Double coset density in exceptional algebraic groups

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Abstract

Let G be a simple algebraic group of exceptional type, defined over an algebraically closed field K of characteristic $p \geq 0$. In this paper, we classify all pairs (X, Y) of reductive subgroups of G which have a dense (X, Y)-double coset in G. In fact, we show that there is a dense (X, Y)-double coset in G precisely when $G = XY$ is a factorisation.

Introduction

Let G be a simple algebraic group of exceptional type, defined over an algebraically closed field K of characteristic $p \geq 0$. In this paper, we classify all pairs (X, Y) of reductive subgroups of G which have a dense (X, Y)-double coset in G. In fact, we show that there is a dense (X, Y)-double coset in G precisely when $G = XY$ is a factorisation. The possible factorisations that can occur have recently been determined in [11, Theorem A].

We now state our main result.

Main Theorem. *Let G be a simple algebraic group of exceptional type, and suppose that X, Y are reductive subgroups of G. Then, either $G = XY$ or there is no dense (X, Y)-double coset in G.*

As a consequence, when X, Y are reductive subgroups of an exceptional algebraic group G, then the number of (X, Y)-double cosets in G is either one or infinite. We hope to obtain a similar theorem for classical algebraic groups in a later paper. (Currently, we can prove this analogue providing X, Y are either maximal connnected subgroups or Levi factors.)

Combining the Main Theorem with [11], we can now describe all triples (G, X, Y), where G is exceptional and X, Y are reductive subgroups of G, for which there is a dense (X, Y)-double coset in G. Let $A_2 < G_2$ be the subgroup generated by all long root subgroups (relative to some fixed maximal torus), and if $p = 3$, let $\tilde{A}_2 < G_2$ be the subgroup generated by all short root subgroups. Similarly in F_4, let D_4 (resp. $\tilde{D}_4$ with $p = 2$) be the subgroup generated by all long (resp. short) root subgroups. Then, if (G, X, Y) is such a triple, either $(G, X^0, Y^0, p) = (G_2, A_2, \tilde{A}_2, 3)$ or $G = F_4, X \geq D_4, Y \geq \tilde{D}_4$ and $p = 2$.

A quasihomogeneous variety is a variety Ω on which an algebraic group X has a dense orbit. Of particular interest is the case that Ω is affine and X is reductive. One natural place to look for such quasihomogeneous affine varieties is in the action of X on the affine variety $\Omega = G/Y$, where $X, Y < G$ are reductive subgroups. The theorem therefore shows that the only examples of quasihomogeneous varieties that occur in this way within exceptional algebraic groups are in fact homogeneous.

In characteristic 0, our result can be deduced from a result of Luna [14]. In fact, Luna proves far more: if X, Y are reductive subgroups of a reductive group G, then the union of all closed (X, Y)-double cosets is dense in G. To deduce the Main Theorem from this, note that if there is a dense (X, Y)-double coset in G then it must also be closed by Luna's result, so that it is actually the only (X, Y)-double coset in G and $G = XY$. Luna's proof involves the construction of étale slices, which is always possible in characteristic 0, but definitely fails in many cases in non-zero characteristic. To prove the theorem, we are inevitably lead to case by case consideration.

The layout of the proof is as follows. In §1, we describe some well known results in invariant theory, and use this to prove the Main Theorem in the case when X and Y are conjugate. Then, in §2, we use Seitz's classification of maximal connected subgroups of exceptional algebraic groups [17] to obtain a list of cases (G, X, Y) with X, Y maximal and satisfying the dimension bound $\dim X + \dim Y \geq \dim G$. We then argue in §3 − §5 that for each of these cases, there can be no dense (X, Y)-double coset in G. Finally, in §6 we complete the proof of the Main Theorem by considering non-maximal subgroups.

1 Double cosets and invariants

Throughout, G will denote an affine algebraic group defined over K. Subgroups of G will always be assumed to be closed without further notice. By a G-module, we mean a rational KG-module, and by a G-variety we mean an algebraic variety V defined over K on which G acts morphically.

When G is semisimple and λ is a dominant weight relative to some fixed root system of G, we shall write $L_G(\lambda)$ for the irreducible G-module of highest weight λ and $\Delta_G(\lambda)$ for the corresponding Weyl module.

If V is a G-variety, the algebra of G-invariants on V is defined to be

$$K[V]^G = \{f \in K[V] \mid f(g.v) = f(v), \text{ for all } v \in V, g \in G\},$$

where $K[V]$ is the algebra of regular functions on V. When G is reductive and V is an affine variety, the Mumford conjecture [9] plays a key role in describing $K[V]^G$; for instance, it implies that $K[V]^G$ is finitely generated. This paper relies on the following well known consequence.

1.1. **Lemma.** *Suppose G is reductive and V is an affine G-variety. If A and B are disjoint, closed G-stable subsets of V, then there exists an invariant $f \in K[V]^G$ with $f(a) = 0$ for all $a \in A$ and $f(b) = 1$ for all $b \in B$. In particular, if G has at least two disjoint closed orbits in V, then there is no dense G-orbit in V.*

Proof. The first statement is proved in [15, Lemma 1.4]. To deduce the second, we can find a non-constant invariant $f \in K[V]^G$. And G-orbit $G.v$ for $v \in V$ lies in the proper closed subset of V defined by the vanishing of $f - f(v)$; hence, $G.v$ is not dense in V. □

We shall apply this to the case of double coset actions. Here, let X and Y be reductive subgroups of an algebraic group G. Then, $X \times Y$ acts on G by $(x, y).g = xgy^{-1}$, for $(x, y) \in X \times Y, g \in G$, and the orbits are (X, Y)-double cosets. By the lemma, if there are two disjoint closed (X, Y)-double cosets in G, then there is no dense (X, Y)-double coset in G. Thus, our strategy for proving the Main Theorem will be to exhibit disjoint closed (X, Y)-double cosets in G whenever $G \neq XY$.

We note that the closure $\overline{XgY}$ of a double coset is a union of double cosets, and a double coset of minimal dimension in $\overline{XgY}$ will be closed. There is a natural closure-preserving bijection between (X, Y)-double cosets in G and X-orbits in G/Y (similarly Y-orbits in G/X); sometimes it is more convenient to work with these.

The next lemma gives a reduction to G simply connected.

1.2. **Lemma.** *Let X, Y be reductive subgroups of a simple algebraic group G. Let $\theta : \tilde{G} \to G$ be the simply connected covering, and set $\tilde{X} = \theta^{-1}X, \tilde{Y} = \theta^{-1}Y$.*

(i) *If there are at least two closed $(\tilde{X}, \tilde{Y})$-double cosets in $\tilde{G}$, then there are at least two closed (X, Y)-double cosets in G.*

(ii) *If there is no dense $(\tilde{X}, \tilde{Y})$-double coset in $\tilde{G}$, then there is no dense (X, Y)-double coset in G.*

Proof. Morphisms of algebraic groups are open maps, so any closed subset of $\tilde{G}$ which is a union of $\ker\theta$-cosets has closed image. Thus, as we may assume $\ker\theta \leq \tilde{X}$, the image of a closed $(\tilde{X}, \tilde{Y})$-double coset in $\tilde{G}$ will be a closed (X, Y)-double coset in G. A similar argument implies (ii). □

The following technique for constructing disjoint closed (X, Y)-double cosets in G will be used repeatedly. Suppose V is a G-module and $v \in V$ is fixed by Y. Let W be an X-variety and $\theta : V \to W$ be an X-equivariant morphism. (For instance, take $W = V/V_0$ for some X-stable subspace $V_0 < V$ with θ the quotient map.) Then, the morphism $\bar{\theta} : G \to W$ defined by $\bar{\theta} : g \mapsto \theta(g.v)$ sends (X, Y)-double cosets in G to X-orbits in W. If we can show that there are two disjoint closed X-orbits in $\bar{\theta}(G)$, then their pre-images will contain two disjoint closed (X, Y)-double cosets in G. To construct closed X-orbits in $\bar{\theta}(G)$, we use the next lemma, which is an easy consequence of the definition of a complete variety:

1.3. **Lemma ([19, p68, Lemma 2]).** *Let G act on a variety V, and let $P < G$ be a subgroup of G such that G/P is complete. If $U \subset V$ is closed and P-stable, then $G.U$ is also closed.*

1.4. **Corollary.** *Let T be a maximal torus of G. Let V be an affine G-variety, and suppose that $v \in V$ is fixed by T. Then, $G.v$ is closed in V.*

Proof. Let $B = UT$ be a Borel subgroup of G, where U is unipotent. Then, $B.v = UT.v = U.v$ which is closed as every orbit of a unipotent group on an affine variety is closed. Hence, $G.v$ is closed by Lemma 1.3. □

1.5. **Corollary.** *Let X, Y be reductive subgroups of G with maximal tori S, T respectively, such that $S \leq T$. Then, XnY is closed in G for all $n \in N_G(T)$.*

Proof. Let $V = G/Y$, an affine variety by [15, Theorem A]. Let $\bar{n} = nY$ be the image of n in V. We just need to show that $X.\bar{n}$ is closed in V. This follows immediately from the previous corollary, since S fixes $\bar{n}$. □

We can now deduce the Main Theorem in the case when X and Y are conjugate. This result was first proved in [4] by means of direct constructions.

1.6. **Proposition.** *Let X be a proper reductive subgroup of a connected reductive algebraic group G. Then, there is no dense (X, X)-double coset in G.*

Proof. We may assume X is connected. Let T be a maximal torus of X. Since all double cosets XnX are closed for $n \in N_G(T)$ by Corollary 1.5, it is sufficient to show that $N_G(T)$ is not contained in X, since then there are certainly two disjoint closed (X, X)-double cosets in G. Now, $N_G(T)$ certainly contains a maximal torus of G, hence the claim follows unless T is also a maximal torus of G. But then, $N_X(T) = N_G(T)$ implies X and G have the same Weyl group, hence the same dimension. Since G is connected, this implies that $X = G$, a contradiction. □

2 Maximal reductive subgroups

Throughout this section, G will denote a simple algebraic group of exceptional type. We need to distinguish between the set of reductive maximal connected subgroups of G and the (possibly larger) set of maximal connected reductive subgroups of G; by the former, we mean maximal connected subgroups of G that happen to be reductive, whilst in the latter, we are also including (potentially) certain sub-parabolic subgroups of G. Let $\mathcal{M} = \mathcal{M}(G)$ be the set of all maximal connected reductive subgroups of G which are either Levi factors of some parabolic subgroup of G or maximal connected subgroups of G. In characteristic 0, $\mathcal{M}$ contains *all* maximal connected reductive subgroups of G, but in non-zero characteristic, this need not be the case: the problem is that there may be reductive subgroups of some parabolic P of G that lie in no Levi factor of P. We begin by considering this complication.

2.1. **Lemma.** *Suppose that $X \in \mathcal{M}(G)$ is a maximal connected subgroup of G, not of maximal rank, with $\dim X$ greater than $3, 14, 22, 35$ or 66, according to whether $G = G_2, F_4, E_6, E_7$ or E_8, respectively. Then, (G, X) is (E_7, A_1F_4), (E_6, F_4), $(E_6, C_4)(p \neq 2)$ or $(F_4, A_1G_2)(p \neq 2)$.*

Proof. This follows from [17, Theorem 1] which classifies all maximal closed connected subgroups of exceptional algebraic groups assuming some mild restrictions on p. By the dimension bound on X, none of these restrictions apply, so the result follows directly. □

We shall need some information on maximal subgroups of certain classical algebraic groups of small rank. Here, we use the notation $H = Cl(V)$ to indicate that H is a classical algebraic group with natural module V (where if $(H, p) = (B_n, 2)$, we take V to be the natural $2n$-dimensional symplectic module). If $H = SO(V)$ or $Sp(V)$, we let N_k be the connected stabilizer of a non-degenerate k-subspace of V; and if $(H, p) = (D_n, 2)$, we let N_1 be the connected stabilizer of a non-singular 1-space.

2.2. **Lemma ([16, Theorem 3]).** *Let $H = Cl(V)$, and suppose that $X \in \mathcal{M}(H)$ is a maximal connected subgroup of H. Then one of the following holds:*
(i) *$X = N_k$ for some k;*
(ii) *$V = U \otimes W$ and $X = Cl(U) \otimes Cl(W)$;*
(iii) *$(H, X) = (SL(V), Sp(V)), (SL(V), SO(V))\,(p \neq 2)$ or $(Sp(V), SO(V))(p = 2)$;*
(iv) *X is simple, and $V \downarrow X$ is irreducible and tensor indecomposable.*

The next lemma gives the necessary technical information to list subgroups occuring in Lemma 2.2(iv) that we shall meet.

2.3. **Lemma.** *Let $H = Cl(V)$ and $X \in \mathcal{M}(H)$ be a simple maximal connected subgroup such that $V \downarrow X$ is irreducible and tensor indecomposable. Assume that* $\dim X \geq 2 \dim V$ *and* $\operatorname{rank} H \leq 6$. *Then, the triple (H, X, λ) is in table 1, where $V \downarrow X = L_X(\lambda)$ (up to duals and field twists).*

Table 1: Irreducible subgroups of large dimension

H	X	λ
SO_8	B_3	λ_3
SO_7	G_2	$\lambda_1 (p \neq 2)$ or $\lambda_2 (p = 3)$
Sp_6	G_2	$\lambda_1 (p = 2)$

Proof. This is essentially [11, Proposition 2.7], with the bound on $\operatorname{rank} H$ reducing the possible cases. We note that although a smaller bound on $\dim V$ applies here, it is easy to see no extra cases occur by the proof in [11]. □

We can now show that $\mathcal{M}(G)$ contains all maximal connected reductive subgroups of G of sufficiently large dimension.

2.4. **Proposition.** *Let X be a maximal connected reductive subgroup of G, and suppose that* $\dim X$ *is greater than* $5, 15, 25, 53$ *or* 111 *if* $G = G_2, F_4, E_6, E_7$ *or* E_8 *respectively. Then, $X \in \mathcal{M}(G)$.*

Proof. If X lies in no parabolic subgroup of G, then X is a reductive maximal connected subgroup as in the conclusion. So, suppose $X < P$, where P is a parabolic subgroup minimal subject to containing X, with Levi factor L and unipotent radical Q.

First note that we may assume X is semisimple. For, otherwise, $X = C_G(Z)$ for some central torus Z, so is of maximal rank. Hence, the root system of X is either a closed subsystem of the root system of G, or lies in the dual of a closed subsystem if $(G, p) = (G_2, 3)$ or $(F_4, 2)$. In the former case, maximal closed subsystems are obtained by deleting nodes from the extended Dynkin diagram of G, and since we are assuming $X < P$, we deduce that X is a Levi subgroup as in the conclusion. In the latter case, maximality implies X is $\tilde{A}_2$ (resp. C_4) if $(G, p) = (G_2, 3)$ (resp. $(F_4, 2)$), contradicting the assumption $X < P$.

So, suppose X is semisimple. We now claim that X is in fact simple of rank greater than $\frac{1}{2} \operatorname{rank} G$, with one exception. To see this, we consider the possibilities for $\bar{X} = X$ modulo Q, a semisimple subgroup of L' isogenous to X. By the dimension bound on X, there are very few possibilities for L'. Moreover, by the minimality assumption on P, $\bar{X}$ lies in no proper parabolic subgroup of L', so either $\bar{X} = L'$ or $\bar{X}$ lies in a reductive maximal connected subgroup of L', which can again be listed using Lemma 2.1 for L' exceptional or Lemma 2.2 and Lemma 2.3 for L' classical. Repeating the argument, one obtains all possibilities for $\bar{X}$ of sufficient dimension, as in the table below. We illustrate the procedure by considering $(G, L') = (E_6, D_5)$; the other cases are similar. So, suppose $L' = D_5$ with natural module V, and that $\dim \bar{X} \geq 26 \geq 2 \dim V = 20$. Hence, by Lemma 2.2, we deduce that either $\bar{X} \leq N_k$ for some k, whence $k = 1$ by the dimension bound, or that $(L', \bar{X})$ is as in Lemma 2.3, giving no further possibilities. So, $\bar{X} \leq N_1 = B_4$. Again, one lists the reductive maximal connected subgroups of B_4 by Lemma 2.2; the only possibility of large enough dimension is $N_1 = D_4$.

But D_4 lies in a parabolic subgroup of L' contradicting the minimality of P. Hence, the only possibilities are $\bar{X} = D_5$ or B_4 as in the table.

G	L'	$\bar{X}$
E_8	E_7	E_7
E_7	D_6	D_6
E_7	E_6	E_6
E_6	D_5	D_5 or B_4
E_6	D_4	D_4
E_6	A_5	A_5
E_6	A_1A_4	A_1A_4
F_4	B_3	B_3
F_4	C_3	C_3

We conclude X is simple of rank greater than $\frac{1}{2}\operatorname{rank} G$ as claimed, except for $(G, X) = (E_6, A_1A_4)$ which we treat later. Now, we can apply [12, Theorem 1], which tells us, since X is isogenous to $\bar{X}$ in the above table, that there is a simple, connected subgroup $Y \leq G$ normalised by a maximal torus of G such that one of the following hold:

(i) $X = Y$;

(ii) $X = Y^\delta$, where δ is a graph automorphism of Y of order 2;

(iii) $G = E_6, X = C_4$ or D_4, $p = 2$ and $X < F_4 < Y = G$.

Since we are assuming X is maximal reductive, (iii) does not occur, and if (ii) holds, $Y = G$; then, the possibilities are $(G, X) = (E_6, F_4)$ or $(E_6, C_4) (p \neq 2)$, contradicting the assumption $X < P$. Finally, if (i) holds, then X is of maximal rank, and the result follows as in the second paragraph.

Now, suppose $(G, X) = (E_6, A_1A_4)$. Then, by [2], Q has an L'-composition series $Q = Q_2 > Q_1 > Q_0 = 1$ such that the composition factors $V_2 = Q_2/Q_1$ and $V_1 = Q_1/Q_0$ are the irreducible L'-modules $L_{A_1}(\lambda_1) \otimes L_{A_4}(\lambda_2)$ and $L_{A_4}(\lambda_4)$ respectively. If the semidirect product $L'Q$ has more than one conjugacy class of closed complements to Q, then the semidirect product $L'V_i$ must have more than one class of closed complements to V_i for some i. This implies by [13, 1.5] that one of $L_{A_4}(\lambda_4)$ or $L_{A_4}(\lambda_2)$ has a rational indecomposable extension by the trivial module, which is not the case by [13, 1.6]. Hence, $L'Q$ has just one conjugacy class of closed complements to Q, so X is conjugate to L'. But then, X lies in A_1A_5 contradicting maximality. □

We now apply this to obtain a list of maximal connected reductive subgroups $X, Y < G$ satisfying the dimension bound $\dim X + \dim Y \geq \dim G$. We write T_1 to denote a 1-dimensional torus. Also, note that E_6 has two classes of Levi subgroup of type T_1D_5; we denote these by T_1D_5 and T_1D_5' in the next proposition.

2.5. **Proposition.** *Let X, Y be maximal connected reductive subgroups of G. Suppose that X and Y are not conjugate and that* $\dim X + \dim Y \geq \dim G$. *If* $(G, p) = (G_2, 3)$ *or* $(F_4, 2)$ *let δ be a non-trivial graph automorphism of G (as an abstract group); otherwise, let $\delta = 1$. Then, either (G, X, Y) or $(G, \delta X, \delta Y)$ is in table 2.*
(In the table we also reference the lemma in which we prove the Main Theorem for these triples.)

Proof. We assume $\dim X \geq \dim Y$, so that $\dim X \geq \frac{1}{2} \dim G$. Thus, X satisfies the dimension bound in Proposition 2.4, so $X \in \mathcal{M}(G)$. If X is not of maximal rank, the only possibility

Table 2: Main case list

G	X	Y	p	Ref
G_2	A_2	$A_1\tilde{A}_1$		5.3
G_2	A_2	$\tilde{A}_2$	$p = 3$	
F_4	B_4	$A_2\tilde{A}_2$		4.8
F_4	B_4	A_1C_3	$p \neq 2$	4.9
F_4	B_4	A_1G_2	$p \neq 2$	4.10
F_4	B_4	C_4	$p = 2$	
E_6	F_4	C_4	$p \neq 2$	5.1
E_6	F_4	A_1A_5		5.2
E_6	F_4	T_1D_5		3.8
E_6	T_1D_5	A_1A_5		3.5
E_6	T_1D_5	C_4	$p \neq 2$	3.8
E_6	T_1D_5	T_1D_5'		3.4
E_7	T_1E_6	A_7		3.5
E_7	T_1E_6	A_1D_6		3.5
E_7	T_1E_6	A_1F_4		3.8
E_8	A_1E_7	D_7		5.2

is $(G, X) = (E_6, F_4)$ by Lemma 2.1. Otherwise, X is of maximal rank; let $\Sigma(X) \subset \Sigma(G)$ be the root systems of X and G respectively, relative to some fixed maximal torus of X. Then, if $\Sigma(X)$ is not a closed subsystem of $\Sigma(G)$, we obtain $(G, X, p) = (G_2, \tilde{A}_2, 3)$ or $(F_4, C_4, 2)$; we eliminate these cases by applying δ to G, to send $\tilde{A}_2 \to A_2$ or $C_4 \to B_4$ respectively. Otherwise, $\Sigma(X)$ is obtained by deleting nodes from the extended Dynkin diagram of G, giving possibilities $(G, X) = (G_2, A_2), (F_4, B_4), (E_6, T_1D_5), (E_7, T_1E_6)$ and (E_8, A_1E_7).

Consequently, $\dim Y$ is greater than $5, 15, 25, 53$ or 111 if $G = G_2, F_4, E_6, E_7$ or E_8 respectively. So, Y is as in Proposition 2.4, and $Y \in \mathcal{M}(G)$. Thus, we can list the possibilities for Y in each case (G, X) as in the previous paragraph, to obtain the conclusion. □

In the next three sections, we verify the Main Theorem for all triples (G, X, Y) listed in table 2.

3 Double cosets involving Levi factors

In this section, G will denote a connected reductive algebraic group with fixed maximal torus T. We shall assume throughout that G is simply connected, which we may do by Lemma 1.2. Let $W = N_G(T)/T$ be the Weyl group of G, and let $X(T)$ be the character group of T. Let Σ be the root system of G relative to T and fix a base $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ for Σ; let $\lambda_1, \ldots, \lambda_n$ be the corresponding fundamental dominant weights. The choice of Π determines a set Σ^+ of positive roots and a Borel subgroup $B = \langle T, U_\alpha \mid \alpha \in \Sigma^+ \rangle$ of G, where U_α is the T-root subgroup corresponding to $\alpha \in \Sigma$. Let U be the unipotent radical of B, so $B = TU$. Let $B^- = TU^-$ be the opposite Borel subgroup to B. For $w \in W$, let $U_w^- < U$ be the subgroup generated by root subgroups U_α such that α is a positive root sent to a negative root by w.

3.1. **Lemma (G. Seitz).** *Let V be a G-module. Let v, v' be vectors in the zero weight space relative to T. Then, v and v' are conjugate under G if and only if they are conjugate under W.*

Proof. Suppose $g.v = v'$ for $g \in G$. By the Bruhat decomposition, we may write $g = unu'$ with $u \in U, n \in N_G(T)$ and $u' \in U_w^-$, where $w = nT \in W$. Then, $nu'n^{-1}.nv = u^{-1}.v'$. The right hand side is a sum of v' and weight vectors whose weights are sums of positive roots, whilst the left hand side is a sum of nv and weight vectors whose weights are sums of negative roots, since $nu'n^{-1} \in U^-$. Hence, $n.v = v'$, so $w.v = v'$, and v and v' are W-conjugate. □

Let $P = P_J$ be the standard parabolic subgroup of G corresponding to the subset J of $I = \{1, \dots, n\}$. So, $P = \langle B, U_{-\alpha_j} \mid j \in J \rangle$ and $L = \langle T, U_{\pm\alpha_j} \mid j \in J \rangle$ is a Levi factor of P. Let $W_L = N_L(T)/T$ be the Weyl group of L, a subgroup of W.

3.2. **Lemma.** *Let X be a connected reductive subgroup of G with maximal torus $S \leq T$; let $W_X = N_X(S)/S$ be the Weyl group of X. Suppose $N_X(S) = N_X(T)$, so that W_X can be identified with a subgroup of W. If there is a dense (X, L)-double coset in G, then $W = W_X W_L$ is a factorisation of W.*

Proof. Let $\lambda = \sum_{i \in I-J} \lambda_i$, and let $V = L_G(\lambda)$. Let $v_0 \in V^U$ be a highest weight vector, and extend to a basis of T-weight vectors $v_0, \dots, v_k$ for V. Let $f_0, \dots, f_k$ be the corresponding dual basis for V^*. Note that f_0 is a B^--stable vector of weight $-\lambda$. We shall consider the action of G on $Z = V \otimes V^*$. Here, $z = v_0 \otimes f_0$ lies in the zero weight space Z_0 of Z relative to T, and W acts on Z_0.

First we show that L fixes z. Certainly, W_L and T both fix z, so by the Bruhat decomposition in L, it suffices to show that $L \cap U^-$ fixes z. Now, $L \cap U^-$ obviously fixes f_0, so we just have to show $L \cap U^-$ fixes v_0. But this follows since no weight of the form $\lambda - \alpha_j$ is a weight of V for $j \in J$ (as $\lambda - \alpha_j$ is W-conjugate to $\lambda + \alpha_j$ which is certainly not a weight of V).

If there is a dense (X, L)-double coset in G, then there is a unique closed (X, L)-double coset in G, so $N_G(T) \subset XL$ by Corollary 1.5. Hence, applying Lemma 3.1, $(XL.z) \cap Z_0 = (XL \cap N_G(T)).z = N_G(T).z = W.z$. Also as L fixes z, $(XL.z) \cap Z_0 = (X.z) \cap Z_0 = W_X.z$. So, $W.z = W_X.z$. The result will therefore follow if we can show $W_L = \mathrm{stab}_W(z)$.

To prove this, we first claim that $W_L = \bigcap_{i \in I-J} \mathrm{stab}_W(\lambda_i)$. For, by [5, p31], W_L is the stabiliser of the subspace C_J of $E = \mathbb{R} \otimes_{\mathbb{Z}} X(T)$ defined by

$$C_J = \{e \in E \mid (e, \alpha_j) = 0 \text{ for all } j \in J, (e, \alpha_i) > 0 \text{ for all } i \in I - J\},$$

where $(\ ,\ )$ is the usual W-invariant inner product defined on E. Clearly, W_L is contained in $\bigcap_{i \in I-J} \mathrm{stab}_W(\lambda_i)$, so we take $w \in \bigcap_{i \in I-J} \mathrm{stab}_W(\lambda_i)$ and show that $w \in W_L$, or equivalently, that $w.C_J = C_J$. Take $e \in C_J$ and write $e = \sum_{i \in I-J} a_i \lambda_i$ with $a_i \in \mathbb{R}^+$, which is always possible by the definition of C_J. For $i \in I - J$, w fixes λ_i. Hence, w fixes e, so $w.C_J = C_J$, and the claim follows. Now, $\mathrm{stab}_W(z) = \mathrm{stab}_W(\langle v_0 \rangle) = \mathrm{stab}_W(\lambda)$. Also, $\mathrm{stab}_W(\lambda) = \bigcap_{i \in I-J} \mathrm{stab}_W(\lambda_i)$. Hence, $\mathrm{stab}_W(z) = W_L$, which is what we wanted. □

3.3. **Remark.** If X is of maximal rank, one can in fact show here that there is a unique closed (X, P)-double coset in G if and only if $W = W_X W_L$ is a factorisation.

3.4. **Corollary.** *If G is a connected reductive algebraic group, and X, Y are arbitrary Levi factors of parabolic subgroups of G, then there is no dense (X, Y)-double coset in G.*

Proof. We apply Lemma 3.2 with $L = Y$. We may conjugate to assume that $S = T$ is a maximal torus X, so that the condition $N_X(S) = N_X(T)$ in Lemma 3.2 is trivially satisfied. It is well known that there are no factorisations $W = W_X W_Y$ of a Weyl group as a product of two proper parabolic subgroups. Hence, the conclusion is immediate from Lemma 3.2. □

3.5. **Corollary.** *The Main Theorem holds if*

$$\begin{aligned}(G, X, Y) &= (E_7, A_7, T_1E_6),\\ (G, X, Y) &= (E_7, A_1D_6, T_1E_6),\\ (G, X, Y) &= (E_6, A_1A_5, T_1D_5).\end{aligned}$$

Proof. Again we apply Lemma 3.2 with $L = Y$. We just need to show $W \neq W_X W_L$. Providing $(G, X, L) \neq (E_7, A_7, T_1E_6)$, this follows immediately by considering orders. For the exception, consider the action of W on $X(T)$. Here, W_L is the stabiliser of λ_7. The W-orbit of λ_7 is of size 56 (for, it is well known that the weights in the 56-dimensional module $L_G(\lambda_7)$ form a single W-orbit). On the other hand, the W_X-orbit of λ_7 is of size 28. To see this, note λ_7 can be written as $\varepsilon_1 + \varepsilon_2$ where $\varepsilon_1, \ldots, \varepsilon_8$ are characters of the diagonal matrices in $SL_8(K)$ with $\varepsilon_i(\mathrm{diag}(t_1, \ldots, t_8)) = t_i$. Hence, $W_X.\lambda_7 = \{\varepsilon_i + \varepsilon_j \mid 1 \leq i < j \leq 8\}$ of order 28. □

We now consider the condition $N_X(S) = N_X(T)$ in slightly more detail. We write $\mathfrak{t}$ and $\mathfrak{g}$ for the Lie algebras of T and G respectively.

3.6. **Lemma.** *Suppose G is simple with $(G, p) \neq (C_n, 2)(n \geq 1)$. Then, $N_G(\mathfrak{t}) = N_G(T)$.*

Proof. As G is simply connected, $\mathfrak{g}$ has a Chevalley basis $\{h_\alpha, e_\beta \mid \alpha \in \Pi, \beta \in \Sigma\}$ as in [5], where $h_\alpha \in \mathfrak{t}$ and e_β spans the β-weight space of $\mathfrak{g}$. We may choose parametrisations $x_\alpha : K \to U_\alpha$ for the root subgroups of G so that the action of $x_\alpha(t)$ on the Chevalley basis is given by the formulas in [5, p64]. In particular, we have $x_\alpha(t).h_\beta = h_\beta - \langle\alpha, \beta\rangle t e_\alpha$ for $\alpha \in \Sigma, \beta \in \Pi, t \in K$, where $\langle\alpha, \beta\rangle$ is the Cartan integer corresponding to α, β.

Certainly, $N_G(T) \leq N_G(\mathfrak{t})$, so take $g \in N_G(\mathfrak{t})$. By the Bruhat decomposition, we may write $g = unu'$ with $u \in U, n \in N_G(T)$ and $u' \in U_w^-$, where $w = nT \in W$. Let $h \in \mathfrak{t}$ and set $h' = g.h$. Then, $nu'n^{-1}.nh = u^{-1}.h'$. Arguing as in Lemma 3.1, we deduce $nh = h', u'h = h$ and $uh' = h'$. This is true for all $h \in \mathfrak{t}$, so $u, u' \in C_U(\mathfrak{t})$. Therefore, it is sufficient to show $C_U(\mathfrak{t}) = \{1\}$.

Let $u \in C_U(\mathfrak{t})$. Then, we may write $u = \prod x_\alpha(t_\alpha)$ where the product is over $\alpha \in \Sigma^+$ in some fixed order, for suitable constants $t_\alpha \in K$. Suppose for a contradiction that $u \neq 1$, and choose $\alpha \in \Sigma^+$ of minimal height such that $t_\alpha \neq 0$. The formulas in [5], together with the minimality assumption on α, imply that the coefficient of e_α in $u.h_\beta$ is $-\langle\alpha, \beta\rangle t_\alpha$. By the assumption on (G, p), we may choose $\beta \in \Sigma$ such that $\langle\alpha, \beta\rangle t_\alpha \neq 0$ in K, contradicting the fact that $u \in C_U(\mathfrak{t})$. □

3.7. **Lemma.** *If $(G, X) = (E_6, F_4), (E_6, C_4)(p \neq 2)$ or (E_7, A_1F_4), then $N_X(S) = N_X(T)$ whenever $S < T$ is a maximal torus of X.*

Proof. We first show $N_X(S)$ normalises $\mathfrak{t}$. Let $\mathfrak{g}_0$ be the zero weight space of $\mathfrak{g}$ with respect to S; $N_X(S)$ certainly normalises $\mathfrak{g}_0$. So, as $\mathfrak{t} \leq \mathfrak{g}_0$, the claim will follow if we can show $\dim \mathfrak{t} = \dim \mathfrak{g}_0$.

By [17, p193], $\mathfrak{g} \downarrow X$ is $\mathrm{Lie}(X)/\Delta_{F_4}(\lambda_4), \mathrm{Lie}(X)/\Delta_{C_4}(\lambda_4)$ or $\mathrm{Lie}(X)/\Delta_{A_1}(\lambda_2) \otimes \Delta_{F_4}(\lambda_4)$ in cases $X = F_4, C_4$ or A_1F_4 respectively (here we write W_1/W_2 to denote an X-module with the same composition factors as $W_1 \oplus W_2$). The dimension of the zero weight space of each of these Weyl modules is known, and the claim is easily verified.

Hence, by Lemma 3.6, $N_X(S) \leq X \cap N_G(\mathfrak{t}) = X \cap N_G(T) = N_X(T)$. Conversely, let $n \in N_X(T)$ and let $s \in S = T \cap X$. Then, $nsn^{-1} \in T \cap X = S$, so n normalises S. □

3.8. **Corollary.** *The Main Theorem holds if*

$$\begin{aligned} (G,X,Y) &= (E_7, A_1F_4, T_1E_6), \\ (G,X,Y) &= (E_6, F_4, T_1D_5), \\ (G,X,Y) &= (E_6, C_4, T_1D_5)(p \neq 2). \end{aligned}$$

Proof. This follows easily from Lemma 3.2 and Lemma 3.7 with $L = Y$; an easy argument involving orders shows $W \neq W_X W_L$ in each case. □

4 The 27-dimensional module for E_6

To verify the Main Theorem for the cases in table 2 with $G = F_4$, we work with the 27-dimensional module for E_6. We need to compute the action of E_6 on this module explicitly in terms of root subgroups. When K is a finite field, there is a large amount of useful information computed by Cohen and Cooperstein [6] on this module. We begin by recalling the description of the 27-dimensional module for E_6 given in [6]; note that although most of [6] refers to finite fields specifically, the first two sections hold over arbitrary fields, so are valid in our case.

4.1. Let V be a 27-dimensional vector space over K whose elements are triples $x = [x_1, x_2, x_3]$ with $x_i \in M_3(K)$. We set

$$\tilde{E} = \{g \in GL(V) \mid \text{there is } \lambda \in K^* \text{ such that, for all } x \in V,\, \mathfrak{D}(g.x) = \lambda\mathfrak{D}(x)\},$$

where $\mathfrak{D} : V \to K$ is the cubic form $\mathfrak{D}(x) = \det x_1 + \det x_2 + \det x_3 - \mathrm{tr}(x_1x_2x_3)$. Then, $E = \tilde{E}'$ is a simply connected simple algebraic group of type E_6, and $\tilde{E}$ is an extension of E by a 1-dimensional torus. Let e^i_{jk} be the element $[x_1, x_2, x_3]$ of V all of whose entries are 0 except the jk-entry of x_i which is 1. Let $e_i = e^1_{ii}$ for $i = 1, 2, 3$ and $e = e_1 + e_2 + e_3$. Let $G = E_e$, a simple algebraic group of type F_4.

Note G preserves the non-degenerate symmetric bilinear form $(\ ,\ )$ given by $(x, y) = \mathrm{tr}(x_1y_1 + x_2y_3 + x_3y_2)$ for $x = [x_1, x_2, x_3], y = [y_1, y_2, y_3] \in V$. Finally, the G-equivariant map $\# : V \to V$, $x \mapsto x^\#$ is defined by the identity

$$\mathfrak{D}(x + ty) = \mathfrak{D}(x) + (x^\#, y)t + (x, y^\#)t^2 + \mathfrak{D}(y)t^3,$$

for $x, y \in V$ and t an indeterminate. Explicitly, the map $\#$ is given by

$$x^\# = [x_1^\# - x_2x_3, x_3^\# - x_1x_2, x_2^\# - x_3x_1]$$

where for $c \in M_3(K)$, $c^\#$ is the adjugate of c (the matrix whose ij-entry is the ji-cofactor of c).

4.2. We now construct a subgroup of G of type A_1C_3, by defining an action of $SL_2(K) \times Sp_6(K)$ on V and verifying that it preserves $\mathfrak{D}$ and fixes e. We realise $Sp_6(K)$ explicitly as $\{g \in SL_6(K) \mid g^T J g = J\}$ where J is the matrix $\begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$. Let V_1 and V_2 be the subspaces of V with elements

$$V_1 = \left[\begin{array}{ccc|ccc|ccc} * & * & * & 0 & 0 & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & * & 0 & 0 & 0 \\ * & * & * & 0 & 0 & * & * & * & * \end{array}\right], V_2 = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & * & * & 0 & * & * & * \\ 0 & 0 & 0 & * & * & 0 & 0 & 0 & 0 \end{array}\right]$$

where $*$ denotes an arbitrary entry. So, $V = V_1 \oplus V_2$; we first define actions of $SL_2(K) \times Sp_6(K)$ on V_1 and V_2 separately. We identify V_1 with the set of alternating 6×6 matrices by the rule

$$\left[\begin{array}{ccc|ccc|ccc} x_{11} & x_{12} & x_{13} & 0 & 0 & y_{13} & 0 & 0 & 0 \\ x_{21} & x_{22} & x_{23} & 0 & 0 & y_{23} & 0 & 0 & 0 \\ x_{31} & x_{32} & x_{33} & 0 & 0 & y_{33} & z_{31} & z_{32} & z_{33} \end{array}\right] \leftrightarrow \begin{pmatrix} 0 & z_{33} & -z_{32} & x_{11} & x_{12} & x_{13} \\ . & 0 & z_{31} & x_{21} & x_{22} & x_{23} \\ . & . & 0 & x_{31} & x_{32} & x_{33} \\ . & . & . & 0 & y_{33} & -y_{23} \\ . & . & . & . & 0 & y_{13} \\ . & . & . & . & . & 0 \end{pmatrix}$$

(where . denotes an entry chosen so that the resulting matrix is alternating). Under this identification, we let $Sp_6(K)$ act on V_1 by $g.M = gMg^T$ where $g \in Sp_6(K)$ and $M \in V_1$ is an alternating 6×6 matrix (so V_1 is just $\bigwedge^2 W$ where W is the natural module for Sp_6). Let $SL_2(K)$ act trivially on V_1. Note e corresponds to the alternating matrix J so the given action certainly fixes e. Next, note that the restrictions of $\mathfrak{D}$ and $(\, , \,)$ to V_1 coincide with two known invariants of $Sp_6(K)$ on V_1, namely the Pfaffian (see [1, p141]) and the symmetric bilinear form on $\bigwedge^2 W$ given by

$$[x \wedge y, u \wedge v] = \langle x, u \rangle \langle y, v \rangle - \langle x, v \rangle \langle y, u \rangle$$

where $x, u, y, v \in W$ and the form $\langle \, , \, \rangle$ is the canonical form defined by matrix J on W. One computes these invariants explicitly (using [1, p142, ex. 6] for the Pfaffian), to verify that $\mathrm{Pf}(x) = -\mathfrak{D}(x)$ and $[x, x'] = (x, x')$, where $x, x' \in V_1$. Hence, the action of $Sp_6(K)$ preserves the restrictions of $\mathfrak{D}$ and $(\, , \,)$ to V_1.

Next, we define an action of $SL_2(K) \times Sp_6(K)$ on V_2. Again, we will identify V_2 with $M_{6,2}(K)$ by the rule

$$\left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & x_1 & -y_1 & 0 & y_4 & y_5 & y_6 \\ 0 & 0 & 0 & x_2 & -y_2 & 0 & x_4 & x_5 & x_6 \\ 0 & 0 & 0 & x_3 & -y_3 & 0 & 0 & 0 & 0 \end{array}\right] \leftrightarrow \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{pmatrix}.$$

Under this identification, we let $SL_2(K) \times Sp_6(K)$ act on V_2 by $(g, h).M = gMh^T$ where $g \in Sp_6(K)$, $h \in SL_2(K)$ and $M \in M_{6,2}(K)$ (so V_2 is just $U \otimes W$ where U and W are the natural modules for SL_2 and Sp_6 respectively). Again, we check this preserves the restrictions of $\mathfrak{D}$ and $(\, , \,)$ to V_2; $\mathfrak{D} = 0$ on V_2, so there is nothing to check here. For $(\, , \,)$ one explicitly computes the usual symmetric bilinear form on $V_2 = U \otimes W$ and shows it coincides with the restriction of $(\ ,\ )$ to V_2.

Finally, we show that the action now defined of $SL_2(K) \times Sp_6(K)$ on V preserves $\mathfrak{D}$, hence yielding an embedding of $SL_2(K) \times Sp_6(K)$ in G. To see this, recall the map $\# : V \to V$, $x \mapsto x^{\#}$ defined by the identity in (4.1). Note $V_1^{\#} \subset V_1$, so as $SL_2(K) \times Sp_6(K)$ preserves $\mathfrak{D}$ and $(\ ,\ )$ on V_1, the action commutes with $\#$ on V_1 at least. It therefore just remains to show that for $v \in V_2, g \in SL_2(K) \times Sp_6(K)$, $(g.v)^{\#} = g.(v^{\#})$; then, the result readily follows from the identity in (4.1). Note $\#$ sends V_2 into V_1. Explicitly, making the usual identifications, $\# : v \mapsto M$ where

$$v = \begin{pmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \\ x_4 & y_4 \\ x_5 & y_5 \\ x_6 & y_6 \end{pmatrix}$$

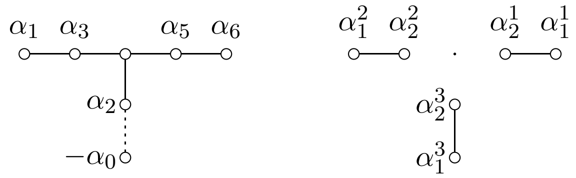
and $M \in M_6(K)$ is the matrix with $M_{ij} = x_j y_i - x_i y_j$. Given this, it is elementary to show $\#$ commutes with the action of $SL_2(K)$. So, let $g \in Sp_6(K)$. Let $x'_a = \sum_i g_{ai} x_i$, $y'_b = \sum_j g_{bj} y_j$. Then, the ab-entry of $(g.v)^{\#}$ is $x'_b y'_a - x'_a y'_b$. On the other hand, the ab-entry of $g.(v^{\#})$ is $\sum_{i,j} g_{ai} g_{bj} (x_j y_i - x_i y_j)$. A quick check shows these are equal, as required. Hence, we have constructed a subgroup of G of type $A_1 C_3$.

4.3. For $g_1, g_2, g_3 \in SL_3(K)$, let $s_{[g_1,g_2,g_3]} \in GL(V)$ be given by

$$s_{[g_1,g_2,g_3]}.x = [g_1 x_1 g_2^{-1}, g_2 x_2 g_3^{-1}, g_3 x_3 g_1^{-1}]$$

for $x = [x_1, x_2, x_3] \in V$. Clearly, $s_{[g_1,g_2,g_3]} \in E$, and the group of all such transformations defines a maximal rank subgroup H of E of type $A_2 A_2 A_2$. Now, H has a maximal torus T consisting of elements $s_{[t_1,t_2,t_3]}$ with each $t_i \in SL_3(K)$ a diagonal matrix; we take this to be a fixed maximal torus of E. For $1 \leq i, j \leq 3$, let $\varepsilon^i_j : T \to K^*$ be the character sending $s_{[t_1,t_2,t_3]} \in T$ to the jj-entry of t_i. Then, $\alpha^i_j = \varepsilon^i_j - \varepsilon^i_{j+1}$ is a root of H (hence of E) for $i = 1, 2, 3, j = 1, 2$ and $\{\alpha^i_j \mid i = 1, 2, 3, j = 1, 2\}$ gives a base for the set of roots of H.

4.4. **Lemma.** *We may choose a base $\Pi = \{\alpha_1, \ldots, \alpha_6\}$ for the roots Σ of E with respect to T such that the roots labelling the extended Dynkin diagram of E and the Dynkin diagram of H correspond as follows (here $\alpha_0 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ is the highest root of Σ):*



Moreover, the weights of V are given in table 3 in terms of the fundamental dominant weights corresponding to the base Π.
(In the table, we list the weights in the form $w = [w_1, w_2, w_3]$ where the jk-entry of w_i gives the weight $a_1\lambda_1 + \cdots + a_6\lambda_6$ of vector e^i_{jk} as $(a_1, \ldots, a_6)$.)

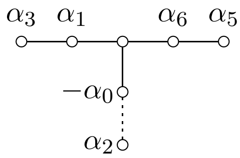
Proof. Note that H is obtained by deleting the node α_4 from the extended Dynkin diagram of E. Hence, we may certainly choose a base $\Pi = \{\alpha_1, \ldots, \alpha_6\}$ for the roots Σ of E such

Table 3: Weights in the 27-dimensional module for E_6

$$
\begin{array}{lcl}
w_1 & = & \left[\begin{array}{ccc} (-1,0,0,0,0,1) & (1,0,-1,0,0,1) & (0,0,1,-1,0,1) \\ (-1,0,0,0,1,-1) & (1,0,-1,0,1,-1) & (0,0,1,-1,1,-1) \\ (-1,0,0,1,-1,0) & (1,0,-1,1,-1,0) & (0,0,1,0,-1,0) \end{array}\right] \\
w_2 & = & \left[\begin{array}{ccc} (1,0,0,0,0,0) & (1,-1,0,0,0,0) & (1,1,0,-1,0,0) \\ (-1,0,1,0,0,0) & (-1,-1,1,0,0,0) & (-1,1,1,-1,0,0) \\ (0,0,-1,1,0,0) & (0,-1,-1,1,0,0) & (0,1,-1,0,0,0) \end{array}\right] \\
w_3 & = & \left[\begin{array}{ccc} (0,0,0,0,0,-1) & (0,0,0,0,-1,1) & (0,0,0,-1,1,0) \\ (0,1,0,0,0,-1) & (0,1,0,0,-1,1) & (0,1,0,-1,1,0) \\ (0,-1,0,1,0,-1) & (0,-1,0,1,-1,1) & (0,-1,0,0,1,0) \end{array}\right]
\end{array}
$$

that $\{\alpha_1^1, \alpha_2^1\} = \{\alpha_5, \alpha_6\}$, $\{\alpha_1^2, \alpha_2^2\} = \{\alpha_1, \alpha_3\}$ and $\{\alpha_1^3, \alpha_2^3\} = \{\alpha_2, -\alpha_0\}$. However, it is not immediately clear which of the 8 possible identifications of roots within these pairs is valid.

First, observe that



is a labelling of the extended Dynkin diagram of E_6 corresponding to a different base for the root system Σ if the central node is labelled with $\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$. Since the Weyl group acts transitively on bases, there is some $w \in W$ inducing the permutation $(\alpha_1\,\alpha_3)(\alpha_5\,\alpha_6)(\alpha_2 - \alpha_0)\ldots$ on the roots (modulo a graph automorphism of E_6). By applying w if necessary, we may therefore assume that α_1^1 corresponds to α_6.

We now construct a triality automorphism of the extended Dynkin diagram of E. Let $\tau \in GL(V)$ be the element sending $[x_1, x_2, x_3] \in V$ to $[x_3, x_1, x_2]$. It is straightforward to check that τ preserves $\mathfrak{D}$, and hence that $\tau \in E$ of order 3. Observe that $\tau s_{[g_1,g_2,g_3]} \tau^{-1} = s_{[g_3,g_1,g_2]}$; in particular, τ normalises T, hence permutes the root subgroups of E. Moreover, τ induces the graph automorphism $\alpha_{jk}^i \mapsto \alpha_{jk}^{\sigma i}$ of the Dynkin diagram of H, where σ is the permutation (123). Hence, there are the following four possibilities for the permutation of the roots $\{-\alpha_0, \alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6\}$ induced by τ: (i) $(\alpha_1 \; -\alpha_0 \; \alpha_6)(\alpha_3 \; \alpha_2 \; \alpha_5)$, (ii) $(\alpha_1 \; \alpha_2 \; \alpha_5)(\alpha_3 \; -\alpha_0 \; \alpha_6)$, (iii) $(\alpha_1 \; \alpha_2 \; \alpha_6)(\alpha_3 \; -\alpha_0 \; \alpha_5)$ or (iv) $(\alpha_1 \; -\alpha_0 \; \alpha_5)(\alpha_3 \; \alpha_2 \; \alpha_6)$. Now, corresponding to the symmetry of the extended Dynkin diagram, we can find an element τ' of the Weyl group of E that induces the permutation $(\alpha_1 \; \alpha_6 \; -\alpha_0)(\alpha_3 \; \alpha_5 \; \alpha_2)$ on this set. In cases (ii)-(iv), an easy computation shows that $\tau' \circ \tau(\alpha_4)$ is not a root (it is not even a $\mathbb{Z}$-linear combination of roots), giving a contradiction. Hence, the first case must hold and $\tau' = \tau^{-1}$. This shows that the roots may be identified as in the conclusion.

We now indicate how to compute the weight of e_{jk}^i in terms of the fundamental weights. The weight of e_{jk}^i is $\varepsilon_j^i - \varepsilon_k^{\sigma i}$ where $\sigma = (123)$. By definition $\alpha_j^i = \varepsilon_j^i - \varepsilon_{j+1}^i$ and $\varepsilon_1^i + \varepsilon_2^i + \varepsilon_3^i = 0$ for each i. Therefore, one can write the weight of e_{jk}^i in terms of the α_j^i, then in terms of the α_i using the identification given, and finally in terms of the λ_i. An elementary (if lengthy) calculation thus gives table 3. $\square$

We now use this information to compute explicit actions of root subgroups $U_{\pm\alpha_i}$ for $i = 1, \ldots, 6$ in terms of the basis elements e_{jk}^i.

4.5. **Lemma.** *We may choose parametrisations $x_\alpha : K \to U_\alpha$ for $\alpha \in \Sigma$, normalised in the standard way, such that the action of $x_\alpha(t)$ on V is given by*

$$x_\alpha(t).e^i_{jk} = \begin{cases} e^i_{jk} + te^i_{bk} & \text{if } (i,j) = (a,c) \\ e^i_{jk} - te^i_{jc} & \text{if } (i,k) = (\sigma^{-1}a, b) \\ e^i_{jk} & \text{otherwise} \end{cases}$$

in the case that $\alpha = \varepsilon^a_b - \varepsilon^a_c$ is a root of H, and also

$$x_{\alpha_4}(t).e^i_{jk} = \begin{cases} e^i_{13} - te^{\sigma^{-1}i}_{32} & \text{if } (j,k) = (1,3) \\ e^i_{23} + te^{\sigma^{-1}i}_{31} & \text{if } (j,k) = (2,3) \\ e^i_{jk} & \text{otherwise} \end{cases}$$
$$x_{-\alpha_4}(t).e^i_{jk} = \begin{cases} e^i_{31} + te^{\sigma i}_{23} & \text{if } (j,k) = (3,1) \\ e^i_{32} - te^{\sigma i}_{13} & \text{if } (j,k) = (3,2) \\ e^i_{jk} & \text{otherwise.} \end{cases}$$

(Here, σ denotes the permutation (123)*.)*

Proof. First, suppose $\alpha = \varepsilon^a_b - \varepsilon^a_c$ is a root of H. Then we may take $x_\alpha(t)$ to be the element $s_{[g_1,g_2,g_3]}$ of H where g_i is the identity matrix I if $i \neq a$ and the matrix $I + tE_{bc}$ if $i = a$. Here, E_{bc} denotes the matrix with a 1 in the bc-entry and zeros elsewhere. The action of $x_\alpha(t)$ on V is then as in the conclusion.

So, suppose $\alpha = \pm\alpha_4$. Note that the root subgroups of $Sp_6(K)$ (with the action as in (4.2)) given by elements

$$r_+(t) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & t \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \qquad r_-(t) = \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & t & 0 & 0 & 1 \end{array}\right)$$

for all $t \in K$ are T-root subgroups of E. The action of these elements on V can be explicitly calculated, to show that $r_+(t)$ (resp. $r_-(t)$) acts on e^i_{jk} as in the formula in the conclusion for $x_{\alpha_4}(t)$ (resp. $x_{-\alpha_4}(t)$). From the computation of weights, one checks that the difference between the weights of e^1_{32} and e^2_{13} is α_4, hence $r_+(K) = U_{\alpha_4}$, and a similar argument shows that $r_-(K) = U_{-\alpha_4}$. Thus, these elements do indeed correspond to the correct root subgroups of E and we may define $x_{\alpha_4}(t) = r_+(t)$ and similarly for $-\alpha_4$.

Finally, that these choices for the parametrisations are normalised in the standard way follows immediately since the corresponding root subgroups in $A_2A_2A_2$ or C_3 are. □

4.6. We can now describe G in terms of root group generators. Let $T_1 = T \cap G$ be a maximal torus of G, and let Σ_1 be the corresponding set of roots. For $\beta \in \Sigma_1$, we shall write Y_β for the corresponding T_1-root subgroup of G. Let

$$\begin{aligned} y_{\beta_0}(t) &= x_{\alpha_0}(t), \\ y_{\beta_1}(t) &= x_{\alpha_2}(t), \\ y_{\beta_2}(t) &= x_{\alpha_4}(t), \\ y_{\beta_3}(t) &= x_{\alpha_3}(t)x_{\alpha_5}(t), \\ y_{\beta_4}(t) &= x_{\alpha_1}(t)x_{\alpha_6}(t), \end{aligned}$$

and define $y_{-\beta_i}(t)$ for $i = 0, \dots, 4$ similarly using the corresponding negative root subgroups of E. Let $Y_{\pm\beta_i} = \{y_{\pm\beta_i}(t) \mid t \in K\}$, a T_1-root subgroup of G for $i = 0, \dots, 4$. By [18, 12.19], we see that the $Y_{\pm\beta_i}$ for $i = 1, \dots, 4$ generate a subgroup of E of type F_4, the centraliser of the standard graph automorphism of E_6 defined in [18, 11.4.5]. But each of these generators fixes e, hence lies in $G = F_4$, so these form a set of root group generators of G corresponding to a base $\Pi_1 = \{\beta_1, \dots, \beta_4\}$ for Σ_1. We let $(T_1, B_1; \Sigma_1, \Pi_1)$ be the corresponding root system.

In particular, we can define a subgroup Y of G of type B_4 obtained by deleting β_4 from the extended Dynkin diagram of G. Explicitly, Y is the subgroup generated by $Y_{\pm\beta_i}$ for $i = 0, \dots, 3$. Finally, an easy calculation shows that each of these generators for Y fix e_1, so we can describe Y geometrically as G_{e_1} (since B_4 is maximal in F_4). We are interested in studying (X, Y)-double cosets in G for certain subgroups X. The next lemma describes $G.e_1$, which corresponds naturally to the coset space G/Y.

4.7. **Lemma.** $G.e_1 = \{x \in V \mid (x, e) = 1, x^{\#} = 0\}$.

Proof. First, suppose $p \neq 0$. In [6, W.3], the result is proved for K a finite field, by a counting argument. We begin by deducing the result for $K = \bar{\mathbb{F}}_p$. For q a power of p, let $\sigma_q \in \Gamma L(V)$ be the Frobenius map $\sum_{i,j,k} a_{ijk} e^i_{jk} \mapsto \sum_{i,j,k} a^q_{ijk} e^i_{jk}$. Then, V^{σ_q} is a vector space over the finite field $\mathbb{F}_q$, and the finite Chevalley groups G^{σ_q} and E^{σ_q} act on V^{σ_q}. Take $x \in V$ such that $(x, e) = 1, x^{\#} = 0$. Then, for large enough q, $x \in V^{\sigma_q}$, so by the result for finite fields, there is some $g \in G^{\sigma_q}$ such that $g.e_1 = x$. Hence, the result holds for $K = \bar{\mathbb{F}}_p$. The result for an arbitrary algebraically closed field of characteristic p follows by [8, Proposition 1.1].

Now we deduce the result for $p = 0$. Again by [8], it is sufficient to prove the result for some algebraically closed field of characteristic 0. Let P be the set of all primes, and choose a non-principal ultrafilter F in the power set of P. Then, the ultraproduct

$$K = \prod_{p \in P} \bar{\mathbb{F}}_p / F$$

is a field. By Loš' Theorem (see [3, Chapter 5]) any first order property in the theory of fields that holds for all but finitely many of the $\bar{\mathbb{F}}_p$ also holds for K. Hence, the properties 'char $K \neq n$' and 'every polynomial over K of degree n has a root in K' hold for all $n \geq 1$, so K is algebraically closed of characteristic 0. Finally, the result of this lemma can be stated as a first order property, and we have shown it holds for all $\bar{\mathbb{F}}_p$; hence, it is true for K completing the proof. □

Let $x_{a,b,c} \in V$ be the element

$$x_{a,b,c} = \left[\begin{array}{ccc|ccc|ccc} a & 0 & 0 & c & 0 & 0 & b & 0 & 0 \\ 0 & b & 0 & 0 & a & 0 & 0 & c & 0 \\ 0 & 0 & c & 0 & 0 & b & 0 & 0 & a \end{array}\right].$$

By Lemma 4.7, $\Omega = \{x_{a,b,c} \mid a + b + c = 1\}$ lies in $G.e_1$.

4.8. **Lemma.** *The Main Theorem holds if*

$$(G, X, Y) \quad = \quad (F_4, A_2\tilde{A}_2, B_4).$$

Proof. Let $X = \{s_{[g_1,g_2,g_3]} \in E \mid g_1, g_2, g_3 \in SL_3(K), g_1 = g_2\}$, clearly a subgroup of G of type $A_2\tilde{A}_2$. We note that $V \downarrow X = V_1 \oplus V_2 \oplus V_3$ as a direct sum of X-modules, where V_i is the subspace spanned by e^i_{jk} for all j,k. Hence, the projection $\theta : V \to V_1$ along this direct sum is an X-equivariant morphism. We define $\bar{\theta} : G \to V_1$ by $\bar{\theta} : g \mapsto \theta(g.e_1)$. Then, $\bar{\theta}$ maps (X,Y)-double cosets to X-orbits in V_1, and it is sufficient to show that X has two disjoint closed orbits in $\bar{\theta}(G) = \theta(G.e_1) \subset V_1$. We have shown that $\theta(G.e_1)$ contains $\theta(\Omega)$. The action of X on V_1 is just the conjugation action of $SL_3(K)$, and Ω projects onto infinitely many non-conjugate semisimple elements of V_1. These have disjoint closed X-orbits, and the result follows. □

4.9. **Lemma.** *The Main Theorem holds if*

$$(G,X,Y) \quad = \quad (F_4, A_1C_3, B_4).$$

Proof. We let X be the subgroup A_1C_3 of G constructed in (4.2). Recall $V \downarrow X = V_1 \oplus V_2$ and that we have identified V_1 with the set of alternating 6×6 matrices. We consider the morphism $\theta : V \to M_6(K)$ obtained by composing the projection $\pi_1 : V_1 \to M_6(K)$ along the direct sum with the map $\pi_2 : V_1 \to M_6(K)$ where $\pi_2 : M \mapsto MJ$ (where J is as defined above) for M an alternating 6×6 matrix. We shall show that θ is an X-equivariant morphism, where the action of X on $M_6(K)$ is just conjugation by the $Sp_6(K)$-factor. For, certainly π_1 is X-equivariant, so we just need to consider π_2. Let M be an alternating 6×6 matrix and $(g,h) \in Sp_6(K) \times SL_2(K)$. Then, $\pi_2((g,h).M) = gMg^TJ = gMJg^{-1} = (g,h).(\pi_2(M))$ as required.

So, define $\bar{\theta} : G \to M_6(K)$ by $\bar{\theta} : g \mapsto \theta(g.e_1)$. As $\bar{\theta}$ sends (X,Y)-double cosets to X-orbits, we just need to show $\bar{\theta}(G)$ contains disjoint closed X-orbits. As before, it is sufficient for this to look at the subset $\theta(\Omega)$ of $\bar{\theta}(G) = \theta(G.e_1)$. One computes $\theta(x_{1+a,0,-a})$ to check that its eigenvalues are $\{0,0,a,a,-1-a,-1-a\}$. Hence, there are infinitely many matrices in $\theta(\Omega)$ such that their SL_6-conjugacy classes have disjoint closures. So, there are infinitely many closed X-orbits in $\bar{\theta}(G)$, as required. □

4.10. **Lemma.** *The Main Theorem holds if*

$$(G,X,Y) \quad = \quad (F_4, A_1G_2, B_4)(p \neq 2).$$

Proof. Let $\tau \in E$ be the element sending $[x_1,x_2,x_3] \in V$ to $[x_3,x_1,x_2]$. We showed in the proof of Lemma 4.4 that τ induces a triality automorphism of the extended Dynkin diagram of E. Let D be the subsystem subgroup of type D_4 generated by root subgroups $U_{\pm\alpha_2}, U_{\pm\alpha_3}, U_{\pm\alpha_4}, U_{\pm\alpha_5}$ of E. Then, τ normalises D and induces a standard triality automorphism on D (as in [18, 11.4.5]). In particular, we deduce by [18, 12.19] that $Z = D^\tau$ is of type G_2 and is generated by the following root elements:

$$y_{\pm\beta_2}(t), y_{\beta_1}(t)y_{\beta_3}(t), y_{-\beta_1}(t)y_{-\beta_3}(t).$$

Now, from the explicit actions defined in Lemma 4.5, the root subgroups $Y_{\pm\beta_2}$ fix e. The elements $y_{\beta_1}(t)y_{\beta_3}(t)$ and $y_{-\beta_1}(t)y_{-\beta_3}(t)$ are transformations of the form $s_{[g_1,g_2,g_3]} \in H$ with $g_1 = g_2$, hence again fix e. Thus, Z fixes e, so $Z < G$ of type G_2.

Set $X = N_G(Z)$, of type A_1G_2. Let A be the A_1 factor of X. We now compute the fixed points of Z on V. A computation involving weights (or using the explicit actions of root subgroups) shows D fixes the space spanned by e_1, hence Z fixes $e_1, \tau e_1$ and $\tau^2 e_1$. Also, Z fixes e, as $Z < G$, hence τe and $\tau^2 e$. Thus, Z centralises the 6-space $V_1 = \langle e^i_{11}, e^i_{22} + e^i_{33} \mid i = 1, 2, 3\rangle$. One checks that the composition factors of $V \downarrow Z$ are $0^6, \lambda_1^3$, hence Z can centralise no larger subspace than V_1 (we are in characteristic $p \neq 2$). As $A = C_G(Z)$, A stablilises V_1. The restriction of (,) to V_1 is non-degenerate, so $V \downarrow X = V_1 \oplus V_1^\perp$ is a direct sum of X-modules.

Let $\theta : V \to V_1$ be projection along the direct sum $V_1 \oplus V_1^\perp$, an X-equivariant morphism. Define $\bar\theta : G \to V_1$ by $\bar\theta : g \mapsto \theta(g.e_1)$. As before, it is sufficient to show there are two disjoint closed X-orbits in $\bar\theta(G)$. Note X preserves Q where $Q(u) = (u, u)$ for $u \in V_1$. Both $\theta(e_1) = e_1$ and $\theta(x_{-1,1,1}) = x_{-1,1,1}$ lie in $\bar\theta(G)$. However, $Q(e_1) = 1$ whilst $Q(x_{-1,1,1}) = 5 \neq 1$, so X has disjoint closed orbits in $\bar\theta(G)$, separated by the values of Q. □

5 The remaining cases

In this section we verify the Main Theorem for the remaining cases (G, X, Y) listed in table 2.

5.1. **Lemma.** *The Main Theorem holds if*

$$(G, X, Y) = (E_6, C_4, F_4)(p \neq 2).$$

Proof. Let G be of type E_6. By [7, 2.7], we may choose a graph automorphism δ of G stabilising a maximal torus T and an involution $h \in T$ fixed by δ such that $Y = G^\delta$ is of type F_4 and $X = G^{\delta h}$ is of type C_4.

Let $\theta(g) = g\delta(g)^{-1}$; then, $\theta \downarrow T : T \to T$ is a homomorphism of algebraic groups, with kernel $T \cap Y$, which is a maximal torus of Y. As Y is not of maximal rank, it follows that $\theta(T)$ is infinite. So, we can certainly pick $t \in T$ such that $h\theta(t)$ is not conjugate to h in G, since h is only conjugate to finitely many elements of T.

To conclude the proof, note that the morphism $g \mapsto h\theta(g)$ sends (X, Y)-double cosets to $\delta(X)$-conjugacy classes, since $h\theta(xgy) = hxgy\delta(xgy)^{-1} = \delta(x)h\theta(g)\delta(x)^{-1}$ for $x \in X, y \in Y$. The image contains h and $h\theta(t)$ which lie in disjoint closed $\delta(X)$-conjugacy classes as they are semisimple elements. Taking pre-images, we see XY and XtY have disjoint closures, as required. □

5.2. **Lemma.** *The Main Theorem holds if*

$$\begin{aligned}(G, X, Y) &= (E_6, F_4, A_1A_5),\\ (G, X, Y) &= (E_8, D_8, A_1E_7).\end{aligned}$$

Proof. Here, Y is of maximal rank; let T be a maximal torus of Y. By conjugating, we may assume T contains a maximal torus of X. Let A be the A_1-factor of Y. Since X contains long root subgroups, we may even assume that $A < X$. By Corollary 1.5 it is sufficient to show that $N_G(T) \not\leq XY$ in each case. Consider the action of G on G by conjugation. Here, $XY.A \subset X$ since Y normalises A and $A < X$. For $\alpha \in \Sigma$, the root system of G relative to T, let U_α be the corresponding T-root subgroup of G. Let α be such that $U_\alpha < A$. There is certainly a root β such that $U_\beta \not\leq X$. Now, the Weyl group of G acts transitively on Σ, so we may pick $n \in N_G(T)$ such that $nU_\alpha n^{-1} = U_\beta$; then, $n.A$ is not contained in X, so $n \notin XY$ as required. □

5.3. **Lemma.** *The Main Theorem holds if*

$$(G, X, Y) \quad = \quad (G_2, A_1\tilde{A}_1, A_2).$$

Proof. Let V be a non-degenerate 8-dimensional orthogonal space. We may embed $G = G_2 < B_3 < SO(V)$, where $V \downarrow B_3 = L_{B_3}(\lambda_3)$ and $G = \mathrm{stab}_{B_3}(z)$ for some non-singular vector z. Let $W = \langle z\rangle^{\perp}$, a 7-dimensional G-module. If $p \neq 2$, then $z \notin W$ and $V = \langle z\rangle \oplus W$; if $p = 2$, $z \in W$ and W is not irreducible as a G-module.

Fix a maximal torus T of G so that $\langle z\rangle$ is of weight 0 relative to T. Let $Y = A_2$ be the subgroup generated by the long root subgroups of G relative to T, and $X = A_1\tilde{A}_1$ be the subgroup generated by $A = \langle U_{\pm(3\alpha+2\beta)}\rangle$ and $\tilde{A} = \langle U_{\pm\alpha}\rangle$, where α (resp. β) is the short (resp. long) root in a base for the root system of G relative to T. The weights of V relative to T are $\{0^2, \pm\alpha, \pm(\alpha+\beta), \pm(2\alpha+\beta)\}$. Choose $v \in V_0$, the zero weight space of V relative to T, so that $\langle v, z\rangle = V_0$; when $p \neq 2$, we can assume $v \in W$.

Now, consider $V \downarrow X$ and $V \downarrow Y$. Let W^+ be the span of weight spaces corresponding to weights $\alpha, \alpha+\beta$ and $-(2\alpha+\beta)$; let W^- be the span of the weight spaces corresponding to the negatives of these weights. Then, a calculation involving weights shows $V \downarrow Y = \langle v, z\rangle \oplus W^+ \oplus W^-$, where the subspace $\langle v, z\rangle$ is fixed pointwise. Hence, $Y = \mathrm{stab}_G(v)^0$. Similarly, let W_1 be the span of weight spaces corresponding to weights $\pm(\alpha+\beta)$ and $\pm(2\alpha+\beta)$; let W_2 be the span of weight spaces corresponding to weights $\pm\alpha$. Then, $V \downarrow X = W_1 \oplus \langle W_2, v, z\rangle$, where $W_1 = L_{A_1}(\lambda_1) \otimes L_{\tilde{A}_1}(\lambda_1)$.

Let $\theta : V \to W_1$ be projection along the direct sum $W_1 \oplus \langle W_2, v, z\rangle$, an X-equivariant morphism. Define $\bar{\theta} : G \to W_1$ by $\bar{\theta} : g \mapsto \theta(g.v)$, so that $\bar{\theta}$ sends (X,Y)-double cosets to X-orbits in W_1. We claim that $\bar{\theta}(G) = W_1$. When $p \neq 2$, we know $SO(W) = G_2N_1$ by [11], so G acts transitively on vectors $w \in W$ with $(w,w) = (v,v)$, noting v is non-degenerate. Given $w_1 \in W_1$, we may choose $\lambda \in K$ so that $(w_1 + \lambda v, w_1 + \lambda v) = (v,v)$. Then, we may choose $g \in G$ so that $g.v = w_1 + \lambda v$; hence, $\bar{\theta}(g) = w_1$ as claimed. Now, suppose $p = 2$, when the action of G on W induces an action on the 6-dimensional symplectic space $W/\langle z\rangle$. We know $Sp_6 = G_2SO_6$ by [11]; applying a bijective morphism $Sp_6 \to SO_7$, we see that G acts transitively on complements to $\langle z\rangle$ in W. Equivalently, G acts transitively on non-degenerate 2-spaces in V containing z. Now, given $w_1 \in W_1$, $\langle v + w_1, z\rangle$ is a non-degenerate 2-space, so there exists $g \in G$ such that $g.v = \lambda(v+w_1)+\mu z$, for $\lambda, \mu \in K$. But, V/W is a 1-dimensional G-module, so $g.v = v + w$ for some $w \in W$. Hence, $\lambda = 1$ and $\bar{\theta}(g) = w_1$, proving the claim.

Thus, it is sufficient to show that X has two disjoint closed orbits in W_1. But, $W_1 = L_{A_1}(\lambda_1) \otimes L_{\tilde{A}_1}(\lambda_1)$, which can be constructed as $M_2(K)$ with $SL_2(K)\times SL_2(K)$-action given by $(g,h).M = gMh^T$ for $(g,h) \in SL_2(K)\times SL_2(K), M \in M_2(K)$. It is clear that this action preserves determinants, hence there are infinitely many closed orbits separated by the values of det. □

We summarise the results so far.

5.4. **Proposition.** *The Main Theorem holds if X and Y are maximal reductive connected subgroups of G.*

Proof. If there is a dense (X,Y)-double coset in G, then $\dim X + \dim Y \geq \dim G$, and if X and Y are conjugate then the result holds by Proposition 1.6. Moreover, if δ is a graph automorphism of G, the result clearly holds for (G,X,Y) if and only if it holds for $(G,\delta X,\delta Y)$, so we only need to consider the possibilites for X, Y up to graph automorphisms. Hence, it is sufficient to consider X and Y as in table 2 by Proposition 2.5. The cases

$(G, X, Y) = (G_2, A_2, \tilde{A}_2)$ and (F_4, B_4, C_4) are factorisations by [11]. For the remaining cases, we have shown in §3 – §5 that the Main Theorem holds. □

6 Completion of proof

We can now complete the proof of the Main Theorem. We may assume that X, Y are connected. So, let X, Y be arbitrary connected reductive subgroups of G. Embed $X \leq \bar{X}$ and $Y \leq \bar{Y}$ where $\bar{X}, \bar{Y}$ are maximal reductive connected subgroups of G. The result then follows by Proposition 5.4, unless $G = \bar{X}\bar{Y}$ is a factorisation but $G \neq XY$.

This can only occur if $(G, \bar{X}, \bar{Y}, p) = (G_2, A_2, \tilde{A}_2, 3)$ or $(F_4, B_4, C_4, 2)$ by [11]. In the former case, the largest reductive subgroup of A_2 is A_1T_1 of dimension 4, so here $\dim X + \dim Y < \dim G$ and there is no dense double coset. Thus, we need to consider $G = F_4, X \leq C_4$ and $Y \leq B_4$ with $p = 2$. Moreover, one of $X \not\geq \tilde{D}_4$ or $T \not\geq D_4$ must hold (otherwise $G = XY$ by [11]), and applying a graph automorphism if necessary, we will assume the former.

Thus, we take $Y = B_4, X < C_4$ with $X \not\geq \tilde{D}_4$, and claim there is no dense (X, Y)-double coset. Note $\dim X \geq \dim G - \dim Y = 16$. We list the maximal reductive subgroups of $C_4 = Sp(V)$ of at least dimension 16 using Lemma 2.2 and Lemma 2.3 (noting $\dim X \geq 2\dim V$), to deduce one of the following holds

(i) $X < SO(V) = \tilde{D}_4$;

(ii) $X \leq N_2 = A_1C_3$;

(iii) $X \leq N_4 = C_2C_2$;

(iv) X lies in an $\tilde{A}_3$-parabolic;

(v) X lies in a C_3-parabolic.

Case (ii) is proved in Lemma 4.9, and in case (iii), X is conjugate to a subgroup of Y, so this follows by Proposition 1.6. In case (iv), dimension implies $X = T_1\tilde{A}_3$, whence X is a Levi subgroup that lies in $\tilde{D}_4$, as in case (i). Finally, in case (v), we may assume X is semisimple, else X is normalised by a maximal torus of G and we are in one of cases (i)-(iv). Then, we can argue as in Proposition 2.4 to show that $X = B_3$ or C_3, then apply [12] to show that X is again normalised by a maximal torus of G, so we are in (i)-(iv) again.

Hence, we are left with case (i), when X is a subgroup of $\tilde{D}_4$. Again we list the maximal reductive subgroups of $\tilde{D}_4 = SO(V)$, to deduce

(i) $X \leq B_3$ where $V \downarrow B_3 = L_{B_3}(\lambda_3)$;

(ii) $X \leq N_1 = B_3$;

(iii) X lies in an $\tilde{A}_3$-parabolic, when we deduce by dimension that X is a Levi factor of type $T_1\tilde{A}_3$.

Now, let $Z = \tilde{D}_4$. Let $\theta : Z/Z \cap Y \to G/Y$ be the morphism $\theta : z(Z \cap Y) \mapsto zY$, a bijection since $G = ZY$. By [10, p56, ex. 4], θ is an open map, hence a closed map as θ is bijective. Thus, it is sufficient to show there are two disjoint closed $(X, Z \cap Y)$-double cosets in Z. Now, $(Z \cap Y)^0 = \tilde{A}_1\tilde{A}_1\tilde{A}_1\tilde{A}_1$. We need to consider three cases $X = B_3, N_1$ or $T_1\tilde{A}_3$. In each case, X contains root subgroups of Z, but there exist root subgroups of Z lying outside of X, so we may repeat the argument of Lemma 5.2 to deduce the result.

This completes the proof of the Main Theorem.

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