CORRIGENDA TO 'HECKE-CLIFFORD SUPERALGEBRAS, CRYSTALS OF TYPE $A_{2\ell}^{(2)}$ AND MODULAR BRANCHING RULES FOR $\widehat{S}_n$'

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1. We are grateful to Steffen Koenig and Steffen Oppermann for pointing out that there is a gap in the proof of Lemma 5.22 of [1]. We do not know at the moment whether Lemma 5.22 is correct or not. However, we claim that it is not needed anywhere in the paper if the following changes are made.

a) Drop Lemma 5.22.

b) Amend Lemmas 5.23 and 6.16 as follows.

Lemma 5.23. *Take $i, j \in I$ with $i \neq j$ and set $k = -\langle h_i, \alpha_j \rangle$. Let M be an irreducible module in $\mathrm{Rep}_I\, \mathcal{H}_n$.*

(i) *There exists a unique integer a with $0 \leq a \leq k$ such that for every $m \geq 0$ we have*
$$\varepsilon_i(\tilde{f}_i^m \tilde{f}_j M) = m + \varepsilon_i(M) - a.$$

(ii) *Assume $m \geq k$. Then a copy of $\tilde{f}_i^m \tilde{f}_j M$ appears in the cosocle of*
$$\mathrm{ind}\, \tilde{f}_i^{m-k} M \circledast L(i^a j i^{k-a}).$$

(iii) *Assume $0 \leq m < k \leq m + \varepsilon$. Then a copy of $\tilde{f}_i^m \tilde{f}_j M$ appears in the cosocle of*
$$\mathrm{ind}\, \tilde{e}_i^{k-m} M \circledast L(i^a j i^{k-a}).$$

Proof. Let $\varepsilon = \varepsilon_i(M)$ and write $M = \tilde{f}_i^\varepsilon N$ for irreducible $N \in \mathrm{Rep}_I\, \mathcal{H}_{n-\varepsilon}$ with $\varepsilon_i(N) = 0$. It suffices to prove (i) for any fixed choice of m, the conclusion for all other $m \geq 0$ then follows immediately by (5.11). So take $m \geq 0$ with $k \leq m + \varepsilon$. Note that $\tilde{f}_i^m \tilde{f}_j M = \tilde{f}_i^m \tilde{f}_j \tilde{f}_i^\varepsilon N$ is a quotient of
$$\begin{cases} \mathrm{ind}\, N \circledast L(i^\varepsilon) \circledast L(j) \circledast L(i)^{\circledast k} \circledast L(i^{m-k}) & \text{if } m \geq k, \\ \mathrm{ind}\, N \circledast L(i^{m+\varepsilon-k}) \circledast L(i)^{\circledast (k-m)} \circledast L(j) \circledast L(i^m) & \text{if } m < k, \end{cases}$$
which by Lemma 5.19 has a filtration with factors isomorphic to
$$\begin{cases} F_a := \mathrm{ind}\, N \circledast L(i^\varepsilon) \circledast L(i^a j i^{k-a}) \circledast L(i^{m-k}) & \text{if } m \geq k, \\ F_a := \mathrm{ind}\, N \circledast L(i^{m+\varepsilon-k}) \circledast L(i^a j i^{k-a}) & \text{if } m < k, \end{cases}$$
for $0 \leq a \leq k$, each appearing with some multiplicity. So $\tilde{f}_i^m \tilde{f}_j M$ is a quotient of some such factor, and to prove (i) it remains to show that $\varepsilon_i(L) = \varepsilon + m - a$ for any irreducible quotient L of F_a. The inequality $\varepsilon_i(L) \leq \varepsilon + m - a$ is clear from the Shuffle Lemma. On the other hand, by transitivity of induction and Lemma 5.21, $F_a \cong \mathrm{ind}\, N \circledast (\mathrm{ind}\, L(i^a j i^{k-a}) \circledast L(i^{\varepsilon+m-k}))$.

So by Frobenius reciprocity, the irreducible module $N \circledast (\operatorname{ind} L(i^a j i^{k-a}) \circledast L(i^{\varepsilon+m-k}))$ is contained in $\operatorname{res}_{n-\varepsilon,m+1+\varepsilon} L$. Hence $\varepsilon_i(L) \geq \varepsilon + m - a$.

To complete the proof of (ii) and (iii), by Lemma 5.21, we also have $F_a \cong \operatorname{ind} N \circledast L(i^{\varepsilon+m-k}) \circledast L(i^a j i^{k-a})$, and by the Shuffle Lemma, the only composition factors K of F_a with $\varepsilon_i(K) = \varepsilon + m - a$ come from its quotient

$$\operatorname{ind} \tilde{f}_i^{m-k+\varepsilon} N \circledast L(i^a j i^{k-a}).$$

The latter is $\operatorname{ind} \tilde{f}_i^{m-k} M \circledast L(i^a j i^{k-a})$ if $m \geq k$ and $\operatorname{ind} \tilde{e}_i^{k-m} M \circledast L(i^a j i^{k-a})$ otherwise. $\square$

Lemma 6.16 *Let $i, j \in I$ with $i \neq j$. Let M be an irreducible module in $\operatorname{Rep} \mathcal{H}_n^\lambda$ such that $\varphi_j(M) > 0$. Then, $\varphi_i(\tilde{f}_j M) - \varepsilon_i(\tilde{f}_j M) \leq \varphi_i(M) - \varepsilon_i(M) - \langle h_i, \alpha_j \rangle$.*

Proof. Let $\varepsilon = \varepsilon_i(M), \varphi = \varphi_i(M)$ and $k = -\langle h_i, \alpha_j \rangle$. By Lemma 5.23, there exist unique $a, b \geq 0$ with $a + b = k$ such that $\varepsilon_i(\tilde{f}_j M) = \varepsilon - a$. We need to show that $\varphi_i(\tilde{f}_j M) \leq \varphi + b$, which follows if we can show that $\operatorname{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ for all $m > \varphi + b$. We claim that

$$\varepsilon_i^*(\tilde{f}_i^m \tilde{f}_j M) \geq \varepsilon_i^*(\tilde{f}_i^{m-b} M)$$

for all $m > \varphi + b$. Given the claim, we know by the definition of φ, Corollary 6.13 and Lemma 6.15 that $\varepsilon_i^*(\tilde{f}_i^{m-b} M) > \langle h_i, \lambda \rangle$ for all $m > \varphi + b$. So the claim implies that $\varepsilon_i^*(\tilde{f}_i^m \tilde{f}_j M) > \langle h_i, \lambda \rangle$ for all $m > \varphi + b$, hence by Corollary 6.13 once more, $\operatorname{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ as required.

To prove the claim, note that $k \leq m + \varepsilon$, so by Lemma 5.23(ii),(iii) that there is a surjection

$$\operatorname{ind}_{n-\varepsilon,\varepsilon+m-k,k+1}^{n+m+1} N \circledast L(i^{\varepsilon+m-k}) \circledast L(i^a j i^b) \twoheadrightarrow \tilde{f}_i^m \tilde{f}_j M,$$

where $N = \tilde{e}_i^\varepsilon M$. By Lemma 5.19, $\operatorname{res}_{a,b+1}^{a+b+1} L(i^a j i^b) \cong L(i^a) \circledast L(j i^b)$. Hence there is a surjection $\operatorname{ind}_{a,b+1}^{a+b+1} L(i^a) \circledast L(j i^b) \twoheadrightarrow L(i^a j i^b)$. Combining, we have proved existence of a surjection

$$\operatorname{ind}_{n-\varepsilon,\varepsilon+m-b,b+1}^{n+m+1} N \circledast L(i^{\varepsilon+m-b}) \circledast L(j i^b) \twoheadrightarrow \tilde{f}_i^m \tilde{f}_j M.$$

Hence by Frobenius reciprocity there is a non-zero map

$$(\operatorname{ind}_{n-\varepsilon,\varepsilon+m-b}^{n+m-b} N \circledast L(i^{\varepsilon+m-b})) \circledast L(j i^b) \to \operatorname{res}_{n+m-b,b+1}^{n+m+1} \tilde{f}_i^m \tilde{f}_j M.$$

Since the left-hand module has irreducible cosocle $\tilde{f}_i^{m-b} M \circledast L(j i^b)$, we deduce that $\tilde{f}_i^m \tilde{f}_j M$ has a constituent isomorphic to $\tilde{f}_i^{m-b} M$ on restriction to the subalgebra $\mathcal{H}_{n+m-b} \subseteq \mathcal{H}_{n+m+1}$. This implies the claim. $\square$

2. Anton Cox has pointed out that the classification of blocks in §8-d is incomplete. More precisely, Theorem 8.12 (though correct) is not sufficient to deduce Corollary 8.13. Therefore the validity of this corollary remains an open problem. In all odd levels in the degenerate case, the corollary has been established by a different method by Oliver Ruff [2].

3. Shunsuke Tsuchioka has pointed out that the proof of the surjectivity of the map π in Theorem 7.17 is incomplete over the ground ring $\mathbb{Z}$. This is easily fixed using Theorem 7.9 as explained in detail below; see also the proof of [3, Theorem 6.14] for a variation on the same argument in a closely related context.

Fix $n \geq 0$ and let M be an irreducible module in $\mathrm{Rep}_I \mathcal{H}_n$. Choose $\lambda \in P_+$ so that every irreducible in $\mathrm{Rep}_I \mathcal{H}_n$ with the same central character as M factors through to the quotient algebra $\mathcal{H}_n^\lambda$. Applying Theorem 7.9, we can find a homogeneous element $u_M \in U_{\mathbb{Z}}^-$ such that $u_M[\mathbf{1}_{\boldsymbol{\lambda}}] = [P_M]$. Let $\tau : U_{\mathbb{Z}}^- \xrightarrow{\sim} U_{\mathbb{Z}}^+$ be the algebra anti-isomorphism with $\tau(f_i^{(r)}) = e_i^{(r)}$ for all i, r. Then we claim that

$$\pi(\tau(u_M)) = \delta_M,$$

from which the surjectivity of π is clear.

To prove the claim, it suffices by weight space considerations to show that $\pi(\tau(u_M))([L]) = \delta_{[M],[L]}$ for every irreducible $L \in \mathrm{Rep}_I \mathcal{H}_n$ with the same central character as M. For this, we have in $K(\lambda)$ that

$$\pi(\tau(u_M))([L])[\mathbf{1}_{\boldsymbol{\lambda}}] = \tau(u_M)[L],$$

as is easy to check from the definitions in the case that u_M is a monomial, then follows in general by linearity. Finally we compute using Lemma 7.6:

$$([\mathbf{1}_{\boldsymbol{\lambda}}], \tau(u_M)[L]) = (u_M[\mathbf{1}_{\boldsymbol{\lambda}}], [L]) = ([P_M], [L]) = \delta_{[M],[L]}.$$

Hence $\pi(\tau(u_M))([L]) = \delta_{[M],[L]}$ as claimed.

4. Tsuchioke has also pointed out that in §6-c we asserted that for M irreducible of type `Q`, the odd involution θ_M of M lifts to a unique odd involution of its projective cover. However there is no uniqueness in general here, so the word “unique” should be omitted. This does not cause any problems in the subsequent development.

References

[1] J. Brundan and A. Kleshchev, Hecke-Clifford superalgebras, crystals of type $A_{2\ell}^{(2)}$ and modular branching rules for $\widehat{S}_n$, *Representation Theory* **5** (2001), 317–403.

[2] O. Ruff, Centers of cyclotomic Sergeev superalgebras; `arXiv:0811.3991`.

[3] S. Tsuchioka, Hecke-Clifford superalgebras and crystals of type $D_l^{(2)}$; `arXiv:0907.4936`.

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