition, this means a graded 2-representation $\mathcal{V}$ such that, for each $k \in \mathbb{Z}$, the weight subcategory $\mathcal{V}_k$ is additive and (Q, Π)-complete, and the underlying ordinary category $\underline{\mathcal{V}}_k$ is idempotent complete. Any graded 2-representation $\mathcal{V}$ can be upgraded to a Karoubian graded 2-representation by passing to its graded super Karoubi envelope $\mathrm{gsKar}(\mathcal{V})$.

Given a graded Karoubian 2-representation $\mathcal{V}$, the underlying graded 2-superfunctor from $\mathfrak{U}(\mathfrak{sl}_2)$ to $\mathcal{V}$ extends canonically to a graded 2-superfunctor from the graded super Karoubi envelope $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$ to $\mathcal{V}$. The direct sum over all $k \in \mathbb{Z}$ of the images under this graded 2-superfunctor of the 1-morphisms $E^{(d)}1_k$ and $F^{(d)}1_k$ from (13.34) and (13.35) give graded superfunctors

$$E^{(d)}, F^{(d)} : \mathcal{V} \rightarrow \mathcal{V}. \tag{14.1}$$

By (13.36), we have that

$$E^d \simeq \bigoplus_{w \in S_d} \Pi^{\ell(w)} Q^{2\ell(w)-\binom{d}{2}} E^{(d)}, \qquad F^d \simeq \bigoplus_{w \in S_d} \Pi^{\ell(w)} Q^{2\ell(w)-\binom{d}{2}} F^{(d)}. \tag{14.2}$$

Lemma 13.9 implies that $Q^{-d(k+d)}\Pi^{\binom{d}{2}}F^{(d)}|_{\mathcal{V}_{k+2d}}$ is right adjoint to $E^{(d)}|_{\mathcal{V}_k}$ and $Q^{d(k+d)}\Pi^{d(k+d)+\binom{d}{2}}E^{(d)}|_{\mathcal{V}_{k-2d}}$ is right adjoint to $F^{(d)}|_{\mathcal{V}_k}$, with units and counits of adjunction that are defined by images of 2-morphisms in $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$.

A graded 2-representation $\mathcal{V}$ is said to be *integrable* if E and F are locally nilpotent, i.e., for any $k \in \mathbb{Z}$ and any $M \in \mathcal{V}_k$ there is some $n \geq 0$ such that $E^n M = F^n M = 0$. Also, for $\ell \in \mathbb{N}$, a *lowest weight object of weight* $-\ell$ means an object $M \in \mathcal{V}_{-\ell}$ such that $FM = 0$.

Example 14.5. Suppose that $\ell \in \mathbb{N}$. By Theorem 13.2, there is an integrable graded Karoubian 2-representation

$$OH^\ell\text{-pgsmod} := \bigoplus_{n=0}^{\ell} OH^\ell_n\text{-pgsmod} \tag{14.3}$$

with the weight k subcategory $(OH^\ell\text{-pgsmod})_k$ equal to OH^ℓ_n-pgsmod if $k = 2n - \ell$ for $0 \leq n \leq \ell$, or the trivial (zero) graded supercategory otherwise. Other data is as follows.

- The graded superfunctors E and F are $Q^{-n}U^\ell_n \otimes_{OH^\ell_n} -$ on the weight subcategory OH^ℓ_n-pgsmod and $Q^{3n+1-\ell}V^\ell_n \otimes_{OH^\ell_{n+1}} -$ on the weight subcategory OH^ℓ_{n+1}-pgsmod, respectively, assuming $0 \leq n < \ell$. On all other weight subcategories, E and F are zero.
- The graded supernatural transformations x and τ are defined by the supernatural transformations $\rho_{(1);n}(x_1)\otimes\mathrm{id}$ viewed as elements $\mathrm{gsEnd}\left(Q^{-n}U^\ell_n\otimes_{OH^\ell_n}-\right)_{2,\bar{1}}$ and the supernatural transformations $-\rho_{(1^2);n}(\tau_1)\otimes\mathrm{id}$ viewed as elements of $\mathrm{gsEnd}\left(Q^{-2n-1}U^\ell_{n+1}\otimes_{OH^\ell_{n+1}} Q^{-n}U^\ell_n\otimes_{OH^\ell_n}-\right)_{-2,\bar{1}}$, respectively, for all admissible n.
- The graded supernatural transformations η and ε are given by the appropriate counit and unit from Theorem 11.3.
- The homomorphisms induced by (13.32) and (13.33) are equal to (11.18) and (11.19) thanks to Remark 13.8.
- For $0 \leq n \leq n + d \leq \ell$, we have that $E^{(d)}|_{OH^\ell_n\text{-pgsmod}} \simeq Q^{-dn}U^\ell_{(d);n}\otimes_{OH^\ell_n} -$ and $F^{(d)}|_{OH^\ell_{n+d}\text{-pgsmod}} \simeq Q^{-d(\ell-3n-2d+1)}V^\ell_{n;(d)}\otimes_{OH^\ell_{n+d}} -$; cf. Theorem 12.1.

We point out also by Theorem 12.1 that $K_0(OH^\ell\text{-pg}\underline{\text{smod}})$ is naturally identified with the $\mathbf{U}_{q,\pi}(\mathfrak{sl}_2)$-module $\mathbf{V}(-\ell)$, and OH^ℓ_0 is a lowest weight object of weight $-\ell$.

Now we come to one of the key constructions introduced by Rouquier in [R1] in the purely even case, the construction of *cyclotomic quotients*. For any $\ell \in \mathbb{Z}$, there is a graded 2-representation $\mathcal{R}(\ell)$ with

$$\mathcal{R}(\ell)_k := \mathcal{H}om_{\mathfrak{U}(\mathfrak{sl}_2)}(\ell, k) \tag{14.4}$$

for $k \in \mathbb{Z}$, viewed as a graded 2-representation of the graded 2-supercategory $\mathfrak{U}(\mathfrak{sl}_2)$ in an obvious way. For example, the graded superfunctor $E|_{\mathcal{R}(\ell)_k} : \mathcal{R}(\ell)_k \to \mathcal{R}(\ell)_{k+2}$ is defined by horizontally composing on the left with the 1-morphism $E1_k$, and the supernatural transformation $x : E|_{\mathcal{R}(\ell)_k} \Rightarrow E|_{\mathcal{R}(\ell)_k}$ is induced by the 2-endomorphism $\uparrow\!\!\circ\; k$: $E1_k \Rightarrow E1_k$. The graded 2-representation $\mathcal{R}(\ell)$ has the following universal property.

Lemma 14.6. *Given any graded 2-representation $\mathcal{V}$ and any $M \in \mathcal{V}_\ell$ there is a canonical strongly equivariant graded superfunctor $\omega_M : \mathcal{R}(\ell) \to \mathcal{V}$ taking the object 1_ℓ of $\mathcal{R}(\ell)_\ell$ to M.*

We define the *universal graded 2-representation of lowest weight* $-\ell \in \mathbb{Z}$, denoted $\mathcal{V}(-\ell)$, to be the quotient 2-representation $\mathcal{R}(-\ell)/I$, where I here is the sub-2-representation of $\mathcal{R}(-\ell)$ generated by $\downarrow\; -\ell$ (the identity endomorphism of the object $F1_{-\ell}$). We denote the lowest weight object of $\mathcal{V}(-\ell)_{-\ell}$ arising from the object $1_{-\ell} \in \mathcal{R}(-\ell)_{-\ell}$ by $\overline{1}_{-\ell}$, and call this the *canonical lowest weight object*. It is a generating object for $\mathcal{V}(-\ell)$. The identity endomorphism of $\overline{1}_{-\ell}$ is equal to the image of the bottom bubble ${}_{-\ell-1}\circlearrowleft_{-\ell}$, i.e., the image of $1 \in R$ under the homomorphism (13.12). If $\ell < 0$, this bubble is not a fake bubble, so it belongs to I. This shows that the graded supercategory $\mathcal{V}(-\ell)$ is trivial if $\ell < 0$. Thus, $\mathcal{V}(-\ell)$ is only interesting if $\ell \in \mathbb{N}$, i.e., it is a dominant weight for $\mathfrak{sl}_2$. The following, the universal property of $\mathcal{V}(-\ell)$, follows immediately from Lemma 14.6 and the universal property of quotients.

Lemma 14.7. *Let $\mathcal{V}$ be any graded 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$, $\ell \in \mathbb{N}$ and $M \in \mathcal{V}_{-\ell}$ be a lowest weight object. The superfunctor $\omega_M : \mathcal{R}(-\ell) \to \mathcal{V}$ from Lemma 14.6 induces a strongly equivariant graded superfunctor $\Omega_M : \mathcal{V}(-\ell) \to \mathcal{V}$ taking $\bar{1}_{-\ell}$ to M.*

There is a more sophisticated version of Lemma 14.7, which is analogous to [R1, Prop. 5.6]. To formulate this, we need one more preliminary lemma.

Lemma 14.8. *The homomorphism*[11] $\beta_{-\ell} : R \to \mathrm{End}_{\mathcal{R}(-\ell)}(1_{-\ell})$ *from* (13.12) *induces an isomorphism* $\bar{\beta}_{-\ell} : R_\ell \xrightarrow{\sim} \mathrm{End}_{\mathcal{V}(-\ell)}(\bar{1}_{-\ell})$.

Proof. The bubble ${}_{r-\ell-1}\circlearrowleft_{-\ell}$ belongs to I for $r > \ell$. Up to a sign, the composition of $\beta_{-\ell}$ with the canonical map $\mathrm{End}_{\mathcal{R}(-\ell)}(1_{-\ell}) \twoheadrightarrow \mathrm{End}_{\mathcal{V}(-\ell)}(\bar{1}_{-\ell})$ takes $\dot{\varepsilon}_r \in R$ to the the image of this bubble, which is zero. We deduce that this homomorphism factors through the quotient R_ℓ of R to induce $\bar{\beta}_{-\ell}$. Moreover, $\bar{\beta}_{-\ell}$ is surjective since $\beta_{-\ell}$ is surjective by the "easy" part of Theorem 13.5.

To show that $\bar{\beta}_{-\ell}$ is also injective, we use the following diagram of graded supercategories and superfunctors:

$$\begin{array}{ccc}
\mathcal{E}nd_{\mathfrak{U}(\mathfrak{sl}_2)}(-\ell) & \xrightarrow{\Psi_\ell} & OH_0^\ell\text{-gsMod-}OH_0^\ell \\
\Big\downarrow{\scriptstyle \mathrm{Ev}_{\bar{1}_{-\ell}}} & & \Big\downarrow{\scriptstyle -\otimes_{OH_0^\ell} OH_0^\ell} \\
\mathcal{V}(-\ell)_{-\ell} & \xrightarrow[\Omega_{OH_0^\ell}]{} & OH_0^\ell\text{-gsMod}
\end{array}$$

Here, the top map comes from Theorem 13.2, the left hand vertical superfunctor is given by evaluating on the object $\bar{1}_{-\ell}$, and the right hand vertical superfunctor is given by tensoring with the lowest weight object OH_0^ℓ. The way the bottom superfunctor $\Omega_{OH_0^\ell}$ is defined in Lemma 14.7 ensures that this diagram commutes strictly. It follows that the middle square in the following diagram commutes:

$$\begin{array}{ccccccccc}
R & \xrightarrow{\beta_{-\ell}} & \mathrm{End}_{\mathcal{R}(-\ell)_{-\ell}}(1_{-\ell}) & \xrightarrow{\Psi_\ell} & \mathrm{End}_{OH_0^\ell\text{-}OH_0^\ell}(OH_0^\ell) & & & & \\
\Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle \mathrm{can}} & & \Big\downarrow{\scriptstyle \theta\mapsto\theta\otimes\mathrm{id}} & & & & \\
R_\ell & \xrightarrow[\bar{\beta}_{-\ell}]{} & \mathrm{End}_{\mathcal{V}(-\ell)_{-\ell}}(\bar{1}_{-\ell}) & \xrightarrow[\Omega_{OH_0^\ell}]{} & \mathrm{End}_{OH_0^\ell\text{-}}(OH_0^\ell) & \xrightarrow[\phi\mapsto\phi(1)]{} & OH_0^\ell & \xrightarrow[\alpha_0^\ell]{\sim} & R_\ell
\end{array}$$

Corollary 13.4 shows that the composition $R \to R_\ell$ around the northeast boundary of this diagram is equal to the canonical quotient map. Hence, the composition $R_\ell \to R_\ell$ of the three maps at the bottom of the diagram is the identity. This implies that $\bar{\beta}_{-\ell}$ is injective. □

Any morphism space $\mathrm{Hom}_{\mathcal{R}(-\ell)}(X, Y)$ in $\mathcal{R}(-\ell)$ can be viewed as a right R-supermodule so that $\mathring{c} \in R$ acts by horizontally composing on the right with $\beta_{-\ell}(\mathring{c})$. This induces a structure of right R_ℓ-supermodule on any morphism space $\mathrm{Hom}_{\mathcal{V}(-\ell)}(X, Y)$; cf. the first paragraph of the proof of Lemma 14.8. Given a graded R_ℓ-superalgebra A, we let $\mathcal{V}(-\ell) \otimes_{R_\ell} A$ be the graded supercategory with the same objects as $\mathcal{V}(-\ell)$ and morphism spaces $\mathrm{Hom}_{\mathcal{V}(-\ell)\otimes_{R_\ell} A}(X, Y) := \mathrm{Hom}_{\mathcal{V}(-\ell)}(X, Y) \otimes_{R_\ell} A$. This is naturally a graded 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$ in its own right.

[11] In fact, $\beta_{-\ell}$ is itself an isomorphism thanks to Theorem 13.5, but this is not relevant for the present lemma.

Theorem 14.9. *Let $\mathcal{V}$ be any graded 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$, $\ell \in \mathbb{N}$ and $M \in \mathcal{V}_{-\ell}$ be any lowest weight object. The strongly equivariant graded superfunctor $\Omega_M : \mathcal{V}(-\ell) \to \mathcal{V}$ from Lemma 14.7 extends to a* fully faithful *strongly equivariant graded superfunctor $\Omega_M \otimes \mathrm{id} : \mathcal{V}(-\ell) \otimes_{R_\ell} \mathrm{End}_{\mathcal{V}}(M) \to \mathcal{V}$.*

Proof. Let $A := \mathrm{End}_{\mathcal{V}}(M)$ for short. The graded superfunctor Ω_M extends to $\Omega_M \otimes \mathrm{id}$ by the universal property of tensor product. To see that the resulting graded superfunctor is fully faithful, we must show that it defines an isomorphism $\mathrm{Hom}_{\mathcal{V}(-\ell)\otimes_{R_\ell} A}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{V}}(X, Y)$ for objects of any weight subcategory of $\mathcal{V}(-\ell) \otimes_{R_\ell} A$. This is clear if $X = Y = \bar{1}_{-\ell}$. The result in general then follows by the (now standard) technique explained in the proof of [R1, Lem. 5.4, Prop. 5.6]. □

Corollary 14.10. *For $\ell \in \mathbb{N}$, let OH^ℓ-*pgsmod *be the graded Karoubian 2-representation from Example 14.5, and let* gsKar$(\mathcal{V}(-\ell))$ *be the graded super Karoubi envelope of $\mathcal{V}(-\ell)$, which is another graded Karoubian 2-representation. The strongly equivariant graded superfunctor $\Omega_{OH_0^\ell} : \mathcal{V}(-\ell) \to OH^\ell$-*pgsmod *associated to the lowest weight object OH_0^ℓ induces a strongly equivariant graded superequivalence $\Xi_\ell :$* gsKar$(\mathcal{V}(-\ell)) \to OH^\ell$-pgsmod*.*

Proof. In view of Lemma 14.8 and Theorem 14.9, $\Omega_{OH_0^\ell}$ is fully faithful. This extends by the universal property of the graded super Karoubi envelope to give a fully faithful strongly equivariant graded superfunctor Ξ_ℓ : gsKar$(\mathcal{V}(-\ell)) \to OH^\ell$-pgsmod. To see that Ξ_ℓ is a graded superequivalence, it remains to check that it is dense. This follows because

$$E^{(n)} OH_0^\ell \simeq U^\ell_{(n);0} \otimes_{OH_0^\ell} OH_0^\ell \simeq OH_n^\ell,$$

the last isomorphism following since $U^\ell_{(n);0}$ is free of rank 1 as a graded left OH_n^ℓ-supermodule by Lemma 10.4(2). □

We record one more basic lemma, which is analogous to the first part of [R1, Lem. 5.2].

Lemma 14.11. *Let $\mathcal{V}$ be an integrable Karoubian graded 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$. Let N be an object of $\mathcal{V}_k$ for some $k \in \mathbb{Z}$. If $\mathrm{Hom}_{\mathcal{V}_k}(E^n M, N) = 0$ for all $\ell \in \mathbb{N}$, $n \geq 0$ such that $k = 2n - \ell$ and all lowest weight objects $M \in \mathcal{V}_{-\ell}$, then $N = 0$.*

Proof. Suppose that $N \neq 0$. By integrability, there exists $n \geq 0$ such that $F^n N \neq 0$ and $F^{n+1} N = 0$. This means that $M := F^n N$ is a non-zero lowest weight object of $\mathcal{V}_{-\ell}$ for $\ell = k - 2n \in \mathbb{N}$. By assumption, we have that $\mathrm{Hom}_{\mathcal{V}_k}(E^n M, N) = 0$. Hence, by adjunction,

$$\mathrm{End}_{\mathcal{V}_{-\ell}}(M) = \mathrm{Hom}_{\mathcal{V}_{-\ell}}(M, F^n N) \simeq \mathrm{Hom}_{\mathcal{V}_k}(E^n M, N) = 0.$$

It follows that $1_M = 0$, so $M = 0$, which is a contradiction. □

Remark 14.12. There is more still to be done here. For example, Rouquier continues in [R1, Sec. 5.1.4] to construct a Jordan-Hölder series in an arbitrary integrable Karoubian 2-representation, and this result assuredly carries over to our setting. There is also a good theory of *locally finite Abelian* 2-representations of $\mathfrak{U}(\mathfrak{sl}_2)$, including an analog of [CR, Prop. 5.20] which implies that the irreducible objects of such a 2-representation can be given the structure of a crystal in the sense of Kashiwara. It would be worthwhile to extend [CR, Th. 5.27] (which is a special case of Rouquier's "control by K_0" from [R1, Th. 5.22]) to this setting. This would pave the way to more applications involving representations of the supergroup $Q(n)$ and the Lie superalgebra $\mathfrak{q}_n(\mathbb{C})$. In the ordinary case, an alternative approach by-passing control by K_0 was developed in [BSW], which we expect should also have an interesting and non-trivial super analog. Another direction we would like to investigate further is to extend Theorems 13.2 and 13.5 from odd $\mathfrak{sl}_2$ to the super Kac-Moody 2-category associated to "odd $\mathfrak{so}_{2n+1}$", thereby giving an odd analog of the 2-representation of $\mathfrak{sl}_n$ constructed in [KL].

15. The odd analog of the Rickard complex

Let $\mathcal{V}$ be a graded 2-supercategory. The notation $\mathrm{Ch}^b(\mathcal{V})$ denotes the graded supercategory of bounded cochain complexes and chain maps in $\mathcal{V}$; differentials in a cochain complex are assumed to be even of degree 0 but we allow chain maps whose components are inhomogeneous. Also $K^b(\mathcal{V})$ is the homotopy category, which is a graded supercategory with the same objects as $\mathrm{Ch}^b(\mathcal{V})$ and morphisms that are chain homotopy equivalence classes of chain maps; chain homotopies are again required to be even of degree 0. If $\mathcal{V}$ is an integrable graded Karoubian 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$ as in the previous section, both $\mathrm{Ch}^b(\mathcal{V})$ and $K^b(\mathcal{V})$ are themselves integrable graded Karoubian 2-representations of $\mathfrak{U}(\mathfrak{sl}_2)$ in a natural way.

Fix $k \in \mathbb{Z}$. The *odd Rickard complex* Θ_k, so-called because it is the odd analog of the complex in [CR, Sec. 6.2] which was introduced originally by Rickard in the context of symmetric groups, is the following cochain complex in $\mathrm{Ch}\,(\mathcal{H}om_{\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))}(-k,k))$:

$$\begin{cases} \cdots \to Q^d E^{(k+d)} F^{(d)} 1_{-k} \xrightarrow{\partial^{-d}} Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k} \to \cdots \to E^{(k)} 1_{-k} \to 0 \to \cdots & \text{if } k \geq 0 \\ \cdots \to Q^d E^{(k+d)} F^{(d)} 1_{-k} \xrightarrow{\partial^{-d}} Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k} \to \cdots \to Q^{-k} F^{(-k)} 1_{-k} \to 0 \to \cdots & \text{if } k \leq 0, \end{cases}$$

where in both cases $E^{(k+d)} F^{(d)} 1_{-k}$ is in cohomological degree $-d$. The differential

$$\partial^{-d} : Q^d E^{(k+d)} F^{(d)} 1_{-k} \to Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k}$$

is the composition first of the "inclusion" of $Q^d E^{(k+d)} F^{(d)} 1_{-k} \to Q^{k+3d-2} E^{(k+d-1)} EFF^{(d-1)} 1_{-k}$ as a summand[12] of $E^{(k+d-1)} EFF^{(d-1)} 1_{-k}$, then $Q^{k+3d-2} E^{(k+d-1)} \varepsilon F^{(d-1)} : Q^{k+3d-2} E^{(k+d-1)} EFF^{(d-1)} 1_{-k} \to Q^{d-1} E^{(k+d-1)} F^{(d-1)} 1_{-k}$. Note this is even of degree 0 as required. The following checks that it is a cochain complex.

Lemma 15.1. *We have that $\partial^{-d+1} \circ \partial^{-d} = 0$ for all d.*

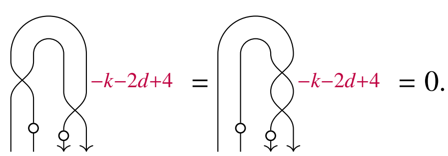
Proof. Ignoring gradings for brevity, it suffices to show that the composition

$$E^{(2)} F^{(2)} 1_{-k-2d+4} \xrightarrow{\text{inc}} E^2 F^2 1_{-k-2d+4} \xrightarrow{E\varepsilon F} E 1_{-k-2d+4} F \xrightarrow{\varepsilon} 1_{-k-2d+4}$$

is zero. The identity endomorphism of $E^{(2)} F^{(2)} 1_{-k-2d+4}$ is

$$\rho_2^{(-k-2d)}(x_1\tau_1)\lambda_2^{(-k-2d)}(\tau_1 x_1) = (\rho_2^{(-k-2d)}(\tau_1)\lambda_2^{(-k-2d)}(\tau_1)) \circ (\rho_2^{(-k-2d)}(x_1)\lambda_2^{(-k-2d)}(x_1)).$$

The composition of this with $\varepsilon \circ (E\varepsilon F)$ is zero:



∎

Remark 15.2. Note Lemma 15.1 plus Theorem 13.2 implies Lemma 12.5. So the proof of that lemma was actually unnecessary (as, by association, was Lemma 11.10) but we included it to make Section 12 independent of the subsequent material.

Suppose now that $\mathcal{V}$ is an integrable graded Karoubian 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$. Given any cochain complex $C \in \mathrm{Ch}^b(\mathcal{V}_{-k})$, we can apply the complex of graded superfunctors that is the image under $\mathcal{V}$ of the odd Rickard complex Θ_k to obtain a double complex. The associated total complex is again bounded

[12] The idempotent endomorphism defining $Q^{k+3d-2} E^{(k+d-1)} EFF^{(d-1)} 1_{-k}$ as a summand of $Q^d E^{k+d} F^d 1_{-k}$ decomposes as the sum of two mutually orthogonal idempotents, one of which is the idempotent defining $Q^{d-1} E^{(k+d)} F^{(d)} 1_{-k}$.

thanks to the integrability assumption. This construction defines a graded superfunctor $\mathrm{Ch}^b(\mathcal{V}_{-k}) \to \mathrm{Ch}^b(\mathcal{V}_k)$. Passing to the quotient $K^b(\mathcal{V})$ of $\mathrm{Ch}^b(\mathcal{V})$, we obtain from this a graded superfunctor

$$\mathcal{V}(\Theta_k) : K^b(\mathcal{V}_{-k}) \to K^b(\mathcal{V}_k). \tag{15.1}$$

Lemma 15.3. *Let $\mathcal{V}$ be the graded 2-representation OH^ℓ-*pgsmod *from Example 14.5. The image of the odd Rickard complex Θ_k under $\mathcal{V}$ recovers the graded superfunctor defined by tensoring with singular Rouqiuer complex from (12.4) shifted globally in degree by an application of Q^{-nk}.*

Proof. This follows using the explicit identification of the divided powers $E^{(d)}$ and $F^{(d)}$ as endofunctors of $\mathcal{V}$ explained in Example 14.5. We just check that the degree shifts match correctly. Let $n = \frac{\ell-k}{2}$ and d be as in Definition 12.2, so $k = \ell - 2n$. In the $-d$th cohomological degree in the odd Rickard complex, we have $Q^d E^{(k+d)} F^{(d)} 1_{-k}$. In the 2-representation $\mathcal{V}$, this acts by tensoring with the graded superbimodule $Q^d(Q^{-(\ell-2n+d)(n-d)} U^\ell_{(k+d);n-d}) \otimes_{OH^\ell_{n-d}} (Q^{-d(\ell-3(n-d)-2d+1)} V^\ell_{n-d;(d)})$, where the degree shifts are as described in Example 14.5. The total grading shift here simplifies to Q^{-nk}, so this is equal to the graded superbimodule $U_{(k+d);n-d} \otimes_{OH^\ell_{n-d}} V_{n-d;(d)}$ in the dth homological degree of the singular Rouquier complex shifted by Q^{-nk}. □

Corollary 15.4. *For $\ell \in \mathbb{N}$, $(\mathrm{gsKar}(\mathcal{V}(-\ell)))(\Theta_k) : K^b(\mathrm{gsKar}(\mathcal{V}(-\ell))_{-k}) \to K^b(\mathrm{gsKar}(\mathcal{V}(-\ell))_k)$ is a graded superequivalence inducing $T : 1_{-k}\mathbf{V}(-\ell) \xrightarrow{\sim} 1_k\mathbf{V}(-\ell)$ at the level of the Grothendieck groups.*

Proof. This follows from Lemma 15.3 together with Corollary 12.4 and Corollary 14.10. □

The proof of the following theorem is based on the argument in [R1, Th. 5.18], the main step really being [R1, Lem. 5.5]. This was itself a generalization of [CR, Th. 6.4] which constructed equivalences between bounded derived categories of locally finite Abelian 2-representations.

Theorem 15.5. *Let $\mathcal{V}$ be an integrable graded Karoubian 2-representation of $\mathfrak{U}(\mathfrak{sl}_2)$. For $k \in \mathbb{Z}$, the graded superfunctor $\mathcal{V}(\Theta_k) : K^b(\mathcal{V}_{-k}) \to K^b(\mathcal{V}_k)$ induced by the odd Rickard complex is a graded superequivalence.*

Proof. By Lemma 13.9, the 1-morphism $Q^d E^{(k+d)} F^{(d)} 1_{-k}$ has a right dual in $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$. Hence, we can form the right dual Θ^k to Θ_k, which is a cochain complex in $\mathrm{Ch}\,(\mathcal{H}om_{\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))}(k,-k))$. The 1-morphism in the dth cohomological degree of Θ^k is the right dual of the 1-morphism in the $(-d)$th cohomological degree of Θ_k, and the differentials in Θ^k are the right mates of the corresponding differentials in Θ_k. Let $\Theta^k \circ \Theta_k$ and $\Theta_k \circ \Theta^k$ be the total complexes associated to the double complexes obtained by composing these cochain complexes. The complex Θ_k is bounded above, and Θ^k is bounded below, but neither is bounded. Consequently, in each cohomological degree, the total complexes $\Theta^k \circ \Theta_k$ and $\Theta_k \circ \Theta^k$ involve *infinite* direct sums of 1-morphisms in $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$, so in fact, one needs to pass to a completion of this graded (Q,Π)-supercategory for it to make sense. This does not cause issues since, on a given object in an integrable graded Karoubian 2-representation, the superfunctors arising from all but finitely many of the summands of these infinite direct sums are zero.

Like Θ_k, the complex Θ^k defines a graded superfunctor denoted $\mathcal{V}(\Theta^k) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_{-k})$. Moreover, $\mathcal{V}(\Theta^k)$ is right adjoint to $\mathcal{V}(\Theta_k)$, with counit and unit of adjunction denoted

$$\mathcal{V}(\varepsilon) : \mathcal{V}(\Theta_k) \circ \mathcal{V}(\Theta^k) \Rightarrow \mathrm{Id}_{K^b(\mathcal{V}_k)}, \qquad \mathcal{V}(\eta) : \mathrm{Id}_{K^b(\mathcal{V}_{-k})} \Rightarrow \mathcal{V}(\Theta^k) \circ \mathcal{V}(\Theta_k).$$

This is explained in more detail in [CR, Sec. 4.1.4]. As the notation $\mathcal{V}(\varepsilon)$ and $\mathcal{V}(\eta)$ suggests, if we identify $\mathcal{V}(\Theta_k) \circ \mathcal{V}(\Theta^k)$ with $\mathcal{V}(\Theta_k \circ \Theta^k)$ and $\mathcal{V}(\Theta^k) \circ \mathcal{V}(\Theta_k)$ with $\mathcal{V}(\Theta^k \circ \Theta_k)$ then these even degree 0 supernatural transformations are induced by corresponding chain maps denoted simply by $\varepsilon : \Theta_k \circ \Theta^k \Rightarrow 1_k$ and $\eta : 1_{-k} \to \Theta^k \circ \Theta_k$ between cochain complexes in the completion of $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$. Although not needed here, these chain maps can be seen quite explicitly; the matrix coefficients of their components are 2-morphisms in $\mathrm{gsKar}(\mathfrak{U}(\mathfrak{sl}_2))$ that arise from the counits and units defining the duality between the 1-morphisms $Q^d E^{(k+d)} F^{(d)} 1_{-k}$ and their right duals.

To prove the theorem, it suffices to show that $\mathcal{V}(\varepsilon)$ and $\mathcal{V}(\eta)$ are isomorphisms. We just explain the argument to see this in the case of $\mathcal{V}(\varepsilon)$, since the case of $\mathcal{V}(\eta)$ is similar. Since a chain map is an isomorphism in $K^b(\mathcal{V}_k)$ if and only if its cone is zero in $K^b(\mathcal{V}_k)$, the even degree 0 graded supernatural transformation $\mathcal{V}(\varepsilon)$ is an isomorphism if and only if $\operatorname{Cone}(\mathcal{V}(\varepsilon)_C) \cong 0$ in $K^b(\mathcal{V}_k)$ for all $C \in K^b(\mathcal{V}_k)$. Now we observe that $\operatorname{Cone}(\mathcal{V}(\varepsilon)_C) = \mathcal{V}(Z)(C)$ where $Z := \operatorname{Cone}(\varepsilon)$ is the cone of $\varepsilon : \Theta_k \circ \Theta^k \Rightarrow 1_k$. Thus, it suffices to show that the graded superfunctor $\mathcal{V}(Z) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_k)$ is zero.

Consider $K^b(\mathcal{V})$ as an integrable Karoubian graded 2-representation in its own right. In this paragraph, we show that $\mathcal{V}(Z)(E^nC) = 0$ in $K^b(\mathcal{V}_k)$ for all $\ell \in \mathbb{N}$, $n \geq 0$ such that $k = 2n - \ell$, and all lowest weight objects $C \in K^b(\mathcal{V}_{-\ell})$. To see this, we apply Lemma 14.7 (with $\mathcal{V}$ replaced by $K^b(\mathcal{V})$) to get a strongly equivariant graded superfunctor $\Omega_C : \mathcal{V}(-\ell) \to K^b(\mathcal{V})$ taking $\bar{1}_{-\ell}$ to C. This extends to a strongly equivariant graded superfunctor $\widehat{\Omega}_C : \operatorname{gsKar}(\mathcal{V}(-\ell)) \to K^b(\mathcal{V})$ by the universal property of graded super Karoubi envelope. Let $K^b(\widehat{\Omega}_C) : K^b(\operatorname{gsKar}(\mathcal{V}(-\ell)_k)) \to K^b(\mathcal{V}_k)$ be the graded superfunctor defined by applying $\widehat{\Omega}_C$ to a complex in $K^b(\operatorname{gsKar}(\mathcal{V}(-\ell)_k))$ to obtain a double complex then taking the associated total complex. Since $\widehat{\Omega}_C$ is strongly equivariant, we have that

$$K^b(\widehat{\Omega}_C) \circ (\operatorname{gsKar}(\mathcal{V}(-\ell)))(Z) \circ \operatorname{inc} \simeq \mathcal{V}(Z) \circ \widehat{\Omega}_C, \tag{15.2}$$

where $\operatorname{inc} : \operatorname{gsKar}(\mathcal{V}(-\ell)) \to K^b(\operatorname{gsKar}(\mathcal{V}(-\ell))$ is the canonical graded superfunctor sending objects to complexes concentrated in cohomological degree 0. By Corollary 15.4, $(\operatorname{gsKar}(\mathcal{V}(-\ell)))(Z) = 0$ in $K^b(\operatorname{gsKar}(\mathcal{V}(-\ell)_k))$, hence, the graded superfunctor on the left hand side of (15.2) takes $E^n\bar{1}_{-\ell}$ to 0. So the graded superfunctor on the right hand side takes $E^n\bar{1}_{-\ell}$ to 0 too. Since we have that $\widehat{\Omega}_C(E^n\bar{1}_{-\ell}) \simeq E^nC$, it follows that $\mathcal{V}(Z)(E^nC) = 0$ as required.

To complete the proof, we let $\mathcal{V}(Z)^\vee$ be a right adjoint to $\mathcal{V}(Z) : K^b(\mathcal{V}_k) \to K^b(\mathcal{V}_k)$, which exists by the general discussion in [CR, Sec. 4.1.4] again. We must show that $\mathcal{V}(Z)(D) = 0$ for any $D \in K^b(\mathcal{V}_k)$, which we do by showing that $\mathcal{V}(Z)^\vee(\mathcal{V}(Z)(D)) = 0$; this is sufficient since it implies that $\operatorname{End}_{K^b(\mathcal{V}_k)}(\mathcal{V}(Z)(D)) = 0$. Using Lemma 14.11, we just need to show that

$$\operatorname{Hom}_{K^b(\mathcal{V}_k)}\left(E^nC, \mathcal{V}(Z)^\vee(\mathcal{V}(Z)(D))\right) = 0$$

for C and n as in the previous paragraph. This follows because by adjunction we have that

$$\operatorname{Hom}_{K^b(\mathcal{V}_k)}\left(E^nC, \mathcal{V}(Z)^\vee(\mathcal{V}(Z)(D))\right) \simeq \operatorname{Hom}_{K^b(\mathcal{V}_k)}(\mathcal{V}(Z)(E^nC), \mathcal{V}(Z)(D))$$

which is zero by the previous paragraph. □

16. Application to representations of spin symmetric groups

Theorem 15.5 can be applied to obtain graded superequivalences between homotopy/derived categories of supermodules over the cyclotomic quiver Hecke superalgebras from [KKT, KKO1, KKO2]. In explaining this, we will mainly cite [KKO2, Sec. 8] which presents the results needed to do this rather concisely. However, we need to reverse the roles of E and F compared to [KKO2] to be consistent with our convention for $\mathfrak{U}(\mathfrak{sl}_2)$ in Section 13, in which we preferred lowest weight modules to highest weight modules.

Fix a Cartan superdatum $(A, P, \Pi, \Pi^\vee)$ as in [KKO2, Sec. 4.1]. So:

- I is an index set with given decomposition $I = I_{\text{even}} \sqcup I_{\text{odd}}$;
- $A = (a_{i,j})_{i,j\in I}$ is a symmetrizable Cartan matrix such that $a_{i,j} \in 2\mathbb{Z}$ for all $i \in I_{\text{odd}}$, $j \in I$;
- P is the weight lattice;
- $\Pi = \{\alpha_i \mid i \in I\}$ is the set of simple roots;
- $\Pi^\vee = \{h_i \mid i \in I\}$ is the set of simple coroots.

Let $d_i\,(i \in I)$ be positive integers chosen so that $d_i a_{i,j} = d_j a_{j,i}$ for all $i, j \in I$. Let P^+ be the corresponding set of dominant weights and $Q^+ := \bigoplus_{i\in I} \mathbb{N}\alpha_i$ be the non-negative part of the root lattice. Finally, let $W < \operatorname{Aut}(P)$ be the Weyl group.

Let $\mathbb{k} = \bigoplus_{d\geq 0} \mathbb{k}_d$ be a positively graded commutative ground ring with $\mathbb{k}_0 = \mathbb{F}$ (our usual algebraically closed ground field) and $\dim_{\mathbb{F}} \mathbb{k}_d < \infty$ for all d. We view $\mathbb{k}$ as a purely even graded $\mathbb{F}$-superalgebra. Given any $\alpha \in Q^+$, there is a corresponding *quiver Hecke superalgebra* R_α which is defined by generators and relations as in [KKO2, Sec. 8.1]; the definition depends on an additional choice of parameters as explained in [KKO2]. Let R^λ_α be the deformed cyclotomic quotient from [KKO2, Def. 8.10] associated to a dominant weight $\lambda \in P^+$ and a choice of monic polynomials a^λ_i $(i \in I)$ as in [KKO2, (8.12)]. We are interested in the graded (Q, Π)-supercategory

$$R^\lambda\text{-pgsmod} := \bigoplus_{\alpha\in Q^+} R^\lambda_\alpha\text{-pgsmod}. \tag{16.1}$$

The constructions in [KKO2, Sec. 8.3] make R^λ-pgsmod into a "supercategorification" of the integrable lowest weight module $V(-\lambda)$ for the covering quantum group $U_{q,\pi}(\mathfrak{g})$ with the given Cartan superdatum. From this, it can be seen that R^λ-pgsmod has the structure of a graded 2-representation of the corresponding graded Kac-Moody 2-supercategory as defined in [BE2], with the Grothendieck group $K_0(R^\lambda\text{-pg}\underline{\text{smod}})$ being identified with the Kostant $\mathbb{Z}[q, q^{-1}]^\pi$-form for $V(-\lambda)$.

To be more precise, we focus now on some fixed $i \in I$ and consider the corresponding $\mathfrak{sl}_2$-subalgebra of $U_{q,\pi}(\mathfrak{g})$. In this generality, we actually need to work now with $q_i := q^{d_i}$ and the grading shift functor $Q_i := Q^{d_i}$ rather than q and Q used in previous sections. This means that when $d_i > 1$ definitions such as Definition 14.1 earlier in the paper should be modified by replacing Q with Q_i and scaling all degrees by d_i too, e.g., x and τ are now of degrees $2d_i$ and $-2d_i$ rather than of degrees 2 and -2. Since the $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings are independent this does not cause any problems. There are graded superfunctors

$$E_i : R^\lambda\text{-pgsmod} \to R^\lambda\text{-pgsmod}, \qquad F_i : R^\lambda\text{-pgsmod} \to R^\lambda\text{-pgsmod}.$$

In terms of the induction and restriction functors denoted F^λ_i and E^λ_i in [KKO2, Sec. 8.3], our E_i is $F^\lambda_i = \bigoplus_{\alpha\in Q_+} F^\lambda_i|_{R^\lambda_\alpha\text{-pgsmod}}$ and our F_i is $\bigoplus_{\alpha\in Q^+} Q_i^{\langle h_i,\alpha-\lambda\rangle-1} E^\lambda_i|_{R^\lambda_\alpha\text{-pgsmod}}$. As well as switching the roles of E and F we have incorporated an additional grading shift into the restriction functors compared to [KKO2]. This is needed because [KKO2] does not follow the standard conventions for covering quantum groups. It ensures that the graded supernatural transformations $\varepsilon : E_iF_i|_{R^\lambda_{\alpha+\alpha_i}\text{-pgsmod}} \Rightarrow \mathrm{Id}_{R^\lambda_{\alpha+\alpha_i}\text{-pgsmod}}$, $\eta : \mathrm{Id}_{R^\lambda_\alpha\text{-pgsmod}} \Rightarrow F_iE_i|_{R^\lambda_\alpha\text{-pgsmod}}$ defined on a graded supermodule by exactly the same underlying functions as for the natural adjunction between restriction and induction are of the correct degree to match the degrees of the rightward cups and caps in (13.9) (also now scaled by d_i). Also in [KKO2, Sec. 8.3], one finds the definition of graded supernatural transformations $x : E_i \Rightarrow E_i$ of degree $2d_i$ and $\tau : E_i^2 \Rightarrow E_i^2$ of degree $-2d_i$, both of which are even if $i \in I_{\text{even}}$ and odd if $i \in I_{\text{odd}}$. (A further complication is that the language of supercategory, superfunctor and supernatural transformation is used differently in [KKO2] compared to here, but the appropriate translation is easy to make; see the table at the end of the introduction in [BE1].)

This construction makes R^λ-pgsmod into a graded integrable Karoubian 2-representation of the ordinary $\mathfrak{sl}_2$ 2-category from [L1, R1] if i is even, or of our reduced odd $\mathfrak{sl}_2$ 2-category $\mathfrak{U}(\mathfrak{sl}_2)$ as in Definition 14.1 if i is odd (with the modified convention for degrees when $d_i > 1$). The last statement is not stated explicitly in [KKO2]—the relevant place is [KKO2, Th. 8.13] but one has to work through the proof which goes back to [KK, Th. 5.2] to see that the isomorphisms are given by the appropriate matrices of supernatural transformations needed to check the difficult relations (13.5) and (13.6). In the odd case, the fact that the odd bubbles act as zero (as required by the final axiom in Definition 14.1) follows because they are zero on the generating lowest weight subcategory R^λ_0-pgsmod as that is purely even.

Theorem 16.1. *In the above setup, for $\alpha \in Q^+$ such that $V(-\lambda)_{\alpha-\lambda} \neq 0$, the even or odd Rickard complex $\Theta_{\langle h_i,\lambda-\alpha\rangle}$ induces a graded superequivalence $K^b(R^\lambda_\alpha\text{-pgsmod}) \to K^b(R^\lambda_{\alpha-\langle h_i,\alpha-\lambda\rangle\alpha_i}\text{-pgsmod})$.*

Proof. This follows from [R1, Th. 5.18] if i is even, with the graded superequivalence being induced by the even analog of the Rickard complex, or from Theorem 15.5 if i is odd. □

There is also a "dual version" of this theorem with R^λ-pgsmod replaced with

$$R^\lambda\text{-gsmod} := \bigoplus_{\alpha\in Q^+} R^\lambda_\alpha\text{-gsmod}. \tag{16.2}$$

The underlying ordinary category is a locally finite Abelian (Q,Π)-category. The results from [KKO2, Sec. 8.3] show that this categorifies the dual Kostant $\mathbb{Z}[q,q^{-1}]^\pi$-form for $V(-\lambda)$. For fixed $i \in I$ again, R^λ-gsmod can be made into a graded 2-representation of the even or reduced odd $\mathfrak{sl}_2$ 2-category exactly as above.

Theorem 16.2. *In the above setup, for $\alpha \in Q^+$ such that $V(-\lambda)_{\alpha-\lambda} \neq 0$, the even or odd Rickard complex $\Theta_{\langle h_i,\lambda-\alpha\rangle}$ induces a graded superequivalence $D^b(R^\lambda_\alpha\text{-gsmod}) \to D^b(R^\lambda_{\alpha-\langle h_i,\alpha-\lambda\rangle\alpha_i}\text{-gsmod})$.*

Proof. Like in the previous theorem, the even or odd Rickard complex $\Theta_{\langle h_i,\lambda-\alpha\rangle}$ induces a graded superequivalence $K^b(R^\lambda_\alpha\text{-gsmod}) \to K^b(R^\lambda_{\alpha-\langle h_i,\alpha-\lambda\rangle\alpha_i}\text{-gsmod})$. The result for derived categories follows since they are localizations of these homotopy categories. □

For a graded superalgebra A, we write $A\otimes C_1$ for the graded superalgebra obtained by tensoring with the rank one Clifford superalgebra generated by an odd degree 0 involution. There is also a variation of Theorem 16.2 with R^λ-gsmod replaced by

$$R^\lambda \otimes C_1\text{-gsmod} := \bigoplus_{\alpha\in Q^+} R^\lambda_\alpha \otimes C_1\text{-gsmod}. \tag{16.3}$$

This can be made into a graded 2-representation which also categorifies the dual Kostant $\mathbb{Z}[q,q^{-1}]^\pi$-form for $V(-\lambda)$, just as R^λ-gsmod did earlier. In particular, for each $i \in I$, we can make $R^\lambda \otimes C_1$-gsmod into a graded 2-representation of the even or reduced odd $\mathfrak{sl}_2$ 2-category exactly as above. This follows by the construction explained in the next paragraph.

There is a general notion of the *Clifford twist* $\mathcal{A}^{\mathtt{CT}}$ of a graded supercategory $\mathcal{A}$, which goes back to [KKT, Lem. 2.3]. By definition, this is the graded supercategory whose objects are pairs (X,ϕ) for $X \in \mathcal{A}$ and an odd degree 0 involution $\phi \in \mathrm{End}_{\mathcal{A}}(X)$. A morphism $f : (X,\phi) \to (Y,\theta)$ is a morphism $f : X \to Y$ in $\mathcal{A}$ such that $\theta \circ f = (-1)^{\mathrm{par}(f)}\phi \circ f$. Degree and parity of morphisms in $\mathcal{A}^{\mathtt{CT}}$ are induced by the ones for $\mathcal{A}$. There are obvious ways to define the Clifford twist $F^{\mathtt{CT}} : \mathcal{A}^{\mathtt{CT}} \to \mathcal{B}^{\mathtt{CT}}$ of a graded superfunctor $F : \mathcal{A} \to \mathcal{B}$, and also the Clifford twist $\alpha^{\mathtt{CT}} : F^{\mathtt{CT}} \Rightarrow G^{\mathtt{CT}}$ of a graded supernatural transformation $\alpha : F \Rightarrow G$ between two graded superfunctors. This makes $\mathtt{CT}$ into a strict graded 2-superfunctor $\mathtt{CT} : \mathfrak{gsCat} \to \mathfrak{gsCat}$. Now if $\mathcal{V}$ is any graded 2-representation of the even or the reduced odd 2-supercategory $\mathfrak{U}(\mathfrak{sl}_2)$, its Clifford twist $\mathcal{V}^{\mathtt{CT}}$ can be made into a graded 2-representation in its own right, with the required graded superfunctors E and F on $\mathcal{V}^{\mathtt{CT}}$ being the Clifford twists of the ones for $\mathcal{V}$, and all of the required graded supernatural transformations being the Clifford twists of the one for $\mathcal{V}$ too. If $\mathcal{V}$ is integrable and Karoubian then so is $\mathcal{V}^{\mathtt{CT}}$.

Theorem 16.3. *In the above setup, for $\alpha \in Q^+$ such that $V(-\lambda)_{\alpha-\lambda} \neq 0$, the even or odd Rickard complex $\Theta_{\langle h_i,\lambda-\alpha\rangle}$ induces a graded superequivalence $D^b(R^\lambda_\alpha \otimes C_1\text{-gsmod}) \to D^b(R^\lambda_{\alpha-\langle h_i,\alpha-\lambda\rangle\alpha_i} \otimes C_1\text{-gsmod})$.*

Proof. This follows by the same arguments as Theorem 16.2. □

Assume henceforth that the characteristic of the ground field $\mathbb{F}$ is $p = 2l + 1 > 2$, and that the Cartan superdatum fixed above is of type $A^{(2)}_{2l}$, with the shortest simple root α_0 being odd and all other simple roots being even. We consider the cyclotomic quiver Hecke superalgebras R^λ_α for $\alpha \in Q_+$ and $\lambda := \Lambda_0$, taking the ring $\Bbbk$ to be the ground field $\mathbb{F}$, and all other choices made as explained in [KLi, Sec. 3.1]. Let $R^\lambda_\alpha \otimes C_1$ be the superalgebra tensor product of R^λ_α with the rank one Clifford superalgebra generated by an odd involution. Now we forget both the $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings on R^λ_α and $R^\lambda_\alpha \otimes C_1$ to view them as ordinary finite-dimensional algebras. For such an algebra A, we write A-mod for the Abelian category of finite-dimensional left A-modules and $D^b(A\text{-mod})$ for its ordinary bounded derived category.

In view of [KLi, Lem. 3.1.39], the following proves [KLi, Conj. 2].

Theorem 16.4. *Suppose that $\alpha, \beta \in Q^+$ are such that $\alpha - \lambda$ and $\beta - \lambda$ are weights of $V(-\lambda)$ in the same W-orbit. The categories $D^b(R^\lambda_\alpha\text{-mod})$ and $D^b(R^\lambda_\beta\text{-mod})$ are equivalent as are $D^b(R^\lambda_\alpha \otimes C_1\text{-mod})$ and $D^b(R^\lambda_\beta \otimes C_1\text{-mod})$.*

Proof. Since the simple reflections generate W, it suffices to prove the theorem in the special case that $\alpha - \lambda$ is a weight of $V(-\lambda)$ and $\beta = \alpha - \langle h_i, \alpha - \lambda\rangle \alpha_i$ for some $i \in I$. The graded superequivalences in Theorems 16.2 and 16.3 are obtained by taking the derived tensor product with the complex of graded superbimodules arising from the appropriate Rickard complex. Similarly, the quasi-inverse graded superequivalences are obtained from the right adjoint of this complex. Now we are forgetting both the $\mathbb{Z}$- and $\mathbb{Z}/2$-gradings, viewing these complexes of graded superbimodules as complexes of ordinary bimodules. The resulting complexes define functors between the ordinary derived categories. Since they are quasi-inverse with all gradings present, they are obviously quasi-inverse without these gradings. □

Corollary 16.5. *Broué's Abelian Defect Group Conjecture holds for double covers of symmetric and alternating groups over any algebraically closed field of positive characteristic.*

Proof. See [KLi, Th. 5.4.12] where this is deduced from [KLi, Conj. 2]. □

References

[BD] J. Brundan and N. Davidson, Categorical actions and crystals, *Contemp. Math.* **684** (2017), 116–159.

[BE1] J. Brundan and A. Ellis, Monoidal supercategories, *Commun. Math. Phys.* **351** (2017), 1045–1089.

[BE2] ———, Super Kac-Moody 2-categories, *Proc. London Math. Soc.* **115** (2017), 925–973.

[BK] J. Brundan and A. Kleshchev, Hecke-Clifford superalgebras, crystals of type $A^{(2)}_{2\ell}$ and modular branching rules for $\widehat{S}_n$, *Represent. Theory* **5** (2001), 317–403.

[BSW] J. Brundan, A. Savage and B. Webster, Heisenberg and Kac-Moody categorification, *Selecta Math.* **26** (2020), Paper No. 74, 62 pp..

[CK] J. Chuang and R. Kessar, Symmetric groups, wreath products, Morita equivalences, and Broué's abelian defect group conjecture, *Bull. London Math. Soc.* **34** (2002), 174–184.

[CR] J. Chuang and R. Rouquier, Derived equivalences for symmetric groups and $\mathfrak{sl}_2$-categorification, *Ann. Math.* **167** (2008), 245–298.

[C] S. Clark, Quantum supergroups IV: the modified form, *Math. Z.* **278** (2014), 493–528.

[CHW] S. Clark, D. Hill and W. Wang, Quantum supergroups I: foundations, *Transform. Groups* **18** (2013), 1019–1053.

[CHW2] ———, Quantum supergroups II: canonical basis, *Represent. Theory* **18** (2014), 278–309.

[CW] S. Clark and W. Wang, Canonical basis for quantum $\mathfrak{osp}(1|2)$, *Lett. Math. Phys.* **103** (2013), 207–231.

[DEL] B. Dupont, M. Ebert and A. Lauda, Super rewriting theory and nondegeneracy of odd categorified $\mathfrak{sl}(2)$; `arXiv:2102.00276`.

[ELV] M. Ebert, A. Lauda and L. Vera, Derived superequivalence for spin symmetric groups and odd $\mathfrak{sl}_2$-categorifications; `arXiv:2203.14153`.

[E] A. Ellis, The odd Littlewood-Richardson rule, *J. Algebraic Combin.* **37** (2013), 777–799.

[EK] A. Ellis and M. Khovanov, The Hopf algebra of odd symmetric functions, *Advances Math.* **231** (2012), 965–999.

[EKL] A. Ellis, M. Khovanov and A. Lauda, The odd nilHecke algebra and its diagrammatics, *IMRN* (2012).

[EL] A. Ellis and A. Lauda, An odd categorification of $U_q(\mathfrak{sl}_2)$, *Advances Math.* **265** (2014), 169–240.

[EvK] A. Evseev and A. Kleshchev, Blocks of symmetric groups, semicuspidal KLR algebras and zigzag Schur-Weyl duality, *Ann. Math.* **188** (2018), 453–512.

[HS] J. Hu and L Shi, Graded dimensions and monomial bases for the cyclotomic quiver Hecke superalgebras; `arXiv:2111.032916`.

[G] I. Grojnowski, Affine $\mathfrak{sl}_p$ controls the representation theory of the symmetric group and related Hecke algebras; `arXiv:math/9907129`.

[KKO1] S.-J. Kang, M. Kashiwara and S.-J. Oh, Supercategorification of quantum Kac-Moody algebras, *Advances Math.* **242** (2013), 116–162.

[KKO2] ______, Supercategorification of quantum Kac-Moody algebras II, *Advances Math.* **265** (2014), 169–240.
[KK] S.-J. Kang and M. Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras, *Invent. Math.* **190** (2012), 699–742.
[KKT] S.-J. Kang, M. Kashiwara and S. Tsuchioka, Quiver Hecke superalgebras, *J. Reine Angew. Math.* **711** (2016), 1–54.
[KS] R. Kessar and M. Schaps, Crossover Morita equivalences for blocks of the covering groups of the symmetric and alternating groups, *J. Group Theory* **9** (2006), 715–730.
[KL] M. Khovanov and A. Lauda, A categorification of quantum $\mathfrak{sl}(n)$, *Quantum Top.* **1** (2010), 1–92.
[KLi] A. Kleshchev and M. Livesey, RoCK blocks for double covers of symmetric groups and quiver Hecke superalgebras, to appear in *Mem. Amer. Math. Soc.*; `arXiv:2201.06870v2`.
[L1] A. Lauda, A categorification of quantum $\mathfrak{sl}(2)$, *Advances Math.* **225** (2010), 3327–3424.
[L2] ______, Categorified quantum $\mathfrak{sl}(2)$ and equivariant cohomology of iterated flag varieties, *Algebr. Represent. Theory* **14** (2011), 253–282.
[Mac] I. Macdonald, *Symmetric Functions and Hall Polynomials*, Oxford Mathematical Monographs, second edition, OUP, 1995.
[M] A. Marcus, Broué's Abelian defect group conjecture for alternating groups, *Proc. Amer. Math. Soc.* **132** (2003), 7–14.
[PS] J. Pike and A. Savage, Twisted Frobenius extensions of graded superrings, *Algebr. Represent. Theory* **19** (2016), 113–133.
[R] D. Rose, A note on the Grothendieck group of an additive category, *Vestn. Chelyab. Gos. Univ. Mat. Mekh. Inform.*, **3** (2015), 135–139.
[R1] R. Rouquier, 2-Kac-Moody algebras; `arXiv:0812.5023`.
[R2] ______, Quiver Hecke algebras and 2-Lie algebras, *Algebra Colloq.* **19** (2012), 359–410.
[V] L. Vera, Faithfulness of simple 2-representations of $\mathfrak{sl}_2$; `arXiv:2011.13003`.

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