HECKE-CLIFFORD SUPERALGEBRAS, CRYSTALS OF TYPE $A_{2\ell}^{(2)}$ AND MODULAR BRANCHING RULES FOR $\widehat{S}_n$

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Abstract. This paper is concerned with the modular representation theory of the affine Hecke-Clifford superalgebra, the cyclotomic Hecke-Clifford superalgebras, and projective representations of the symmetric group. Our approach exploits crystal graphs of affine Kac-Moody algebras.

1. Introduction

In [LLT], Lascoux, Leclerc and Thibon made the startling combinatorial observation that the crystal graph of the basic representation of the affine Kac-Moody algebra $\mathfrak{g} = A_\ell^{(1)}$, determined explicitly by Misra and Miwa [MM], coincides with the modular branching graph for the symmetric group S_n in characteristic $p = \ell + 1$, as in [K_1]. The same observation applies to the modular branching graph for the associated complex Iwahori-Hecke algebras at a primitive $(\ell+1)$th root of unity, see [B]. In this latter case, Lascoux, Leclerc and Thibon conjectured moreover that the coefficients of the canonical basis of the basic representation coincide with the decomposition numbers of the Iwahori-Hecke algebras.

This conjecture was proved by Ariki [A_1][1]. More generally, Ariki established a similar result connecting the canonical basis of an arbitrary integrable highest weight module of $\mathfrak{g}$ to the representation theory of a corresponding cyclotomic Hecke algebra, as defined in [AK]. Note that this work is concerned with the cyclotomic Hecke algebras over the ground field $\mathbb{C}$, but Ariki and Mathas [A_2, AM] were later able to extend the classification of the irreducible modules (but not the result on decomposition numbers) to arbitrary fields. For further developments related to the LLT conjecture, see [LT_1, VV, Sch].

Subsequently, Grojnowski and Vazirani [G_1, G_2, GV, V_1, V_2] have developed a new approach to (amongst other things) the classification of the irreducible modules of the cyclotomic Hecke algebras. The approach is valid over an arbitrary ground field, and is entirely independent of the "Specht module theory" that plays an important role in Ariki's work. Branching rules are built in from the outset (like in [OV]), resulting in an explanation and generalization of the link between modular branching rules and crystal graphs. The methods are purely algebraic, exploiting affine Hecke algebras in the spirit of [BZ, Z_1] and others. On the other hand, Ariki's result on decomposition numbers does not follow, since that ultimately depends on the geometric work of Kazhdan, Lusztig and Ginzburg.

In this article, we use Grojnowski's methods to develop a parallel theory in the twisted case: we replace the affine Hecke algebras with the affine Hecke-Clifford superalgebras of Jones and Nazarov [JN], and the Kac-Moody algebra $A_\ell^{(1)}$ with the twisted algebra $\mathfrak{g} = A_{2\ell}^{(2)}$.

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[1]Grojnowski has informed us that the result can also be proved following his note "Representations of affine Hecke algebras (and affine quantum GL_n) at roots of unity", *Internat. Math. Res. Notices* **5** (1994), 215–217.

In particular, we obtain an algebraic construction purely in terms of the representation theory of Hecke-Clifford superalgebras of the plus part $U_{\mathbb{Z}}^+$ of the enveloping algebra of $\mathfrak{g}$, as well as of Kashiwara's highest weight crystals $B(\infty)$ and $B(\lambda)$ for each dominant weight λ. These emerge as the modular branching graphs of the Hecke-Clifford superalgebras. Note there is at present no analogue of the notion of Specht module in our theory, underlining the importance of Grojnowski's methods. However, we do not obtain an analogue of Ariki's result on decomposition numbers.

As we work over an arbitrary ground field, the results of the article have applications to the modular representation theory of the double covers $\widehat{S}_n$ of the symmeric groups, as was predicted originally by Leclerc and Thibon [LT$_2$], see also [BK$_2$]. In particular, the parametrization of irreducibles, classification of blocks and analogues of the modular branching rules of the symmetric group for the double covers over fields of odd characteristic follow from the special case $\lambda = \Lambda_0$ of our main results. These matters are discussed in the final section of the paper, §9.

Let us now describe the main results in more detail. Let $\mathcal{H}_n$ denote the affine Hecke-Clifford superalgebra of [JN], over an algebraically closed field F of characteristic different from 2 and at defining parameter a primitive $(2\ell+1)$-th root of unity $q \in F^\times$. All results also have analogues in the degenerate case $q = 1$, working instead with the affine Sergeev superalgebra of Nazarov [N], when the field F should be taken to be of characteristic $(2\ell+1)$.

We consider

$$K(\infty) = \bigoplus_{n \geq 0} K(\mathrm{Rep}_I \mathcal{H}_n),$$

the sum of the Grothendieck groups of integral $\mathbb{Z}_2$-graded representations of $\mathcal{H}_n$ for all n (see §4-d for the precise definition). In a familiar way (cf. [Z$_2$]), $K(\infty)$ has a natural structure as a commutative graded Hopf algebra over $\mathbb{Z}$, multiplication being induced by induction and comultiplication being induced by restriction. Hence, the graded dual $K(\infty)^*$ is a cocommutative graded Hopf algebra over $\mathbb{Z}$.

Next, let $\mathfrak{g}$ denote the twisted affine Kac-Moody algebra of type $A_{2\ell}^{(2)}$. We will adopt standard Lie theoretic notation for the root system of $\mathfrak{g}$, summarized in more detail in the main body of the article. In particular, $U_{\mathbb{Q}} = U_{\mathbb{Q}}^- U_{\mathbb{Q}}^0 U_{\mathbb{Q}}^+$ denotes the $\mathbb{Q}$-form of the universal enveloping algebra of $\mathfrak{g}$ generated by the Chevalley generators $e_i, f_i, h_i (i \in I)$. Also $U_{\mathbb{Z}} = U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+$ denotes the Kostant $\mathbb{Z}$-form of $U_{\mathbb{Q}}$. The first main theorem (Theorem 7.17) identifies $K(\infty)^*$ with $U_{\mathbb{Z}}^+$, viewing the latter as a graded Hopf algebra over $\mathbb{Z}$ via the principal grading:

Theorem A. *$K(\infty)^*$ and $U_{\mathbb{Z}}^+$ are isomorphic as graded Hopf algebras.*

We also introduce for the first time for each dominant integral weight $\lambda \in P_+$ a finite dimensional quotient superalgebra $\mathcal{H}_n^\lambda$ of $\mathcal{H}_n$. These superalgebras play the role of the cyclotomic Hecke algebras in Grojnowski's theory. In the special case $\lambda = \Lambda_0$, $\mathcal{H}_n^\lambda$ is the finite Hecke-Clifford superalgebra introduced by Olshanski [O]. Consider the sum of the Grothendieck groups

$$K(\lambda) = \bigoplus_{n \geq 0} K(\mathrm{Rep}\, \mathcal{H}_n^\lambda)$$

of finite dimensional $\mathbb{Z}_2$-graded $\mathcal{H}_n^\lambda$-modules for all n. In this case, we can identify the graded dual $K(\lambda)^*$ with

$$K(\lambda)^* = \bigoplus_{n \geq 0} K(\mathrm{Proj}\, \mathcal{H}_n^\lambda)$$

where $K(\operatorname{Proj}\mathcal{H}_n^\lambda)$ denotes the Grothendieck group of finite dimensional $\mathbb{Z}_2$-graded projectives. It turns out that the natural Cartan map $K(\lambda)^* \to K(\lambda)$ is injective (Theorem 7.10), so we can view both $K(\lambda)^*$ and $K(\lambda)$ as lattices in $K(\lambda)_\mathbb{Q} = \mathbb{Q} \otimes_\mathbb{Z} K(\lambda) = \mathbb{Q} \otimes_\mathbb{Z} K(\lambda)^*$.

Each $K(\lambda)$ has a natural structure of right $K(\infty)$-comodule, with $K(\lambda)^* \subset K(\lambda)$ being a subcomodule. In other words, according to Theorem A, $K(\lambda)^*$ and $K(\lambda)$ are left $U_\mathbb{Z}^+$-modules. The action of the Chevalley generator e_i here is essentially a refinement of the restriction functors from $\mathcal{H}_n^\lambda$ to $\mathcal{H}_{n-1}^\lambda$. We show moreover, by considering refinements of induction, that the action of $U_\mathbb{Z}^+$ extends to an action of all of $U_\mathbb{Z}$ on both $K(\lambda)$ and $K(\lambda)^*$. Hence, on extending scalars, we have an action of $U_\mathbb{Q}$ on $K(\lambda)_\mathbb{Q}$, with $K(\lambda)^* \subset K(\lambda)$ being two different integral forms. The second main theorem (Theorem 7.16) is the following:

Theorem B. *For each $\lambda \in P_+$, $K(\lambda)_\mathbb{Q}$ is the integrable highest weight $U_\mathbb{Q}$-module of highest weight λ, with highest weight vector $[\mathbf{1}_\lambda]$ corresponding to the irreducible $\mathcal{H}_0^\lambda$-module. Moreover $K(\lambda)^* \subset K(\lambda)$ are integral forms for $K(\lambda)_\mathbb{Q}$ with $K(\lambda)^* = U_\mathbb{Z}^-[\mathbf{1}_\lambda]$ and $K(\lambda)$ being the dual lattice under the Shapovalov form.*

Now let $B(\infty)$ denote the set of isomorphism classes of irreducible integral $\mathcal{H}_n$-modules for all $n \geq 0$. This has a natural crystal graph structure, the action of the crystal operators $\tilde{e}_i$ being defined by considering the *socle* of the restriction of an irreducible $\mathcal{H}_n$-module to $\mathcal{H}_{n-1}$. Similarly, writing $B(\lambda)$ for the set of isomorphism classes of irreducible $\mathcal{H}_n^\lambda$-modules for all $n \geq 0$, each $B(\lambda)$ has a natural crystal structure describing branching rules between the algebras $\mathcal{H}_n^\lambda$ and $\mathcal{H}_{n-1}^\lambda$. We stress the crystal structures on $B(\infty)$ and each $B(\lambda)$ are defined purely in terms of the representation theory of the Hecke-Clifford superalgebras. The next main result (Theorems 8.10 and 8.11) identifies the crystals:

Theorem C. *The crystal $B(\infty)$ is isomorphic to Kashiwara's crystal associated to the crystal base of $U_\mathbb{Q}^-$. Moreover, for each $\lambda \in P_+$, the crystal $B(\lambda)$ is isomorphic to Kashiwara's crystal associated to the integrable highest weight $U_\mathbb{Q}$-module of highest weight λ.*

Acknowledgements. We would like to express our debt to the beautiful ideas of Ian Grojnowski in [G$_1$]. Many of the proofs here, and certainly the overall strategy adopted in the article, are exactly as in Grojnowski's work. We would also like to thank Monica Vazirani for explaining [GV] to us, in discussions which initiated the present work.

2. Affine Hecke-Clifford superalgebras

§2-a. **Ground field and parameters.** Let F be an algebraically closed field of characteristic different from 2, and choose $q \in F^\times$ such that

either q is a primitive odd root of unity (including the possibly $q = 1$),
or q is not a root of unity at all.

Let h be the "quantum characteristic", i.e. the smallest positive integer such that

$$q^{1-h} + q^{3-h} + \cdots + q^{h-3} + q^{h-1} = 0$$

or ∞ in case no such integer exists. We refer to the case $q \neq 1$ as the *quantum case* and $q = 1$ as the *degenerate case*. All the results will apply to either situation, but there are sufficiently many differences that for the purpose of exposition we will work in the quantum case in the main body of the text, with a summary of modifications in the degenerate case given at the end of each section whenever necessary. We set

$$\xi = q - q^{-1}$$

as a convenient shorthand.

§2-b. **Modules over superalgebras.** We will use freely the basic notions of superalgebra, referring the reader to [Le, ch.I], [Man, ch.3, §§1–2] and [BK$_2$, §2]. We will denote the *parity* of a homogeneous vector v of a vector superspace by $\bar{v} \in \mathbb{Z}_2$. By a *superalgebra*, we mean a $\mathbb{Z}_2$-graded associative algebra over the fixed field F. If A and B are two superalgebras, then $A \otimes B = A \otimes_F B$ is again a superalgebra with multiplication satisfying

$$(a_1 \otimes b_1)(a_2 \otimes b_2) = (-1)^{\bar{b}_1 \bar{a}_2} a_1 a_2 \otimes b_1 b_2$$

for $a_i \in A, b_i \in B$. Note this and other such expressions only make sense for *homogeneous* b_1, a_2: the intended meaning for arbitrary elements is to be obtained by extending linearly from the homogeneous case.

If A is a superalgebra, an *A-module* means a $\mathbb{Z}_2$-graded left A-module. A morphism $f : M \to N$ of A-modules M and N means a (not necessarily homogeneous) linear map such that $f(am) = (-1)^{\bar{f}\bar{a}} a f(m)$ for all $a \in A, m \in M$. The category of all such A-modules is denoted A-mod. By a *submodule* of an A-module, we always mean a *graded* submodule unless we explicitly say otherwise. We have the *parity change functor*

$$\Pi : A\text{-mod} \to A\text{-mod}. \tag{2.1}$$

For an object M, ΠM is the same underlying vector space but with the opposite $\mathbb{Z}_2$-grading. The new action of $a \in A$ on $m \in \Pi M$ is defined in terms of the old action by $a \cdot m := (-1)^{\bar{a}} am$.

It will occasionally be necessary to consider bimodules over two superalgebras A, B: an (A, B)-bimodule is an A-module M, as in the previous paragraph, which is also a $\mathbb{Z}_2$-graded right B-module such that $(am)b = a(mb)$ for all $a \in A, b \in B, m \in M$. Note a morphism $f : M \to N$ of (A, B)-bimodules means a morphism of A-modules as in the previous paragraph such that $f(mb) = f(m)b$ for all $m \in M, b \in B$. This gives us the category A-mod-B of all (A, B)-bimodules. Also if M is an (A, B)-bimodule, ΠM denotes the A-module defined as in the previous paragraph, with the right B-action on ΠM being the same as the original action on M.

If M is a finite dimensional irreducible A-module, Schur's lemma (e.g. [BK$_2$, §2]) says that $\mathrm{End}_A(M)$ is either one dimensional, or two dimensional on basis id_M, θ_M where θ_M is an odd involution of M, unique up to a sign. In the former case, we call M an *irreducible of type* $\mathtt{M}$, in the latter case M is an *irreducible of type* $\mathtt{Q}$. We will occasionally write $M \simeq N$ (as opposed to the usual $M \cong N$) to indicate that there is actually an *even* isomorphism between A-modules M and N. For instance, if M is an irreducible of type $\mathtt{Q}$, then $M \simeq \Pi M$, while if M is irreducible of type $\mathtt{M}$ then $M \cong \Pi M$ but $M \not\simeq \Pi M$.

Given another superalgebra B and A-, B-modules M, N respectively, $M \otimes N = M \otimes_F N$ has a natural structure of $A \otimes B$-module with

$$(a \otimes b)(m \otimes n) = (-1)^{\bar{b}\bar{m}} am \otimes bn.$$

We will call this the *outer* tensor product of M and N and denote it $M \boxtimes N$. If M and N are finite dimensional irreducibles, $M \boxtimes N$ need not be irreducible (unlike the purely even case). Indeed, if both M and N are of type $\mathtt{M}$, then $M \boxtimes N$ is also irreducible of type $\mathtt{M}$, while if one is of type $\mathtt{M}$ and the other of type $\mathtt{Q}$, then $M \boxtimes N$ is irreducible of type $\mathtt{Q}$. In either of these two cases, it will be convenient to write $M \circledast N$ in place of $M \boxtimes N$. But if both M and N are of type $\mathtt{Q}$, then $M \boxtimes N$ is decomposable: let $\theta_M : M \to M$ be an odd involution of M as an A-module and $\theta_N : N \to N$ be an odd involution of N as a B-module. Then

$$\theta_M \otimes \theta_N : M \boxtimes N \to M \boxtimes N, \quad m \otimes n \mapsto (-1)^{\bar{m}} \theta_M(m) \otimes \theta_N(n)$$

is an even $A \otimes B$-automorphism of $M \boxtimes N$ whose square is -1. Therefore $M \boxtimes N$ decomposes as a direct sum of two $A \otimes B$-modules, namely, the $\pm\sqrt{-1}$-eigenspaces of the linear map $\theta_M \otimes \theta_N$. The map $\theta_M \otimes \mathrm{id}_N$ then gives an odd isomorphism between the two summands as $A \otimes B$-modules. In this case, we pick either summand, and denote it $M \circledast N$: it is an irreducible $A \otimes B$-module of type $\mathtt{M}$. Thus,

$$M \boxtimes N \simeq \begin{cases} M \circledast N \oplus \Pi(M \circledast N), & \text{if } M \text{ and } N \text{ are both of type } \mathtt{Q}, \\ M \circledast N, & \text{otherwise.} \end{cases}$$

We stress that $M \circledast N$ is in general only well-defined up to isomorphism.

Finally, we make some remarks about antiautomorphisms and duality. In this paper, all antiautomorphisms $\tau : A \to A$ of a superalgebra A will be *unsigned*, so satisfy $\tau(ab) = \tau(b)\tau(a)$. If M is a finite dimensional A-module, then we can use τ to make the dual space $M^* = \mathrm{Hom}_F(M, F)$ into an A-module by defining $(af)(m) = f(\tau(a)m)$ for all $a \in A, f \in M^*, m \in M$. We will denote the resulting module by M^τ. We often use the fact that there is a natural even isomorphism

$$\mathrm{Hom}_A(M, N) \to \mathrm{Hom}_A(N^\tau, M^\tau) \tag{2.2}$$

for all finite dimensional A-modules M, N. The isomorphism sends $f \in \mathrm{Hom}_A(M, N)$ to the dual map $f^* \in \mathrm{Hom}_A(N^\tau, M^\tau)$ defined by $(f^*\theta)(m) = \theta(fm)$ for all $\theta \in N^\tau$.

§2-c. **Grothendieck groups.** For a superalgebra A, the category A-mod is a *superadditive category*: each $\mathrm{Hom}_A(M, N)$ is a $\mathbb{Z}_2$-graded abelian group in a way that is compatible with composition. Moreover, the *underlying even category*, i.e. the subcategory consisting of the same objects but only even morphisms, is an abelian category in the usual sense. This allows us to make use of all the basic notions of homological algebra on restricting our attention to even morphisms. For example by a short exact sequence in A-mod, we mean a sequence

$$0 \longrightarrow M_1 \longrightarrow M_2 \longrightarrow M_3 \longrightarrow 0 \tag{2.3}$$

that is a short exact sequence in the underlying even category, so in particular all the maps are necessarily even. Note all functors between superadditive categories that we shall consider send even morphisms to even morphisms, so make sense on restriction to the underlying even subcategory.

Let us write $\mathrm{Rep}\, A$ for the full subcategory of A-mod consisting of all finite dimensional A-modules. We define the *Grothendieck group* $K(\mathrm{Rep}\, A)$ to be the quotient of the free $\mathbb{Z}$-module with generators given by all finite dimensional A-modules by the submodule generated by

(1) $M_1 - M_2 + M_3$ for every short exact sequence of the form (2.3);
(2) $M - \Pi M$ for every A-module M.

We will write $[M]$ for the image of the A-module M in $K(\mathrm{Rep}\, A)$. In an entirely similar way, we define the Grothendieck group $K(\mathrm{Proj}\, A)$, where $\mathrm{Proj}\, A$ denotes the full category of A-mod consisting of finite dimensional projectives.

Now suppose that A and B are two superalgebras. Then the Grothendieck groups $K(\mathrm{Rep}\, A)$ and $K(\mathrm{Rep}\, B)$ are free $\mathbb{Z}$-modules with canonical bases corresponding to the isomorphism classes of irreducible modules. Moreover, there is an isomorphism

$$K(\mathrm{Rep}\, A) \otimes_{\mathbb{Z}} K(\mathrm{Rep}\, B) \to K(\mathrm{Rep}\, A \otimes B), \qquad [L] \otimes [L'] \mapsto [L \circledast L'] \tag{2.4}$$

for irreducible modules $L \in \operatorname{Rep} A, L' \in \operatorname{Rep} B$. This simple observation explains the importance of the operation $\circledast$.

§2-d. **The superalgebras.** We now proceed to define the superalgebras we will be interested in. First, $\mathcal{P}_n$ denotes the algebra $F[X_1^{\pm 1}, \dots, X_n^{\pm n}]$ of Laurent polynomials, viewed as a superalgebra concentrated in degree $\bar{0}$. We record the relations:

$$X_i X_i^{-1} = 1, \; X_i^{-1} X_i = 1, \tag{2.5}$$

$$X_k X_j = X_j X_k, \; X_k^{-1} X_j^{-1} = X_j^{-1} X_k^{-1}, X_k X_j^{-1} = X_j^{-1} X_k, \; X_k^{-1} X_j = X_j X_k^{-1} \tag{2.6}$$

for all $1 \le i \le n, 1 \le j < k \le n$. (Of course, some of these relations are redundant, but the precise form of the relations is used in the proof of Theorem 2.2.)

Let $\mathcal{A}_n$ denote the superalgebra with even generators $X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and odd generators $C_1, \dots, C_n$, where the $X_i^{\pm 1}$ are subject to the polynomial relations (2.5), (2.6), the C_i are subject to the Clifford superalgebra relations

$$C_i^2 = 1, \tag{2.7}$$

$$C_k C_j = -C_j C_k \tag{2.8}$$

for all $1 \le i \le n, 1 \le j < k \le n$, and there are the mixed relations

$$C_i X_j = X_j C_i, \quad C_i X_j^{-1} = X_j^{-1} C_i, \quad C_i X_i = X_i^{-1} C_i, \quad C_i X_i^{-1} = X_i C_i \tag{2.9}$$

for all $1 \le i, j \le n$ with $i \ne j$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $\beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_2^n$, we write X^α and C^β for the monomials $X_1^{\alpha_1} X_2^{\alpha_2} \dots X_n^{\alpha_n}$ and $C_1^{\beta_1} C_2^{\beta_2} \dots C_n^{\beta_n}$, respectively. Then, it is straightforward to show that the elements

$$\{X^\alpha C^\beta \mid \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n\}$$

form a basis for $\mathcal{A}_n$. In particular, $\mathcal{A}_n \cong \mathcal{A}_1 \otimes \dots \otimes \mathcal{A}_1$ (n times), and $\mathcal{P}_n$ can be identified with the subalgebra of $\mathcal{A}_n$ generated by the $X_i^{\pm 1}$.

The symmetric group will be denoted S_n, with basic transpositions $s_1, \dots, s_{n-1}$ and corresponding Bruhat ordering denoted $\le$. We define a left action of S_n on $\mathcal{A}_n$ by algebra automorphisms so that

$$w \cdot X_i = X_{wi}, \qquad w \cdot C_i = C_{wi} \tag{2.10}$$

for each $w \in S_n, i = 1, \dots, n$.

Let $\mathcal{H}_n^{\text{cl}}$ denote the usual (classical) Hecke algebra of S_n, viewed as a superalgebra concentrated in degree $\bar{0}$. This can be defined on generators $T_1, \dots, T_{n-1}$ subject to relations

$$T_i^2 = \xi T_i + 1, \tag{2.11}$$

$$T_i T_j = T_j T_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \tag{2.12}$$

for all admissible i, j with $|i - j| > 2$. Recall $\mathcal{H}_n^{\text{cl}}$ has a basis denoted $\{T_w \mid w \in S_n\}$, where $T_w = T_{i_1} \dots T_{i_m}$ if $w = s_{i_1} \dots s_{i_m}$ is any reduced expression for w. We also record

$$T_i^{-1} = T_i - \xi. \tag{2.13}$$

Now we are ready to define the main object of study: the *affine Hecke-Clifford superalgebra* $\mathcal{H}_n$. This was first introduced by Jones and Nazarov [JN], being the q-analogue of the affine Sergeev superalgebra of Nazarov [N]. By definition, $\mathcal{H}_n$ has even generators $T_1, \dots, T_{n-1}$,

$X_1^{\pm 1}, \dots, X_n^{\pm 1}$ and odd generators $C_1, \dots, C_n$, subject to the same relations as $\mathcal{A}_n$ (2.5), (2.6), (2.7), (2.8), (2.9) and as $\mathcal{H}_n^{\text{cl}}$ (2.11), (2.12), together with the new relations:

$$T_iC_j = C_jT_i, \tag{2.14}$$
$$T_iC_i = C_{i+1}T_i, \tag{2.15}$$
$$T_iX_j = X_jT_i, \qquad T_iX_j^{-1} = X_j^{-1}T_i, \tag{2.16}$$
$$(T_i + \xi C_iC_{i+1})X_iT_i = X_{i+1} \tag{2.17}$$

for all admissible i, j with $j \neq i, i+1$. The relation (2.15) is equivalent to

$$T_iC_{i+1} = C_iT_i - \xi(C_i - C_{i+1}), \tag{2.18}$$

while (2.17) is equivalent to any of the following four statements:

$$T_iX_i = X_{i+1}T_i - \xi(X_{i+1} + C_iC_{i+1}X_i), \tag{2.19}$$
$$T_iX_i^{-1} = X_{i+1}^{-1}T_i + \xi(X_i^{-1} + X_{i+1}^{-1}C_iC_{i+1}), \tag{2.20}$$
$$T_iX_{i+1} = X_iT_i + \xi(1 - C_iC_{i+1})X_{i+1}, \tag{2.21}$$
$$T_iX_{i+1}^{-1} = X_i^{-1}T_i - \xi X_i^{-1}(1 - C_iC_{i+1}) \tag{2.22}$$

for all $i = 1, \dots, n-1$. Using (2.19) and induction on $j \geq 1$, one shows that

$$(T_i + \xi C_iC_{i+1})X_i^j = X_{i+1}^j(T_i - \xi) - \xi \sum_{k=1}^{j-1} \left(X_i^{j-k}X_{i+1}^k + X_i^{-k}X_{i+1}^{j-k}C_iC_{i+1} \right) \tag{2.23}$$

for all $j \geq 1, 1 \leq i < n$. Hence:

$$\begin{aligned}(T_i + \xi C_iC_{i+1})X_i^jC_i =& X_{i+1}^jC_{i+1}(T_i + \xi C_iC_{i+1}) \\ & - \xi \sum_{k=1}^{j-1} \left(X_i^{j-k}X_{i+1}^kC_i - X_i^{-k}X_{i+1}^{j-k}C_{i+1} \right),\end{aligned} \tag{2.24}$$

$$\begin{aligned}(T_i - \xi)X_i^{-j} =& X_{i+1}^{-j}(T_i + \xi C_iC_{i+1}) \\ & + \xi \sum_{k=1}^{j-1} \left(X_i^{-k}X_{i+1}^{k-j} + X_i^{j-k}X_{i+1}^{-k}C_iC_{i+1} \right).\end{aligned} \tag{2.25}$$

Finally, let $\mathcal{H}_n^{\text{fin}}$ denote the subalgebra of $\mathcal{H}_n$ generated by all C_i, T_j for $i = 1, \dots, n, j = 1, \dots, n-1$. Alternatively, as follows easily from Theorem 2.2 below, $\mathcal{H}_n^{\text{fin}}$ can be defined as the superalgebra generated by elements C_i, T_j subject only to the relations (2.7), (2.8), (2.11), (2.12) and (2.14), (2.15). Hence, $\mathcal{H}_n^{\text{fin}}$ is the (finite) *Hecke-Clifford superalgebra* first introduced by Olshanski [O] as the q-analogue of the Sergeev superalgebra of [S$_1$].

§2-e. **Basis theorem.** Now we proceed to study the algebra $\mathcal{H}_n$ in more detail. The first goal is to construct a basis. There are obvious homomorphisms $f : \mathcal{A}_n \to \mathcal{H}_n$ and $g : \mathcal{H}_n^{\text{cl}} \to \mathcal{H}_n$ under which the X_i, C_i or T_j map to the same elements of $\mathcal{H}_n$. We write $X^\alpha C^\beta$ also for the image under f of the basis element $X^\alpha C^\beta$ of $\mathcal{A}_n$, and T_w for the image under g of $T_w \in \mathcal{H}_n^{\text{cl}}$. This notation will be justified shortly, when we show that f and g are both algebra monomorphisms. The following lemma is obvious from the relations:

Lemma 2.1. *Let $f \in \mathcal{A}_n$, $x \in S_n$. Then, in $\mathcal{H}_n$,*

$$T_x f = (x \cdot f)T_x + \sum_{y<x} f_y T_y, \qquad fT_x = T_x(x^{-1} \cdot f) + \sum_{y<x} T_y f'_y$$

for some $f_y, f'_y \in \mathcal{A}_n$.

It follows easily that $\mathcal{H}_n$ is at least *spanned* by all $X^\alpha C^\beta T_w$ for all $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n$ and $w \in S_n$. We wish to prove that these elements are linearly independent too:

Theorem 2.2. *The $\{X^\alpha C^\beta T_w \mid \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n, w \in S_n\}$ form a basis for $\mathcal{H}_n$.*

Proof. Consider instead the algebra $\tilde{\mathcal{H}}_n$ on generators $\tilde{X}_i^{\pm 1}, \tilde{C}_i, \tilde{T}_j$ for $1 \leq i \leq n, 1 \leq j < n$ subject to relations (2.5)–(2.9), (2.11), (2.14)–(2.16) and (2.18)–(2.22). Thus we have all the relations of $\mathcal{H}_n$ *except for* the braid relations (2.12). Using precisely these relations as the reduction system, it is a routine if tedious exercise using Bergman's diamond lemma [Be, 1.2] to prove that $\tilde{\mathcal{H}}_n$ has a basis given by all $\tilde{X}^\alpha \tilde{C}^\beta \tilde{T}$ for all $\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n$ and all words $\tilde{T}$ in the $\tilde{T}_j$ which do not involve a subword of the form $\tilde{T}_j^2$ for any j. Hence, the subalgebra $\widetilde{A}_n$ of $\tilde{\mathcal{H}}_n$ generated by the $\tilde{X}_i^{\pm 1}, \tilde{C}_i$ is isomorphic to $\mathcal{A}_n$. Also let $\tilde{\mathcal{H}}_n^{\mathrm{cl}}$ denote the subalgebra of $\tilde{\mathcal{H}}_n$ generated by the $\tilde{T}_j$, so that $\tilde{\mathcal{H}}_n^{\mathrm{cl}}$ is isomorphic to the algebra on generators $\tilde{T}_1, \dots, \tilde{T}_{n-1}$ subject to relations $\tilde{T}_j^2 = \xi \tilde{T}_j + 1$ for each j.

Now, by definition, $\mathcal{H}_n$ is the quotient of $\tilde{\mathcal{H}}_n$ by the two-sided ideal $\mathcal{I}$ generated by the elements

$$a_{i,j} = \tilde{T}_i \tilde{T}_j - \tilde{T}_j \tilde{T}_i, \quad b_i = \tilde{T}_i \tilde{T}_{i+1} \tilde{T}_i - \tilde{T}_{i+1} \tilde{T}_i \tilde{T}_{i+1}$$

for i, j as in (2.12). Let $\mathcal{J}$ be the two-sided ideal of $\tilde{\mathcal{H}}_n^{\mathrm{cl}}$ generated by the same elements $a_{i,j}, b_i$ for all i, j. Then, by the basis theorem for $\mathcal{H}_n^{\mathrm{cl}}$, $\tilde{\mathcal{H}}_n^{\mathrm{cl}}/\mathcal{J} \cong \mathcal{H}_n^{\mathrm{cl}}$, with basis given by elements T_w for $w \in S_n$ defined in the usual way. It follows immediately that to prove the theorem, it suffices to show that $\mathcal{I} = \widetilde{A}_n \mathcal{J}$ in $\tilde{\mathcal{H}}_n$. In turn, this follows if we can show that $rt = t'r$ for each $r \in \{a_{i,j}, b_i\}$ and each generator t of $\widetilde{A}_n$, where t' is some other element of $\widetilde{A}_n$. Now again this is a routine check: for example, the most complicated case involves verifying that $(\tilde{T}_{i+1} \tilde{T}_i \tilde{T}_{i+1} - \tilde{T}_i \tilde{T}_{i+1} \tilde{T}_i)\tilde{X}_{i+2} = \tilde{X}_i(\tilde{T}_{i+1} \tilde{T}_i \tilde{T}_{i+1} - \tilde{T}_i \tilde{T}_{i+1} \tilde{T}_i)$ using only the relations in $\tilde{\mathcal{H}}_n$. $\square$

So we have a right from now on to *identify* $\mathcal{A}_n$ and $\mathcal{H}_n^{\mathrm{cl}}$ with the corresponding subalgebras of $\mathcal{H}_n$. The theorem also shows that the superalgebra $\mathcal{H}_n^{\mathrm{fin}}$ has basis $\{C^\beta T_w | \beta \in \mathbb{Z}_2^n, w \in S_n\}$. So Theorem 2.2 can be restated as saying that $\mathcal{H}_n$ is a free right $\mathcal{H}_n^{\mathrm{fin}}$-module on basis $\{X^\alpha \mid \alpha \in \mathbb{Z}^n\}$.

As another consequence, it makes sense to consider the tower of superalgebras

$$\mathcal{H}_0 \subset \mathcal{H}_1 \subset \mathcal{H}_2 \subset \dots \subset \mathcal{H}_n \subset \dots$$

where for $i \leq n$, $\mathcal{H}_i$ is identified with the subalgebra of $\mathcal{H}_n$ generated by $C_1, \dots, C_i$, $X_1^{\pm 1}, \dots, X_i^{\pm 1}$, $T_1, \dots, T_{i-1}$. Similarly, we can consider $\mathcal{A}_i \subseteq \mathcal{A}_n, \mathcal{P}_i \subseteq \mathcal{P}_n$, etc....

Finally, we point out that there are obvious variants of the basis of Theorem 2.2, reordering the C's, X's and T's. For instance, $\mathcal{H}_n$ also has $\{T_w X^\alpha C^\beta \mid w \in S_n, \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n\}$ as a basis. This follows using Lemma 2.1.

§2-f. **The center of $\boldsymbol{\mathcal{H}_n}$.** The next theorem was first established in [JN, Prop. 3.2] (for the case $F = \mathbb{C}$).

Theorem 2.3. *The (super)center of $\mathcal{H}_n$ consists of all symmetric polynomials in $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$.*

Proof. Let Z denote the center of $\mathcal{H}_n$, i.e. the z such that $zy = yz$ for all $y \in \mathcal{H}_n$. (The argument applies equally well to the supercenter defined by $zy = (-1)^{\bar{z}\bar{y}}yz$.) One first checks that symmetric polynomials in $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$ *are* central following the argument in the proof of [JN, Prop. 3.2(b)].

Conversely, take $z = \sum_{w \in S_n} f_w T_w \in Z$ where each $f_w \in \mathcal{A}_n$. Let w be maximal with respect to the Bruhat order such that $f_w \neq 0$. Assume for a contradiction that $w \neq 1$. Then, there exists some $i \in \{1, \dots, n\}$ with $wi \neq i$. Consider $(X_i + X_i^{-1})z - z(X_i + X_i^{-1})$. By Lemma 2.1, this looks like $f_w(X_i + X_i^{-1} - X_{wi} - X_{wi}^{-1})T_w$ plus a linear combination of terms of the form $f_x T_x$ for $f_x \in \mathcal{A}_n$ and $x \in S_n$ with $x \not\geq w$ in the Bruhat order. So in view of Theorem 2.2, z is not central, a contradiction.

Hence, we must have that $z \in \mathcal{A}_n$. Considering the form of the center of $\mathcal{A}_n$ one easily shows that z in fact lies in $F[X_1+X_1^{-1}, \dots, X_n+X_n^{-1}]$. To see that z is actually a symmetric polynomial, write $z = \sum_{i,j \geq 0} a_{i,j}(X_1 + X_1^{-1})^i (X_2 + X_2^{-1})^j$ where the coefficients $a_{i,j}$ lie in $F[X_3 + X_3^{-1}, \dots, X_n + X_n^{-1}]$. Applying Lemma 2.1 to $T_1 z = z T_1$ now gives that $a_{i,j} = a_{j,i}$ for each i, j, hence z is symmetric in $X_1 + X_1^{-1}$ and $X_2 + X_2^{-1}$. Similar argument shows that z is symmetric in $X_i + X_i^{-1}$ and $X_{i+1} + X_{i+1}^{-1}$ for all $i = 1, \dots, n-1$ to complete the proof. □

§2-g. **Parabolic subalgebras.** Suppose that $\mu = (\mu_1, \dots, \mu_u)$ is a composition of n, i.e. a sequence of positive integers summing to n. Let $S_\mu \cong S_{\mu_1} \times \cdots \times S_{\mu_u}$ denote the corresponding Young subgroup of S_n, and $\mathcal{H}^{\mathrm{cl}}_\mu \subseteq \mathcal{H}^{\mathrm{cl}}_n$ denote its Hecke algebra. So $\mathcal{H}^{\mathrm{cl}}_\mu \cong \mathcal{H}^{\mathrm{cl}}_{\mu_1} \otimes \cdots \otimes \mathcal{H}^{\mathrm{cl}}_{\mu_u}$ is the subalgebra of $\mathcal{H}^{\mathrm{cl}}_n$ generated by the T_j for which $s_j \in S_\mu$.

We define the parabolic subalgebra $\mathcal{H}_\mu$ of the affine Hecke-Clifford superalgebra $\mathcal{H}_n$ in a similar way: it is the subalgebra of $\mathcal{H}_n$ generated by $\mathcal{A}_n$ and all T_j for which $s_j \in S_\mu$. It follows easily from Theorem 2.2 that the elements

$$\{X^\alpha C^\beta T_w \mid \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n, w \in S_\mu\}$$

form a basis for $\mathcal{H}_\mu$. In particular, $\mathcal{H}_\mu \cong \mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_u}$. Note that the parabolic subalgebra $\mathcal{H}_{(1,1,\dots,1)}$ is precisely the subalgebra $\mathcal{A}_n$.

We will need the usual induction and restriction functors between $\mathcal{H}_n$ and $\mathcal{H}_\mu$. These will be denoted simply

$$\mathrm{ind}^n_\mu : \mathcal{H}_\mu\text{-mod} \to \mathcal{H}_n\text{-mod}, \qquad \mathrm{res}^n_\mu : \mathcal{H}_n\text{-mod} \to \mathcal{H}_\mu\text{-mod}, \tag{2.26}$$

the former being the tensor functor $\mathcal{H}_n \otimes_{\mathcal{H}_\mu} ?$ which is left adjoint to res^n_μ. More generally, we will consider induction and restriction between nested parabolic subalgebras, with obvious notation. We will also occasionally consider the restriction functor

$$\mathrm{res}^n_{n-1} : \mathcal{H}_n\text{-mod} \to \mathcal{H}_{n-1}\text{-mod} \tag{2.27}$$

where $\mathcal{H}_{n-1}$ denotes the subalgebra of $\mathcal{H}_n$ generated by $X_i^{\pm 1}, C_i$ and T_j for $i = 1, \dots, n-1, j = 1, \dots, n-2$.

§2-h. **Mackey theorem.** Let μ, ν be compositions of n. We let D_ν denote the set of minimal length left S_ν-coset representatives in S_n, and D_μ^{-1} denote the set of minimal length right S_μ-coset representatives. Then $D_{\mu,\nu} := D_\mu^{-1} \cap D_\nu$ is the set of minimal length (S_μ, S_ν)-double coset representatives in S_n. We recall some well-known properties, see e.g. [DJ, §1]. First, for $x \in D_{\mu,\nu}$, $S_\mu \cap x S_\nu x^{-1}$ and $x^{-1} S_\mu x \cap S_\nu$ are Young subgroups of S_n. So we can define compositions $\mu \cap x\nu$ and $x^{-1}\mu \cap \nu$ of n from

$$S_\mu \cap x S_\nu x^{-1} = S_{\mu \cap x\nu} \quad \text{and} \quad x^{-1} S_\mu x \cap S_\nu = S_{x^{-1}\mu \cap \nu}.$$

Then, for $x \in D_{\mu,\nu}$, every $w \in S_\mu x S_\nu$ can be written as $w = uxv$ for unique elements $u \in S_\mu$ and $v \in S_\nu \cap D^{-1}_{x^{-1}\mu\cap\nu}$. Moreover, when this is done, $\ell(w) = \ell(u) + \ell(x) + \ell(v)$. This fact implies the following well-known lemma which is essentially equivalent to the Mackey theorem for $\mathcal{H}_n^{\mathrm{cl}}$ (see e.g. [DJ, Theorem 2.7]):

Lemma 2.4. *For $x \in D_{\mu,\nu}$, the subspace $\mathcal{H}_\mu^{\mathrm{cl}} T_x \mathcal{H}_\nu^{\mathrm{cl}}$ of $\mathcal{H}_n^{\mathrm{cl}}$ has basis $\{T_w \mid w \in S_\mu x S_\nu\}$.*

This is our starting point for proving a version of the Mackey theorem for $\mathcal{H}_n$.

Lemma 2.5. *For $x \in D_{\mu,\nu}$, the subspace $\mathcal{H}_\mu T_x \mathcal{H}_\nu^{\mathrm{cl}}$ of $\mathcal{H}_n$ has basis $\{X^\alpha C^\beta T_w \mid \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n, w \in S_\mu x S_\nu\}$. Moreover,*

$$\mathcal{H}_n = \bigoplus_{x \in D_{\mu,\nu}} \mathcal{H}_\mu T_x \mathcal{H}_\nu^{\mathrm{cl}}.$$

Proof. Since $\mathcal{H}_\mu = \mathcal{A}_n \mathcal{H}_\mu^{\mathrm{cl}}$, Lemma 2.4 implies at once that the given $\{X^\alpha C^\beta T_w\}$ span $\mathcal{H}_\mu T_x \mathcal{H}_\nu^{\mathrm{cl}}$. But they are linearly independent too by Theorem 2.2, proving the first statement. The second follows immediately using Theorem 2.2 once more. □

Now fix some total order $\prec$ refining the Bruhat order $<$ on $D_{\mu,\nu}$. For $x \in D_{\mu,\nu}$, set

$$\mathcal{B}_{\preceq x} = \bigoplus_{y \in D_{\mu,\nu},\ y \preceq x} \mathcal{H}_\mu T_y \mathcal{H}_\nu^{\mathrm{cl}}, \tag{2.28}$$

$$\mathcal{B}_{\prec x} = \bigoplus_{y \in D_{\mu,\nu},\ y \prec x} \mathcal{H}_\mu T_y \mathcal{H}_\nu^{\mathrm{cl}}, \tag{2.29}$$

$$\mathcal{B}_x = \mathcal{B}_{\preceq x} / \mathcal{B}_{\prec x}. \tag{2.30}$$

It follows immediately from Lemma 2.1 that $\mathcal{B}_{\preceq x}$ (resp. $\mathcal{B}_{\prec x}$) is invariant under right multiplication by $\mathcal{A}_n$. Hence, since $\mathcal{H}_\nu = \mathcal{H}_\nu^{\mathrm{cl}} \mathcal{A}_n$, we have defined a filtration of $\mathcal{H}_n$ as an $(\mathcal{H}_\mu, \mathcal{H}_\nu)$-bimodule. We want to describe the quotients $\mathcal{B}_x$ more explicitly as $(\mathcal{H}_\mu, \mathcal{H}_\nu)$-bimodules.

Lemma 2.6. *For each $x \in D_{\mu,\nu}$, there exists an algebra isomorphism*

$$\varphi = \varphi_{x^{-1}} : \mathcal{H}_{\mu\cap x\nu} \to \mathcal{H}_{x^{-1}\mu\cap\nu}$$

with $\varphi(T_w) = T_{x^{-1}wx}$, $\varphi(X_i) = X_{x^{-1}i}$, and $\varphi(C_i) = C_{x^{-1}i}$ for $w \in S_{\mu\cap x\nu}$, $1 \le i \le n$.

Proof. The isomorphism $\psi : S_{\mu\cap x\nu} \to S_{x^{-1}\mu\cap\nu}, u \mapsto x^{-1}ux$ is length preserving. Equivalently, $x^{-1}(i+1) = (x^{-1}i) + 1$ for each i with $s_i \in S_{\mu\cap x\nu}$. Using this, it is straightforward to check that the map φ defined as above respects the defining relations on generators. □

Let N be a left $\mathcal{H}_{x^{-1}\mu\cap\nu}$-module. By twisting the action with the isomorphism $\varphi_{x^{-1}} : \mathcal{H}_{\mu\cap x\nu} \to \mathcal{H}_{x^{-1}\mu\cap\nu}$ from Lemma 2.6, we get a left $\mathcal{H}_{\mu\cap x\nu}$-module, which will be denoted ${}^xN$. Now we can identify the module $\mathcal{B}_x$ introduced above.

Lemma 2.7. *View $\mathcal{H}_\mu$ as an $(\mathcal{H}_\mu, \mathcal{H}_{\mu\cap x\nu})$-bimodule and $\mathcal{H}_\nu$ as an $(\mathcal{H}_{x^{-1}\mu\cap\nu}, \mathcal{H}_\nu)$-bimodule in the natural ways. Then, ${}^x\mathcal{H}_\nu$ is an $(\mathcal{H}_{\mu\cap x\nu}, \mathcal{H}_\nu)$-bimodule and*

$$\mathcal{B}_x \simeq \mathcal{H}_\mu \otimes_{\mathcal{H}_{\mu\cap x\nu}} {}^x\mathcal{H}_\nu$$

as an $(\mathcal{H}_\mu, \mathcal{H}_\nu)$-bimodule.

Proof. We define a bilinear map $\mathcal{H}_\mu \times \mathcal{H}_\nu \to \mathcal{B}_x = \mathcal{B}_{\preceq x}/\mathcal{B}_{\prec x}$ by $(u, v) \mapsto uT_xv + \mathcal{B}_{\prec x}$. For $y \in S_\mu \cap xS_\nu x^{-1}$,

$$T_yT_x = T_{yx} = T_{xx^{-1}yx} = T_xT_{x^{-1}yx}$$

which is all that is required to check that the map is $\mathcal{H}_{\mu\cap x\nu}$-balanced. Hence there is an induced $(\mathcal{H}_\mu, \mathcal{H}_\nu)$-bimodule map $\Phi : \mathcal{H}_\mu \otimes_{\mathcal{H}_{\mu\cap x\nu}} {}^x\mathcal{H}_\nu \to \mathcal{B}_x$. Finally, to prove that Φ is bijective, note that

$$\{X^\alpha C^\beta T_u \otimes T_v \mid \alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n, u \in S_\mu, v \in S_\nu \cap D^{-1}_{x^{-1}\mu\cap\nu}\}$$

is a basis of the induced module $\mathcal{H}_\mu \otimes_{\mathcal{H}_{\mu\cap x\nu}} {}^x\mathcal{H}_\mu$ as a vector space. In view of Lemma 2.5, the image of these elements under Φ is a basis of $\mathcal{B}_x$. □

Now we can prove the Mackey theorem.

Theorem 2.8. ("Mackey Theorem") *Let M be an $\mathcal{H}_\nu$-module. Then $\mathrm{res}^n_\mu \mathrm{ind}^n_\nu M$ admits a filtration with subquotients $\simeq$ to $\mathrm{ind}^\mu_{\mu\cap x\nu} {}^x(\mathrm{res}^\nu_{x^{-1}\mu\cap\nu} M)$, one for each $x \in D_{\mu,\nu}$. Moreover, the subquotients can be taken in any order refining the Bruhat order on $D_{\mu,\nu}$, in particular $\mathrm{ind}^\mu_{\mu\cap\nu}\mathrm{res}^\nu_{\mu\cap\nu} M$ appears as a submodule.*

Proof. This follows from Lemma 2.7 and the isomorphism

$$(\mathcal{H}_\mu \otimes_{\mathcal{H}_{\mu\cap x\nu}} {}^x\mathcal{H}_\nu) \otimes_{\mathcal{H}_\nu} M \simeq \mathrm{ind}^{\mathcal{H}_\mu}_{\mathcal{H}_{\mu\cap x\nu}} {}^x(\mathrm{res}^{\mathcal{H}_\nu}_{\mathcal{H}_{x^{-1}\mu\cap\nu}} M),$$

which is easy to check. □

§2-i. **Some (anti)automorphisms.** A check of relations shows that $\mathcal{H}_n$ posesses an automorphism σ and an antiautomorphism τ defined on the generators as follows:

$$\sigma : T_i \mapsto -T_{n-i} + \xi, \qquad C_j \mapsto C_{n+1-j}, \quad X_j \mapsto X_{n+1-j}; \tag{2.31}$$

$$\tau : T_i \mapsto T_i + \xi C_i C_{i+1}, \quad C_j \mapsto C_j, \qquad X_j \mapsto X_j, \tag{2.32}$$

for all $i = 1, \dots, n-1, j = 1, \dots, n$.

If M is a finite dimensional $\mathcal{H}_n$-module, we can use τ to make the dual space M^* into an $\mathcal{H}_n$-module denoted M^τ, see §2-b. Note τ leaves invariant every parabolic subalgebra of $\mathcal{H}_n$, so also induces a duality on finite dimensional $\mathcal{H}_\mu$-modules for each composition μ of n.

Instead, given any $\mathcal{H}_n$-module M, we can twist the action with σ to get a new module denoted M^σ. More generally, for any composition $\nu = (\nu_1, \dots, \nu_u)$ of n we denote by ν^* the composition with the same non-zero parts but taken in the opposite order. For example $(3,2,1)^* = (1,2,3)$. Then σ induces an isomorphism of parabolic subalgebras $\mathcal{H}_{\nu^*} \to \mathcal{H}_\nu$. So if M is an $\mathcal{H}_\nu$-module, we can inflate through σ to get an $\mathcal{H}_{\nu^*}$-module denoted M^σ. If $M = M_1 \boxtimes \cdots \boxtimes M_u$ is an outer tensor product module over $\mathcal{H}_\nu$ then $M^\sigma \cong M_u^\sigma \boxtimes \cdots \boxtimes M_1^\sigma$. The same holds if each M_i is irreducible and $\boxtimes$ is replaced with $\circledast$. These observations imply:

Lemma 2.9. *Let $M \in \mathcal{H}_m$-mod and $N \in \mathcal{H}_n$-mod. Then*

$$(\mathrm{ind}^{m+n}_{m,n} M \boxtimes N)^\sigma \cong \mathrm{ind}^{m+n}_{n,m} N^\sigma \boxtimes M^\sigma.$$

Moreover, if M and N are irreducible, the same holds for $\circledast$ in place of $\boxtimes$.

§2-j. **Duality.** Thoughout this subsection, let μ be a composition of n and set $\nu = \mu^*$. Let $d \in D_{\mu,\nu}$ be the longest double coset representative. Note that $\mu \cap d\nu = \mu$ and $d^{-1}\mu \cap \nu = \nu$, so $S_\mu d S_\nu = S_\mu d = d S_\nu$. There is an isomorphism

$$\varphi = \varphi_{d^{-1}} : \mathcal{H}_\mu \to \mathcal{H}_\nu, \tag{2.33}$$

see Lemma 2.6. As in §2-h, for an $\mathcal{H}_\nu$-module M, ${}^dM$ denotes the $\mathcal{H}_\mu$-module obtained by pulling back the action through φ. We begin by considering the classical situation, adopting the obvious analogous notation for modules over $\mathcal{H}^{\mathrm{cl}}_\mu, \mathcal{H}^{\mathrm{cl}}_\nu$.

Lemma 2.10. *Define a linear map* $\theta^{\mathrm{cl}} : \mathcal{H}_n^{\mathrm{cl}} \to {}^d\mathcal{H}_\nu^{\mathrm{cl}}$ *by*

$$\theta^{\mathrm{cl}}(T_w) = \begin{cases} T_{d^{-1}w} & \text{if } w \in dS_\nu, \\ 0 & \text{otherwise,} \end{cases}$$

for each $w \in S_n$. *Then:*

(i) θ^{cl} *is an even homomorphism of* $(\mathcal{H}_\mu^{\mathrm{cl}}, \mathcal{H}_\nu^{\mathrm{cl}})$*-bimodules;*

(ii) $\ker \theta^{\mathrm{cl}}$ *contains no non-zero left ideals of* $\mathcal{H}_n^{\mathrm{cl}}$*;*

(iii) *the map*

$$f^{\mathrm{cl}} : \mathcal{H}_n^{\mathrm{cl}} \to \mathrm{Hom}_{\mathcal{H}_\mu^{\mathrm{cl}}}(\mathcal{H}_n^{\mathrm{cl}}, {}^d\mathcal{H}_\nu^{\mathrm{cl}}), \qquad h \mapsto h\theta^{\mathrm{cl}}$$

is an even isomorphism of $(\mathcal{H}_n^{\mathrm{cl}}, \mathcal{H}_\nu^{\mathrm{cl}})$*-bimodules.*

Proof. (i) Since $d^{-1}\mu \cap \nu = \nu$, $\mathcal{H}_\mu^{\mathrm{cl}} T_d \mathcal{H}_\nu^{\mathrm{cl}} = T_d \mathcal{H}_\nu^{\mathrm{cl}}$ is isomorphic as an $(\mathcal{H}_\mu^{\mathrm{cl}}, \mathcal{H}_\nu^{\mathrm{cl}})$-bimodule to ${}^d\mathcal{H}_\nu^{\mathrm{cl}}$, the isomorphism being simply the map $T_w \mapsto T_{d^{-1}w}$ for $w \in dS_\nu$, compare Lemma 2.7. Now (i) follows because θ^{cl} is simply this isomorphism composed with the projection from $\mathcal{H}_n^{\mathrm{cl}}$ to $\mathcal{H}_\mu^{\mathrm{cl}} T_d \mathcal{H}_\nu^{\mathrm{cl}}$ along the bimodule decomposition $\mathcal{H}_n^{\mathrm{cl}} = \bigoplus_{x \in D_{\mu,\nu}} \mathcal{H}_\mu^{\mathrm{cl}} T_x \mathcal{H}_\nu^{\mathrm{cl}}$.

(ii) We show by downward induction on $\ell(x)$ that $\theta^{\mathrm{cl}}(\mathcal{H}_n^{\mathrm{cl}} t) \neq 0$ whenever we are given $x \in D_\nu$ and

$$t = \sum_{y \in D_\nu \text{ with } \ell(y) \leq \ell(x)} T_y h_y$$

with each $h_y \in \mathcal{H}_\nu^{\mathrm{cl}}$ and $h_x \neq 0$. Since d is the longest element of D_ν, the induction starts with $x = d$: in this case, the conclusion is clear as $\theta^{\mathrm{cl}}(t) = h_x \neq 0$. So now suppose $x < d$ and that the claim has been proved for all higher $x \in D_\nu$. Pick a basic transposition s such that $sx > x$ and $sx \in D_\nu$. Then,

$$T_s t = \sum_{y \in D_\nu \text{ with } \ell(y) \leq \ell(sx),} T_y h'_y$$

for $h'_y \in \mathcal{H}_\nu^{\mathrm{cl}}$ with $h'_{sx} = h_x \neq 0$. But now the induction hypothesis shows that $\theta^{\mathrm{cl}}(\mathcal{H}_n^{\mathrm{cl}} t) = \theta^{\mathrm{cl}}(\mathcal{H}_n^{\mathrm{cl}} T_s t) \neq 0$.

(iii) We remind the reader that $h\theta^{\mathrm{cl}} : \mathcal{H}_n^{\mathrm{cl}} \to {}^d\mathcal{H}_\mu^{\mathrm{cl}}$ denotes the map with $(h\theta^{\mathrm{cl}})(t) = (-1)^{\bar{h}\bar{t}} \theta^{\mathrm{cl}}(th)$ (the sign being + always in this case). Given this and (i), it is straightforward to check that f^{cl} is a homomorphism of $(\mathcal{H}_n^{\mathrm{cl}}, \mathcal{H}_\nu^{\mathrm{cl}})$-bimodules. To see that it is an isomorphism, it suffices by dimension to show that it is injective. Suppose h lies in the kernel. Then, $(f^{\mathrm{cl}}(h))(t) = \theta^{\mathrm{cl}}(th) = 0$ for all $t \in \mathcal{H}_n^{\mathrm{cl}}$. Hence $h = 0$ by (ii). □

Now we extend this result to $\mathcal{H}_n$, recalling the definition of φ from (2.33).

Lemma 2.11. *Define a linear map* $\theta : \mathcal{H}_n \to {}^d\mathcal{H}_\nu$ *by*

$$\theta(fT_w) = \begin{cases} \varphi(f) T_{d^{-1}w} & \text{if } w \in dS_\nu, \\ 0 & \text{otherwise,} \end{cases}$$

for each $f \in \mathcal{A}_n, w \in S_n$. *Then:*

(i) θ *is an even homomorphism of* $(\mathcal{H}_\mu, \mathcal{H}_\nu)$*-bimodules;*

(ii) *the map*

$$f : \mathcal{H}_n \to \mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu), \qquad h \mapsto h\theta$$

is an even isomorphism of $(\mathcal{H}_n, \mathcal{H}_\nu)$*-bimodules.*

Proof. (i) According to a special case of Lemma 2.7, the top factor $\mathcal{B}_d$ in the bimodule filtration of $\mathcal{H}_n$ defined in (2.30) is isomorphic to ${}^d\mathcal{H}_\nu$ as an $(\mathcal{H}_\mu, \mathcal{H}_\nu)$-bimodule. The map θ is simply the composite of this isomorphism with the quotient map $\mathcal{H}_n \to \mathcal{B}_d$.

(ii) Recall that $\{T_w \mid w \in D_\mu^{-1}\}$ forms a basis for $\mathcal{H}_n$ as a free left $\mathcal{H}_\mu$-module, and ${}^d\mathcal{H}_\nu$ is isomorphic to $\mathcal{H}_\mu$ as a left $\mathcal{H}_\mu$-module. It follows that the maps

$$\{\psi_w \mid w \in D_\mu^{-1}\}$$

form a basis for $\mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu)$ as a free right $\mathcal{H}_\nu$-module, where $\psi_w : \mathcal{H}_n \to {}^d\mathcal{H}_\nu$ is the unique left $\mathcal{H}_\mu$-module homomorphism with $\psi_w(T_u) = \delta_{w,u}.1$ for all $u \in D_\mu^{-1}$.

The analogous maps $\psi_w^{\mathrm{cl}} \in \mathrm{Hom}_{\mathcal{H}_\mu^{\mathrm{cl}}}(\mathcal{H}_n^{\mathrm{cl}}, {}^d\mathcal{H}_\nu^{\mathrm{cl}})$ defined by $\psi_w^{\mathrm{cl}}(T_u) = \delta_{w,u}.1$ for $u \in D_\mu^{-1}$ form a basis for $\mathrm{Hom}_{\mathcal{H}_\mu^{\mathrm{cl}}}(\mathcal{H}_n^{\mathrm{cl}}, {}^d\mathcal{H}_\nu^{\mathrm{cl}})$ as a free right $\mathcal{H}_\nu^{\mathrm{cl}}$-module. So in view of Lemma 2.10(iii), we can find a basis $\{a_w \mid w \in D_\mu^{-1}\}$ for $\mathcal{H}_n^{\mathrm{cl}}$ viewed as a right $\mathcal{H}_\nu^{\mathrm{cl}}$-module such that $f^{\mathrm{cl}}(a_w) = \psi_w^{\mathrm{cl}}$ for each $w \in D_\mu^{-1}$, i.e.

$$\theta^{\mathrm{cl}}(T_u a_w) = \begin{cases} 1 & \text{if } u = w, \\ 0 & \text{otherwise} \end{cases}$$

for every $u \in D_\mu^{-1}$. But $\mathcal{H}_\nu = \mathcal{H}_\nu^{\mathrm{cl}}\mathcal{A}_n$, so the elements $\{a_w \mid w \in D_\mu^{-1}\}$ also form a basis for $\mathcal{H}_n$ as a right $\mathcal{H}_\nu$-module, and $f(a_w) = \psi_w$ since $\theta = \theta^{\mathrm{cl}}$ on $\mathcal{H}_n^{\mathrm{cl}}$. Thus f maps a basis of $\mathcal{H}_n$ to a basis of $\mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu)$ (as free right $\mathcal{H}_\nu$-modules), hence f is an isomorphism of $(\mathcal{H}_n, \mathcal{H}_\nu)$-bimodules. □

Corollary 2.12. *There is a natural isomorphism* $\mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^dM) \simeq \mathcal{H}_n \otimes_{\mathcal{H}_\nu} M$ *of* $\mathcal{H}_n$*-modules, for every left* $\mathcal{H}_\nu$*-module* M*.*

Proof. Let $f : \mathcal{H}_n \to \mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu)$ be the bimodule isomorphism constructed in Lemma 2.11. Then, there are natural isomorphisms

$$\mathcal{H}_n \otimes_{\mathcal{H}_\nu} M \xrightarrow{f\otimes \mathrm{id}} \mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu) \otimes_{\mathcal{H}_\nu} M \simeq \mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, {}^d\mathcal{H}_\nu \otimes_{\mathcal{H}_\nu} M) \simeq \mathrm{Hom}_{\mathcal{H}_n}(\mathcal{H}_n, {}^dM),$$

the second isomorphism depending on the fact that $\mathcal{H}_n$ is a free left $\mathcal{H}_\mu$-module, see e.g. [AF, 20.10]. □

Recall the duality (2.32) on finite dimensional $\mathcal{H}_n$- (resp. $\mathcal{H}_\nu$-) modules.

Corollary 2.13. *There is a natural isomorphism* $\mathrm{ind}_\nu^n(M^\tau) \simeq (\mathrm{ind}_\mu^n({}^dM))^\tau$ *for every finite dimensional* $\mathcal{H}_\nu$*-module* M*.*

Proof. One routinely checks using (2.2) that the functor $\tau \circ \mathrm{ind}_\mu^n \circ \tau$ (from the category of finite dimensional $\mathcal{H}_\mu$-modules to finite dimensional $\mathcal{H}_n$-modules) is right adjoint to res_μ^n. Hence, it is isomorphic to $\mathrm{Hom}_{\mathcal{H}_\mu}(\mathcal{H}_n, ?)$ by uniqueness of adjoint functors. Now combine this natural isomorphism with the previous corollary (with μ and ν swapped and d replaced by d^{-1}). □

We finally record a special case, which is the analogue of [G_1, Proposition 5.8]:

Theorem 2.14. *Given a finite dimensional* $\mathcal{H}_m$*-module* M *and a finite dimensional* $\mathcal{H}_n$*-module* N*,*

$$(\mathrm{ind}_{m,n}^{m+n} M \boxtimes N)^\tau \cong \mathrm{ind}_{n,m}^{m+n}(N^\tau \boxtimes M^\tau).$$

Moreover, if M *and* N *are irreducible, the same is true with* $\boxtimes$ *replaced by* $\circledast$*.*

§2-k. **Modifications in the degenerate case.** Now we summarize the necessary changes in the degenerate case $q = 1$. So now F is an algebraically closed field of odd characteristic h

and $q = 1$. The superalgebras $\mathcal{P}_n, \mathcal{A}_n, \mathcal{H}_n, \mathcal{H}_\mu, \mathcal{H}_n^{\text{fin}}$ need to be replaced with their degenerate analogues (but we keep the same symbols).

First of all, $\mathcal{P}_n$ becomes the ordinary polynomial algebra $F[x_1, \dots, x_n]$ concentrated in degree $\bar{0}$. Then, $\mathcal{A}_n$ is replaced by the algebra on even generators $x_1, \dots, x_n$ and odd generators $c_1, \dots, c_n$, where the x_i satisfy the relations of a polynomial ring and the c_i satisfy the same Clifford relations as before (2.7), (2.8). The relations (2.9) become instead:

$$c_i x_i = -x_i c_i, \quad c_i x_j = x_j c_i \tag{2.34}$$

for $1 \le i, j \le n$ with $i \ne j$. The algebra $\mathcal{H}_n^{\text{cl}}$ is replaced by the group algebra FS_n of the symmetric group, always writing simply w in place of T_w before.

Now the affine Hecke-Clifford superalgebra becomes the *affine Sergeev superalgebra* of [N]. This is the superalgebra $\mathcal{H}_n$ defined on even generators $x_1, \dots, x_n, s_1, \dots, s_{n-1}$ and odd generators $c_1, \dots, c_n$. The relations between the x's and c's are as in the new $\mathcal{A}_n$, the relations between the s_i are the usual relations of the symmetric group, i.e. (2.11), (2.12) with q there equal to 1, and there are new relations replacing (2.14)–(2.17):

$$s_i c_i = c_{i+1} s_i, \qquad s_i c_{i+1} = c_i s_i, \qquad s_i c_j = c_j s_i, \tag{2.35}$$

$$s_i x_i = x_{i+1} s_i - 1 - c_i c_{i+1}, \qquad s_i x_{i+1} = x_i s_i + 1 - c_i c_{i+1}, \qquad s_i x_j = x_j s_i \tag{2.36}$$

for all admissible i, j with $j \ne i, i+1$. We record the useful formula, being the analogue of (2.23):

$$s_i x_i^j = x_{i+1}^j s_i - \sum_{k=0}^{j-1} (x_i^{j-k-1} x_{i+1}^k + x_i^{j-k-1} (-x_{i+1})^k c_i c_{i+1}) \tag{2.37}$$

for all $j \ge 1, 1 \le i < n$. Finally, $\mathcal{H}_n^{\text{fin}}$ is replaced by the subalgebra of $\mathcal{H}_n$ generated by the c_i, s_j: it is alternatively the twisted tensor product of a Clifford algebra with the group algebra of the symmetric group, i.e. the original *Sergeev superalgebra* considered in [S$_1$], also see [BK$_2$, §3].

The automorphism $\sigma : \mathcal{H}_n \to \mathcal{H}_n$ and antiautomorphism $\tau : \mathcal{H}_n \to \mathcal{H}_n$ are now defined by:

$$\sigma : s_i \mapsto -s_{n-i}, \quad c_j \mapsto c_{n+1-j}, \quad x_j \mapsto x_{n+1-j}; \tag{2.38}$$

$$\tau : s_i \mapsto s_i, \qquad c_j \mapsto c_j, \qquad x_j \mapsto x_j, \tag{2.39}$$

for all $i = 1, \dots, n-1, j = 1, \dots, n$.

The basis theorem, proved in an analogous way to Theorem 2.2, now says that $\mathcal{H}_n$ has a basis given by $\{x^\alpha c^\beta w \mid \alpha \in \mathbb{Z}_{\ge 0}^n, \beta \in \mathbb{Z}_2^n, w \in S_n\}$. The center of $\mathcal{H}_n$, proved as in Theorem 2.3, is now the set of all symmetric polynomials in $x_1^2, x_2^2, \dots, x_n^2$ (see also [N, Prop. 3.1]). Parabolic subalgebras $\mathcal{H}_\mu$ of $\mathcal{H}_n$ are defined in the same way as before. The Mackey Theorem and Theorem 2.14 are proved in an entirely similar way.

3. Cyclotomic Hecke-Clifford superalgebras

§3-a. **Cyclotomic Hecke-Clifford superalgebras.** Keep all the notation from the previous section. Suppose now that $f \in F[X_1] \subset \mathcal{H}_n$ is a polynomial of the form

$$a_d X_1^d + a_{d-1} X_1^{d-1} + \cdots + a_1 X_1 + a_0 \tag{3.1}$$

for coefficients $a_i \in F$ with $a_d = 1$ and $a_i = a_0 a_{d-i}$ for each $i = 0, 1, \dots, d$. This assumption implies that

$$C_1 f = a_0 X_1^{-d} f C_1. \tag{3.2}$$

Define $\mathcal{I}_f$ to be the two-sided ideal of $\mathcal{H}_n$ generated by f and let

$$\mathcal{H}_n^f := \mathcal{H}_n / \mathcal{I}_f. \tag{3.3}$$

We call $\mathcal{H}_n^f$ the *cyclotomic Hecke-Clifford superalgebra* corresponding to f. It is the analogue in our setting of the cyclotomic Hecke algebra of [AK].

§3-b. **Basis theorem.** The goal in this subsection is to describe an explicit basis for $\mathcal{H}_n^f$, analogous to the Ariki-Koike basis [AK] for cyclotomic Hecke algebras. We introduce some further notation for the proof. Set $f_1 = f$ and for $i = 2, \dots, n$, define inductively $f_i = (T_{i-1} + \xi C_{i-1} C_i) f_{i-1} T_{i-1}$. The first lemma follows easily by induction from (2.23):

Lemma 3.1. *For $i = 1, \dots, n$,*

$$f_i = X_i^d + (\text{terms lying in } \mathcal{P}_{i-1} X_i^e \mathcal{H}_i^{\text{fin}} \text{ for } 0 < e < d) + u_i$$

where $u_i \in \mathcal{H}_i^{\text{fin}}$ is a unit.

Given $Z = \{z_1 < \dots < z_u\} \subseteq \{1, \dots, n\}$, let $f_Z = f_{z_1} f_{z_2} \dots f_{z_u} \in \mathcal{H}_n$. Define

$$\Pi_n = \{(\alpha, Z) \mid Z \subseteq \{1, \dots, n\}, \alpha \in \mathbb{Z}^n \text{ with } 0 \leq \alpha_i < d \text{ whenever } i \notin Z\}, \tag{3.4}$$

$$\Pi_n^+ = \{(\alpha, Z) \in \Pi_n \mid Z \neq \varnothing\}. \tag{3.5}$$

Lemma 3.2. *$\mathcal{H}_n$ is a free right $\mathcal{H}_n^{\text{fin}}$-module on basis $\{X^\alpha f_Z \mid (\alpha, Z) \in \Pi_n\}$.*

Proof. Define a total order $\prec$ on $\mathbb{Z}$ so that

$$\lfloor \tfrac{d}{2} \rfloor \prec \lfloor \tfrac{d}{2} \rfloor - 1 \prec \lfloor \tfrac{d}{2} \rfloor + 1 \prec \lfloor \tfrac{d}{2} \rfloor - 2 \prec \lfloor \tfrac{d}{2} \rfloor + 2 \prec \dots.$$

We have a corresponding reverse lexicographic ordering on $\mathbb{Z}^n$: $\alpha \prec \alpha'$ if and only if $\alpha_n = \alpha'_n, \dots, \alpha_{k+1} = \alpha'_{k+1}, \alpha_k \prec \alpha'_k$ for some $k = 1, \dots, n$. It is important that $\mathbb{Z}^n$ has a smallest element with respect to this total order. Define a function $\gamma : \Pi_n \to \mathbb{Z}^n$ by $\gamma(\alpha, Z) := (\gamma_1, \dots, \gamma_n)$ where

$$\gamma_i = \begin{cases} \alpha_i & \text{if } i \notin Z \text{ or } \alpha_i < 0, \\ \alpha_i + d & \text{if } i \in Z \text{ and } \alpha_i \geq 0. \end{cases}$$

We claim that for $(\alpha, Z) \in \Pi_n$,

$$X^\alpha f_Z = X^{\gamma(\alpha, Z)} u + (\text{terms lying in } X^\beta \mathcal{H}_n^{\text{fin}} \text{ for } \beta \prec \gamma(\alpha, Z)),$$

where u is some unit in $\mathcal{H}_n^{\text{fin}}$. In other words, $X^\alpha f_Z = X^{\gamma(\alpha, Z)} u +$(lower terms). Since $\gamma : \Pi_n \to \mathbb{Z}^n$ is a bijection and we already know that the $\{X^\alpha \mid \alpha \in \mathbb{Z}^n\}$ form a basis for $\mathcal{H}_n$ viewed as a right $\mathcal{H}_n^{\text{fin}}$-module by Theorem 2.2, the claim immediately implies the lemma.

To prove the claim, proceed by induction on n. If $n = 1$, the statement is quite obvious. Now assume $n > 1$ and the statement has been proved for $(n-1)$. We need to consider $X^\alpha f_Z$ for $(\alpha, Z) \in \Pi_n$. If $n \notin Z$, then the conclusion follows without difficulty from the induction hypothesis, so assume $n \in Z$. Let $\alpha' = (\alpha_1, \dots, \alpha_{n-1})$ and $Z' = Z - \{n\}$. Then by induction

$$X^{\alpha'} f_{Z'} = X^{\gamma(\alpha', Z')} u' + (\text{terms lying in } X^{\beta'} \mathcal{H}_{n-1}^{\text{fin}} \text{ for } \beta' \prec \gamma(\alpha', Z')),$$

where u' is some unit in $\mathcal{H}_{n-1}^{\text{fin}}$. Multiplying on the left by $X_n^{\alpha_n}$ and on the right by f_n, using Lemma 3.1, one deduces that

$$X^\alpha f_Z = X_n^{\alpha_n+d} X^{\gamma(\alpha',Z')} u' + X_n^{\alpha_n} X^{\gamma(\alpha',Z')} u' u_n + (\text{terms lying in } X^\beta \mathcal{H}_n^{\text{fin}} \text{ for } \beta \prec \gamma(\alpha, Z))$$

where both u' and $u'u_n$ are units in $\mathcal{H}_n^{\text{fin}}$. Noting that

$$X^{\gamma(\alpha,Z)} = \begin{cases} X_n^{\alpha_n+d} X^{\gamma(\alpha',Z')} & \text{if } \alpha_n \geq 0, \\ X_n^{\alpha_n} X^{\gamma(\alpha',Z')} & \text{if } \alpha_n < 0, \end{cases}$$

this is of the desired form. □

Lemma 3.3. (i) *For* $1 \leq i \leq n$,

$$C_i f_i \mathcal{H}_n^{\text{fin}} \subseteq X_i^{-d} f_i \mathcal{H}_n^{\text{fin}} + \sum_{k=1}^{i-1} \mathcal{P}_i f_k \mathcal{H}_n^{\text{fin}}.$$

(ii) *For* $1 \leq i \leq j \leq n$,

$$\mathcal{P}_j (T_{j-1} - \xi) \dots (T_{i+1} - \xi)(T_i - \xi) X_i^{-d} f_i \mathcal{H}_n^{\text{fin}} \subseteq \sum_{k=1}^{j} \mathcal{P}_j f_k \mathcal{H}_n^{\text{fin}}.$$

(iii) *For* $n > 1$, $\mathcal{H}_{n-1}^{\text{fin}} f_n \mathcal{H}_n^{\text{fin}} = f_n \mathcal{H}_n^{\text{fin}}$.

Proof. (i) Proceed by induction on i, the case $i = 1$ being immediate from (3.2). For $i > 1$,

$$\begin{aligned} C_i f_i \mathcal{H}_n^{\text{fin}} &= C_i (T_{i-1} + \xi C_{i-1} C_i) f_{i-1} \mathcal{H}_n^{\text{fin}} = (T_{i-1} - \xi) C_{i-1} f_{i-1} \mathcal{H}_n^{\text{fin}} \\ &\subseteq (T_{i-1} - \xi) X_{i-1}^{-d} f_{i-1} \mathcal{H}_n^{\text{fin}} + \sum_{k=1}^{i-2} (T_{i-1} - \xi) \mathcal{P}_{i-1} f_k \mathcal{H}_n^{\text{fin}} \\ &\subseteq X_i^{-d} (T_{i-1} + \xi C_{i-1} C_i) f_{i-1} \mathcal{H}_n^{\text{fin}} + \sum_{k=1}^{i-1} \mathcal{P}_i f_k \mathcal{H}_n^{\text{fin}} = X_i^{-d} f_i \mathcal{H}_n^{\text{fin}} + \sum_{k=1}^{i-1} \mathcal{P}_i f_k \mathcal{H}_n^{\text{fin}}, \end{aligned}$$

applying Lemma 2.1, the relations in $\mathcal{H}_n$ especially (2.25), and the induction hypothesis.

(ii) Proceed by induction on $(j - i)$, the conclusion being immediate in case $j = i$. If $j > i$, an application of (2.25), combined with (i) to commute C_i past f_i, gives that

$$\mathcal{P}_j (T_{j-1} - \xi) \dots (T_i - \xi) X_i^{-d} f_i \mathcal{H}_n^{\text{fin}} \subseteq \mathcal{P}_j (T_{j-1} - \xi) \dots (T_{i+1} - \xi) X_{i+1}^{-d} f_{i+1} \mathcal{H}_n^{\text{fin}} + \sum_{k=1}^{j} \mathcal{P}_j f_k \mathcal{H}_n^{\text{fin}}.$$

Now apply the induction hypothesis.

(iii) By considering the antiautomorphism τ of $\mathcal{H}_{n-1}^{\text{fin}}$, one sees that $\mathcal{H}_{n-1}^{\text{fin}}$ is generated by the elements C_i for $1 \leq i \leq n-1$ and $(T_j + \xi C_j C_{j+1})$ for $1 \leq j < n-1$, and the latter satisfy the braid relations. We show that each of these generators of $\mathcal{H}_{n-1}^{\text{fin}}$ leave $f_n \mathcal{H}_n^{\text{fin}}$ invariant.

First, consider $C_i f_n \mathcal{H}_n^{\text{fin}}$ for $1 \leq i \leq n-1$. Expanding the definition of f_n and commuting C_i past the leading terms, it equals

$$(T_{n-1} + \xi C_{n-1} C_n) \dots C_i (T_i + \xi C_i C_{i+1}) f_i \mathcal{H}_n^{\text{fin}}.$$

By the relations, $C_i(T_i + \xi C_i C_{i+1}) = (T_i + \xi C_i C_{i+1}) C_{i+1}$. Now the conclusion in this case follows immediately since $C_{i+1} f_i \mathcal{H}_n^{\text{fin}} = f_i \mathcal{H}_n^{\text{fin}}$.

Next, consider $(T_j + \xi C_j C_{j+1}) f_n \mathcal{H}_n^{\mathrm{fin}}$ for $1 \leq j < n-1$. Expanding and commuting again gives that it equals

$$(T_{n-1} + \xi C_{n-1} C_n) \dots (T_j + \xi C_j C_{j+1})(T_{j+1} + \xi C_{j+1} C_{j+2})(T_j + \xi C_j C_{j+1}) f_j \mathcal{H}_n^{\mathrm{fin}}.$$

Applying the braid relation gives

$$(T_{n-1} + \xi C_{n-1} C_n) \dots (T_{j+1} + \xi C_{j+1} C_{j+2})(T_j + \xi C_j C_{j+1})(T_{j+1} + \xi C_{j+1} C_{j+2}) f_j \mathcal{H}_n^{\mathrm{fin}}$$

which again equals $f_n \mathcal{H}_n^{\mathrm{fin}}$ as required. $\square$

Lemma 3.4. $\mathcal{I}_f = \displaystyle\sum_{i=1}^{n} \mathcal{P}_n f_i \mathcal{H}_n^{\mathrm{fin}}$.

Proof. Let $\mathcal{H}_{2\dots n}^{\mathrm{fin}}$ denote the subalgebra of $\mathcal{H}_n^{\mathrm{fin}}$ generated by $C_2, \dots, C_n, T_2, \dots, T_{n-1}$, so $\mathcal{H}_{2\dots n}^{\mathrm{fin}} \cong \mathcal{H}_{n-1}^{\mathrm{fin}}$. We first claim that

$$\mathcal{H}_n^{\mathrm{fin}} \subseteq \sum_{i=1}^{n} (T_{i-1} + \xi C_{i-1} C_i) \dots (T_1 + \xi C_1 C_2) \mathcal{H}_{2\dots n}^{\mathrm{fin}} + \sum_{i=1}^{n} (T_{i-1} - \xi) \dots (T_1 - \xi) C_1 \mathcal{H}_{2\dots n}^{\mathrm{fin}}.$$

To prove this, it suffices to show that $T_w C^\beta$ lies in the right hand side for each $w \in S_n, \beta \in \mathbb{Z}_2^n$. Proceed by induction on the Bruhat order on $w \in S_n$, the case $w = 1$ being trivial. For the induction step, we can find $1 \leq i \leq n$ and $u \in S_{2\dots n}$ such that

$$T_w C^\beta = T_{i-1} \dots T_1 T_u C^\beta = T_{i-1} \dots T_1 C_1^{\beta_1} T_u C^{\beta'}$$

where $\beta' = (\bar{0}, \beta_2, \dots, \beta_n)$. Now an argument using Lemma 2.1 gives that

$$T_w C^\beta = \begin{cases} (T_{i-1} + \xi C_{i-1} C_i) \dots (T_1 + \xi C_1 C_2) T_u C^{\beta'} & \text{if } \beta_1 = \bar{0}, \\ (T_{i-1} - \xi) \dots (T_1 - \xi) C_1 T_u C^{\beta'} & \text{if } \beta_1 = \bar{1}, \end{cases}$$

modulo lower terms of the form $T_{w'} C^{\beta''}$ with $w' < w$. The claim follows.

Now let $\mathcal{J} = \sum_{i=1}^{n} \mathcal{P}_n f_i \mathcal{H}_n^{\mathrm{fin}}$. Clearly $\mathcal{J} \subseteq \mathcal{I}_f$. So it suffices for the lemma to show that $\mathcal{I}_f \subseteq \mathcal{J}$. Noting that $\mathcal{H}_n = \mathcal{P}_n \mathcal{H}_n^{\mathrm{fin}}$ and using the result in the previous paragraph, we get that

$$\begin{aligned}\mathcal{I}_f &= \mathcal{H}_n f_1 \mathcal{H}_n = \mathcal{H}_n f_1 \mathcal{P}_n \mathcal{H}_n^{\mathrm{fin}} = \mathcal{H}_n f_1 \mathcal{H}_n^{\mathrm{fin}} = \mathcal{P}_n \mathcal{H}_n^{\mathrm{fin}} f_1 \mathcal{H}_n^{\mathrm{fin}} \\ &\subseteq \sum_{i=1}^{n} \mathcal{P}_n (T_{i-1} + \xi C_{i-1} C_i) \dots (T_1 + \xi C_1 C_2) f_1 \mathcal{H}_n^{\mathrm{fin}} + \sum_{i=1}^{n} \mathcal{P}_n (T_{i-1} - \xi) \dots (T_1 - \xi) C_1 f_1 \mathcal{H}_n^{\mathrm{fin}}.\end{aligned}$$

Now, each $\mathcal{P}_n (T_{i-1} + \xi C_{i-1} C_i) \dots (T_1 + \xi C_1 C_2) f_1 \mathcal{H}_n^{\mathrm{fin}} = \mathcal{P}_n f_i \mathcal{H}_n^{\mathrm{fin}} \subseteq \mathcal{J}$. Also each $\mathcal{P}_n (T_{i-1} - \xi) \dots (T_1 - \xi) C_1 f_1 \mathcal{H}_n^{\mathrm{fin}}$ is contained in $\mathcal{J}$ thanks to Lemma 3.3(i),(ii). $\square$

Lemma 3.5. $\mathcal{I}_f = \displaystyle\sum_{(\alpha, Z) \in \Pi_n^+} X^\alpha f_Z \mathcal{H}_n^{\mathrm{fin}}$.

Proof. We proceed by induction on n, the case $n = 1$ being almost obvious. So now suppose that $n > 1$. Clearly we can assume that $d > 0$. It will be convenient to write $\mathcal{I}_f'$ for the two-sided ideal of $\mathcal{H}_{n-1}$ generated by f, so

$$\mathcal{I}_f' = \sum_{(\alpha', Z') \in \Pi_{n-1}^+} X^{\alpha'} f_{Z'} \mathcal{H}_{n-1}^{\mathrm{fin}} \tag{3.6}$$

by the induction hypothesis. Let $\mathcal{J} = \sum_{(\alpha, Z) \in \Pi_n^+} X^\alpha f_Z \mathcal{H}_n^{\mathrm{fin}}$. Obviously $\mathcal{J} \subseteq \mathcal{I}_f$. So in view of Lemma 3.4, it suffices to show that $X^\alpha f_i \mathcal{H}_n^{\mathrm{fin}} \subseteq \mathcal{J}$ for each $\alpha \in \mathbb{Z}^n$ and each $i = 1, \dots, n$.

Consider first $X^\alpha f_n \mathcal{H}_n^{\rm fin}$. Write $X^\alpha = X_n^{\alpha_n} X^\beta$ for $\beta \in \mathbb{Z}^{n-1}$. Expanding X^β in terms of the basis of $\mathcal{H}_{n-1}$ from Lemma 3.2, we see that

$$X^\alpha f_n \mathcal{H}_n^{\rm fin} \subseteq \sum_{(\alpha',Z')\in\Pi_{n-1}} X_n^{\alpha_n} X^{\alpha'} f_{Z'} \mathcal{H}_{n-1}^{\rm fin} f_n \mathcal{H}_n^{\rm fin}.$$

This is contained in $\mathcal{J}$ thanks to Lemma 3.3(iii).

Finally, consider $X^\alpha f_i \mathcal{H}_n^{\rm fin}$ with $i < n$. Write $X^\alpha = X_n^{\alpha_n} X^\beta$ for $\beta \in \mathbb{Z}^{n-1}$. By the induction hypothesis,

$$X^\alpha f_i \mathcal{H}_n^{\rm fin} = X_n^{\alpha_n} X^\beta f_i \mathcal{H}_n^{\rm fin} \subseteq \sum_{(\alpha',Z')\in\Pi_{n-1}^+} X_n^{\alpha_n} X^{\alpha'} f_{Z'} \mathcal{H}_n^{\rm fin}.$$

Now we need to consider the cases $\alpha_n \geq 0$ and $\alpha_n < d$ separately. The argument is entirely similar in each case, so suppose in fact that $\alpha_n \geq 0$. Then we show by induction on α_n that $X_n^{\alpha_n} X^{\alpha'} f_{Z'} \mathcal{H}_n^{\rm fin} \subseteq \mathcal{J}$ for each $(\alpha', Z') \in \Pi_{n-1}^+$. This is immediate if $\alpha_n < d$, so take $\alpha_n \geq d$ and consider the induction step. Expanding f_n using Lemma 3.1, the set

$$X_n^{\alpha_n - d} X^{\alpha'} f_{Z'} f_n \mathcal{H}_n^{\rm fin} \subseteq \mathcal{J}$$

looks like the desired $X_n^{\alpha_n} X^{\alpha'} f_{Z'} \mathcal{H}_n^{\rm fin}$ plus a sum of terms lying in $X_n^{\alpha_n-d+e} \mathcal{I}_f' \mathcal{H}_n^{\rm fin}$ with $0 \leq e < d$. It now suffices to show that each such $X_n^{\alpha_n-d+e} \mathcal{I}_f' \mathcal{H}_n^{\rm fin} \subseteq \mathcal{J}$. But by (3.6),

$$X_n^{\alpha_n-d+e} \mathcal{I}_f' \mathcal{H}_n^{\rm fin} \subseteq \sum_{(\alpha',Z')\in\Pi_{n-1}^+} X_n^{\alpha_n-d+e} X^{\alpha'} f_{Z'} \mathcal{H}_n^{\rm fin}$$

and each such term lies in $\mathcal{J}$ by induction, since $0 \leq \alpha_n - d + e < \alpha_n$. □

Theorem 3.6. *The canonical images of the elements*

$$\{X^\alpha C^\beta T_w \mid \alpha \in \mathbb{Z}^n \textit{ with } 0 \leq \alpha_1, \dots, \alpha_n < d, \beta \in \mathbb{Z}_2^n, w \in S_n\}$$

form a basis for $\mathcal{H}_n^f$.

Proof. By Lemmas 3.2 and 3.5, the elements $\{X^\alpha f_Z \mid (\alpha, Z) \in \Pi_n^+\}$ form a basis for $\mathcal{I}_f$ viewed as a right $\mathcal{H}_n^{\rm fin}$-module. Hence Lemma 3.2 implies that the elements

$$\{X^\alpha \mid \alpha \in \mathbb{Z}^n \text{ with } 0 \leq \alpha_1, \dots, \alpha_n < d\}$$

form a basis for a complement to $\mathcal{I}_f$ in $\mathcal{H}_n$ viewed as a right $\mathcal{H}_n^{\rm fin}$-module. The theorem follows at once. □

§3-c. **Cyclotomic Mackey theorem.** We will need a special case of a Mackey theorem for cyclotomic Hecke-Clifford superalgebras. Let $f \in \mathcal{P}_1$ be a polynomial of degree $d > 0$ of the special form (3.1). Let $\mathcal{H}_n^f$ denote the corresponding cyclotomic Hecke-Clifford superalgebra. Given any $y \in \mathcal{H}_n$, we will write $\tilde{y}$ for its canonical image in $\mathcal{H}_n^f$. Thus, Theorem 3.6 says that the elements

$$\{\tilde{X}^\alpha \tilde{C}^\beta \tilde{T}_w \mid \alpha \in \mathbb{Z}^n \text{ with } 0 \leq \alpha_1, \dots, \alpha_n < d, \beta \in \mathbb{Z}_2^n, w \in S_n\}$$

form a basis for $\mathcal{H}_n^f$.

It is obvious from Theorem 3.6 that the subalgebra of $\mathcal{H}_{n+1}^f$ spanned by the $\tilde{X}^\alpha \tilde{C}^\beta \tilde{T}_w$ for $\alpha \in \mathbb{Z}^n$ with $0 \leq \alpha_1, \dots, \alpha_n < d$, $\beta \in \mathbb{Z}_2^n$ and $w \in S_n$ is isomorphic to $\mathcal{H}_n^f$. We will write

$\operatorname{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$ and $\operatorname{res}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$ for the induction and restriction functors between $\mathcal{H}_n^f$ and $\mathcal{H}_{n+1}^f$, to avoid confusion with the affine analogue from (2.27). So,

$$\operatorname{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} M = \mathcal{H}_{n+1}^f \otimes_{\mathcal{H}_n^f} M.$$

Lemma 3.7. *For all* $1 \le i < n, j \ge 0, k \in \mathbb{Z}_2$,

$$X_{i+1}^j C_{i+1}^k T_i - T_i X_i^j C_i^k \in \sum_{h=1}^{j} \left(X_{i+1}^h \mathcal{A}_i + X_{i+1}^{h-1} C_{i+1} \mathcal{A}_i \right).$$

Proof. Rearrange (2.23) and (2.24). $\square$

Lemma 3.8. (i) $\mathcal{H}_{n+1}^f$ *is a free right* $\mathcal{H}_n^f$*-module on basis*

$$\{\tilde{X}_j^a \tilde{C}_j^b \tilde{T}_j \dots \tilde{T}_n \mid 0 \le a < d, b \in \mathbb{Z}_2, 1 \le j \le n+1\}.$$

(ii) $\mathcal{H}_{n+1}^f = \mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f \oplus \bigoplus_{0 \le a < d, b \in \mathbb{Z}_2} \tilde{X}_{n+1}^a \tilde{C}_{n+1}^b \mathcal{H}_n^f$ *as an* $(\mathcal{H}_n^f, \mathcal{H}_n^f)$*-bimodule.*

(iii) *For any* $0 \le a < d$, *there are isomorphisms*

$$\tilde{X}_{n+1}^a \mathcal{H}_n^f \simeq \mathcal{H}_n^f, \qquad \tilde{X}_{n+1}^a \tilde{C}_{n+1} \mathcal{H}_n^f \simeq \Pi \mathcal{H}_n^f, \qquad \mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f \simeq \mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_n^f,$$

as $(\mathcal{H}_n^f, \mathcal{H}_n^f)$*-bimodules.*

Proof. For (i), by Theorem 3.6 and dimension considerations, we just need to check that $\mathcal{H}_{n+1}^f$ is generated as a right $\mathcal{H}_n^f$-module by the given elements. For this, it suffices to show that all elements of the form

$$\tilde{X}^\alpha \tilde{C}^\beta \tilde{T}_j \dots \tilde{T}_n \qquad (\alpha \in \mathbb{Z}^n, \beta \in \mathbb{Z}_2^n, 0 \le \alpha_1, \dots, \alpha_{n+1} < d)$$

lie in the right $\mathcal{H}_n^f$-module generated by $\{\tilde{X}_k^a \tilde{C}_k^b \tilde{T}_k \dots \tilde{T}_n \mid 0 \le a < d, b \in \mathbb{Z}_2, j \le k \le n+1\}$. This involves considering terms of the form $\tilde{X}_k^a \tilde{C}_k^b \tilde{T}_{k-1} \dots \tilde{T}_n$ for $j < k \le n+1$ and $0 \le a < d, b \in \mathbb{Z}_2$, for which Lemma 3.7 is useful.

For (ii),(iii), define a map $\mathcal{H}_n^f \times \mathcal{H}_n^f \to \mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f, (u, v) \mapsto u \tilde{T}_n v$. This is $\mathcal{H}_{n-1}^f$-balanced, so induces a well-defined epimorphism

$$\Phi : \mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_n^f \to \mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f$$

of $(\mathcal{H}_n^f, \mathcal{H}_n^f)$-bimodules. We know from (i) that $\mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_n^f$ is a free right $\mathcal{H}_n^f$-module on basis $\tilde{X}_j^a \tilde{C}_j^b \tilde{T}_j \dots \tilde{T}_{n-1} \otimes 1$ for $1 \le j \le n, 0 \le a < d, b \in \mathbb{Z}_2$. But Φ maps these elements to $\tilde{X}_j^a \tilde{C}_j^b \tilde{T}_j \dots \tilde{T}_{n-1} \tilde{T}_n$ which, again using (i), form a basis for $\mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f$ as a free right $\mathcal{H}_n^f$-module. This shows that $\mathcal{H}_n^f \tilde{T}_n \mathcal{H}_n^f \simeq \mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_n^f$. Now the remaining parts of (ii), (iii) are obvious consequences of (i). $\square$

We have now decomposed $\mathcal{H}_{n+1}^f$ as an $(\mathcal{H}_n^f, \mathcal{H}_n^f)$-bimodule. So the same argument as for Theorem 2.8 easily gives:

Theorem 3.9. *Let* M *be an* $\mathcal{H}_n^f$*-module. Then, there is a natural isomorphism*

$$\operatorname{res}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} \operatorname{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} M \simeq (M \oplus \Pi M)^{\oplus d} \oplus \operatorname{ind}_{\mathcal{H}_{n-1}^f}^{\mathcal{H}_n^f} \operatorname{res}_{\mathcal{H}_{n-1}^f}^{\mathcal{H}_n^f} M$$

of $\mathcal{H}_n^f$*-modules.*

§3-d. **Duality.** We wish next to prove that the induction functor $\mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$ commutes with the τ-duality. We need a little preliminary work.

Lemma 3.10. *For $1 \leq i \leq n$ and $a \geq 0$,*

$$(\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})\dots(\tilde{T}_i + \xi\tilde{C}_i\tilde{C}_{i+1})\tilde{X}_i^a\tilde{T}_i\dots\tilde{T}_n = \tilde{X}_{n+1}^a + (*) \tag{3.7}$$

where $(*)$ is a term lying in $\mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f + \sum_{k=1}^{a-1}\left(\tilde{X}_{n+1}^k\mathcal{H}_n^f + \tilde{X}_{n+1}^{k-1}\tilde{C}_{n+1}\mathcal{H}_n^f\right)$, and

$$(\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})\dots(\tilde{T}_i + \xi\tilde{C}_i\tilde{C}_{i+1})\tilde{X}_i^a\tilde{C}_i\tilde{T}_i\dots\tilde{T}_n = \tilde{X}_{n+1}^a\tilde{C}_{n+1} + (**) \tag{3.8}$$

where $(**)$ is a term lying in $\mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f + \sum_{k=1}^{a}\left(\tilde{X}_{n+1}^k\mathcal{H}_n^f + \tilde{X}_{n+1}^{k-1}\tilde{C}_{n+1}\mathcal{H}_n^f\right)$.

Proof. We prove (3.7) and (3.8) simultaneously by induction on $n = i, i+1, \dots$. In case $n = i$, they follow from a calculation involving (2.23) or (2.24) respectively, together with Lemma 3.7 to commute $\tilde{X}_{n+1}^a$ and $\tilde{X}_{n+1}^a\tilde{C}_{n+1}$ past $\tilde{T}_n$. The induction step is similar, noting that both $(\tilde{T}_{n+1} + \xi\tilde{C}_{n+1}\tilde{C}_{n+2})$ and $\tilde{T}_{n+1}$ centralize $\mathcal{H}_n^f$. □

Lemma 3.11. *For any $s \in \mathcal{H}_n^f$,*

$$\tilde{X}_{n+1}^d s = -a_0 s + \left(\text{a term lying in } \mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f + \sum_{k=1}^{d-1}\left(\tilde{X}_{n+1}^k\mathcal{H}_n^f + \tilde{X}_{n+1}^{k-1}\tilde{C}_{n+1}\mathcal{H}_n^f\right)\right).$$

Proof. Recall the polynomial $f \in F[X_1]$ from (3.1). Its image $\tilde{f} \in \mathcal{H}_n^f$ is zero. Hence,

$$(\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})\dots(\tilde{T}_1 + \xi\tilde{C}_1\tilde{C}_2)\tilde{f}\tilde{T}_1\dots\tilde{T}_n = 0.$$

But a calculation using (3.7) shows that the left hand side equals $\tilde{X}_{n+1}^d + a_0$ modulo terms of the given form. The lemma follows easily on multiplying on the right by s. □

Lemma 3.12. *There exists an even $(\mathcal{H}_n^f, \mathcal{H}_n^f)$-bimodule homomorphism $\theta : \mathcal{H}_{n+1}^f \to \mathcal{H}_n^f$ such that* $\ker\theta$ *contains no non-zero left ideals of $\mathcal{H}_{n+1}^f$.*

Proof. By Lemma 3.8(ii), we know that

$$\mathcal{H}_{n+1}^f = \mathcal{H}_n^f \oplus \tilde{C}_{n+1}\mathcal{H}_n^f \oplus \bigoplus_{a=1}^{d-1}\left(\tilde{X}_{n+1}^a\mathcal{H}_n^f \oplus \tilde{X}_{n+1}^a\tilde{C}_{n+1}\mathcal{H}_n^f\right) \oplus \mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f$$

as an $(\mathcal{H}_n^f, \mathcal{H}_n^f)$-bimodule. Let $\theta : \mathcal{H}_{n+1}^f \to \mathcal{H}_n^f$ be the projection onto the first summand of this bimodule decomposition. We just need to show that if $y \in \mathcal{H}_{n+1}^f$ has the property that $\theta(hy) = 0$ for all $h \in \mathcal{H}_{n+1}^f$, then $y = 0$. Using Lemma 3.8(i), we may write

$$y = \sum_{a=0}^{d-1}\left(\tilde{X}_{n+1}^a s_a + \tilde{X}_{n+1}^a\tilde{C}_{n+1}t_a\right) + \sum_{a=0}^{d-1}\sum_{j=1}^{n}\left(\tilde{X}_j^a\tilde{T}_j\dots\tilde{T}_n u_{a,j} + \tilde{X}_j^a\tilde{C}_j\tilde{T}_j\dots\tilde{T}_n v_{a,j}\right)$$

for $s_a, t_a, u_{a,j}, v_{a,j} \in \mathcal{H}_n^f$. Consider $\theta(\tilde{X}_{n+1}^{d-1}\tilde{C}_{n+1}y)$. An application of Lemma 3.7 reveals that this equals t_{d-1}, hence $t_{d-1} = 0$. Next consider $\theta(\tilde{X}_{n+1}y)$. Using Lemma 3.11 as well as Lemma 3.7 this time, we get that $s_{d-1} = 0$. Now consider similarly $\theta(\tilde{X}_{n+1}^{d-2}\tilde{C}_{n+1}y)$,

$\theta(\tilde{X}_{n+1}^2 y)$, $\theta(\tilde{X}_{n+1}^{d-3}\tilde{C}_{n+1}y), \theta(\tilde{X}_{n+1}^3 y), \dots$ in turn to deduce $t_{d-2} = s_{d-2} = t_{d-3} = s_{d-3} = \dots = 0$.

We have now reduced to the case that

$$y = \sum_{a=0}^{d-1}\sum_{j=1}^{n}\left(\tilde{X}_j^a\tilde{T}_j\dots\tilde{T}_n u_{a,j} + \tilde{X}_j^a\tilde{C}_j\tilde{T}_j\dots\tilde{T}_n v_{a,j}\right).$$

Now consider $y' := (\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})y$. Note

$$(\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})\tilde{T}_{n-1}\tilde{T}_n = \tilde{T}_{n-1}\tilde{T}_n(\tilde{T}_{n-1} + \xi\tilde{C}_{n-1}\tilde{C}_n)$$

so the terms of y with $j < n$ yield terms of y' which lie in $\mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f$. Hence, by Lemma 3.10 too,

$$y' = \tilde{X}_{n+1}^{d-1}\tilde{C}_{n+1}v_{d-1,n} + (*)$$

where $(*)$ is a term lying in $\mathcal{H}_n^f\tilde{T}_n\mathcal{H}_n^f + \sum_{k=1}^{d-1}\left(\tilde{X}_{n+1}^k\mathcal{H}_n^f + \tilde{X}_{n+1}^{k-1}\tilde{C}_{n+1}\mathcal{H}_n^f\right)$. Now multiplying y' by $\tilde{X}_{n+1}^{d-1}\tilde{C}_{n+1}$ and applying θ, as in the previous paragraph, gives that $v_{d-1,n} = 0$. Hence in fact, by Lemma 3.10 once more, we have that

$$y' = \tilde{X}_{n+1}^{d-1}u_{d-1,n} + (*)$$

and now one gets $u_{d-1,n} = 0$ on multiplying by $\tilde{X}_{n+1}$ and applying θ, again as in the previous paragraph. Continuing in this way gives that all $u_{a,n} = v_{a,n} = 0$.

Now repeat the argument in the previous paragraph again, this time considering $y' := (\tilde{T}_n + \xi\tilde{C}_n\tilde{C}_{n+1})(\tilde{T}_{n-1} + \xi\tilde{C}_{n-1}\tilde{C}_n)y$, to get that all $u_{a,n-1} = v_{a,n-1} = 0$. Continuing in this way eventually gives the desired conclusion: $y = 0$. □

Now we are ready to prove the main result of the subsection:

Theorem 3.13. *There is a natural isomorphism $\mathcal{H}_{n+1}^f \otimes_{\mathcal{H}_n^f} M \simeq \mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, M)$ for all $\mathcal{H}_n^f$-modules M.*

Proof. We show that there is an even isomorphism $\varphi : \mathcal{H}_{n+1}^f \to \mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, \mathcal{H}_n^f)$ of $(\mathcal{H}_{n+1}^f, \mathcal{H}_n^f)$-bimodules. The lemma then follows on applying the functor $? \otimes_{\mathcal{H}_n^f} M$: one obtains natural isomorphisms

$$\mathcal{H}_{n+1}^f \otimes_{\mathcal{H}_n^f} M \xrightarrow{\varphi\otimes\mathrm{id}} \mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, \mathcal{H}_n^f) \otimes_{\mathcal{H}_n^f} M \simeq \mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, M).$$

Note the existence of the second isomorphism here uses the fact that $\mathcal{H}_{n+1}^f$ is a projective left $\mathcal{H}_n^f$-module, see [AF, 20.10].

To construct φ, let θ be as in Lemma 3.12, and define $\varphi(h)$ to be the map $h\theta$, for each $h \in \mathcal{H}_{n+1}^f$. One easily checks that $\varphi : \mathcal{H}_{n+1}^f \to \mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, \mathcal{H}_n^f)$ is then a well-defined homomorphism of $(\mathcal{H}_{n+1}^f, \mathcal{H}_n^f)$-bimodules. To see that it is an isomorphism, it suffices by dimensions to check it is injective. Suppose $\varphi(h) = 0$ for some $h \in \mathcal{H}_{n+1}^f$. Then for every $x \in \mathcal{H}_{n+1}^f$, $\theta(xh) = 0$, i.e. the left ideal $\mathcal{H}_{n+1}^f h$ is contained in $\ker\theta$. So Lemma 3.12 implies $h = 0$. □

Corollary 3.14. *$\mathcal{H}_n^f$ is a Frobenius superalgebra, i.e. there is an even isomorphism of left $\mathcal{H}_n^f$-modules $\mathcal{H}_n^f \simeq \mathrm{Hom}_F(\mathcal{H}_n^f, F)$ between the left regular module and the F-linear dual of the right regular module.*

Proof. Proceed by induction on n. For the induction step,

$$\begin{aligned}\mathcal{H}_n^f &\simeq \mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_{n-1}^f \simeq \mathcal{H}_n^f \otimes_{\mathcal{H}_{n-1}^f} \mathrm{Hom}_F(\mathcal{H}_{n-1}^f, F) \\ &\simeq \mathrm{Hom}_{\mathcal{H}_{n-1}^f}(\mathcal{H}_n^f, \mathrm{Hom}_F(\mathcal{H}_{n-1}^f, F)) \simeq \mathrm{Hom}_F(\mathcal{H}_{n-1}^f \otimes_{\mathcal{H}_{n-1}^f} \mathcal{H}_n^f, F) \\ &\simeq \mathrm{Hom}_F(\mathcal{H}_n^f, F),\end{aligned}$$

applying Theorem 3.13 and adjointness of tensor and Hom. □

For the next corollary, recall the duality induced by τ (2.32) on finite dimensional $\mathcal{H}_n$-modules. Since τ leaves the two-sided ideal $\mathcal{I}_f$ invariant, it induces a duality also denoted τ on finite dimensional $\mathcal{H}_n^f$-modules.

Corollary 3.15. *The exact functor* $\mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$ *is both left and right adjoint to* $\mathrm{res}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$. *Moreover, it commutes with duality in the sense that there is a natural isomorphism*

$$\mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}(M^\tau) \simeq (\mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} M)^\tau$$

for all finite dimensional $\mathcal{H}_n^f$*-modules* M.

Proof. The fact that $\mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} = \mathcal{H}_{n+1}^f \otimes_{\mathcal{H}_n^f} ?$ is right adjoint to $\mathrm{res}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f}$ is immediate from Theorem 3.13, since $\mathrm{Hom}_{\mathcal{H}_n^f}(\mathcal{H}_{n+1}^f, ?)$ is right adjoint to restriction by adjointness of tensor and Hom. But on finite dimensional modules, a standard check using (2.2) shows that the functor $\tau \circ \mathrm{ind}_{\mathcal{H}_n^f}^{\mathcal{H}_{n+1}^f} \circ \tau$ is also right adjoint to restriction. Now the remaining part of the corollary follows by uniqueness of adjoint functors. □

§3-e. **Modifications in the degenerate case.** In the degenerate case, $\mathcal{H}_n^f$ becomes the *cyclotomic Sergeev superalgebra* defined for $f \in F[x_1] \subset \mathcal{H}_n$ a polynomial of the form

$$x_1^d + a_{d-2}x_1^{d-2} + a_{d-4}x_1^{d-4} + \dots,$$

i.e. the powers of x_1 appearing are either all even or all odd and the leading coefficient is 1. By definition, $\mathcal{H}_n^f = \mathcal{H}_n/\mathcal{I}_f$ where $\mathcal{I}_f$ is the two-sided ideal generated by f. The basis theorem says that $\mathcal{H}_n^f$ has basis given by the images of

$$\{x^\alpha c^\beta w \mid \alpha \in \mathbb{Z}_{\geq 0}^n \text{ with } 0 \leq \alpha_i < d, \beta \in \mathbb{Z}_2^n, w \in S_n\},$$

where $\mathcal{I}_f$ is the two-sided ideal of $\mathcal{H}_n$ generated by f.

The proof is entirely similar to that of Theorem 3.6; actually in this case it is much more straightforward. To give a little more detail, one defines the element $f_i = s_{i-1} \dots s_1 f s_1 \dots s_{i-1}$ for each $i = 1, \dots, n$ then defines f_Z as before for $Z \subseteq \{1, \dots, n\}$. By (2.37),

$$f_i = x_i^d + (\text{a linear combination of terms lying in } \mathcal{P}_{i-1}x_i^e\mathcal{H}_i^{\mathrm{fin}} \text{ for } 0 \leq e < d).$$

Given this one easily proves as in Lemma 3.2 that the

$$\{x^\alpha f_Z \mid Z \subseteq \{1, \dots, n\}, \alpha \in \mathbb{Z}_{\geq 0}^n \text{ with } 0 \leq \alpha_i < d \text{ if } i \notin Z\}$$

form a basis for $\mathcal{H}_n$ viewed as a right $\mathcal{H}_n^{\mathrm{fin}}$-module. Moreover, arguing as for Lemma 3.5 the

$$\{x^\alpha f_Z \mid \varnothing \neq Z \subseteq \{1, \dots, n\}, \alpha \in \mathbb{Z}_{\geq 0}^n \text{ with } 0 \leq \alpha_i < d \text{ if } i \notin Z\}$$

form a basis for $\mathcal{I}_f$ as a right $\mathcal{H}_n^{\mathrm{fin}}$-module. Finally, the proof is completed as in Theorem 3.6.

Theorem 3.9 goes through without significant alteration. Note a suitable decomposition of $\mathcal{H}^f_{n+1}$ as an $(\mathcal{H}^f_n, \mathcal{H}^f_n)$-bimodule is

$$\mathcal{H}^f_{n+1} = \bigoplus_{0 \le a < d, b \in \mathbb{Z}_2} x^a_{n+1} c^b_{n+1} \mathcal{H}^f_n \oplus \mathcal{H}^f_n s_n \mathcal{H}^f_n. \tag{3.9}$$

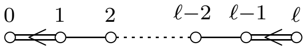
To prove that induction commutes with duality, i.e. Theorem 3.13, there is a slight twist in proving the analogue of Lemma 3.12: the map θ should be taken to be the projection

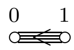
$$\theta : \mathcal{H}^f_{n+1} \to x^{d-1}_{n+1} \mathcal{H}^f_n \simeq \mathcal{H}^f_n$$

along the direct sum decomposition (3.9).

4. The category of integral representations

§4-a. **Affine Kac-Moody algebra.** Now we introduce some standard Lie theoretic notation. Let us treat the case $h \neq \infty$ first, when we let $\ell = (h-1)/2$ and $\mathfrak{g}$ denote the twisted affine Kac-Moody algebra of type $A^{(2)}_{2\ell}$ (over $\mathbb{C}$), see [Kc, ch. 4, table Aff 2]. In particular we label the Dynkin diagram by the index set $I = \{0, 1, \dots, \ell\}$ as follows:



if $\ell \geq 2$, and


if $\ell = 1$.

The weight lattice is denoted P, the simple roots are $\{\alpha_i \,|\, i \in I\} \subset P$ and the corresponding simple coroots are $\{h_i \mid i \in I\} \subset P^*$. The Cartan matrix $(\langle h_i, \alpha_j \rangle)_{0 \le i,j \le \ell}$ is

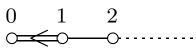
$$\begin{pmatrix} 2 & -2 & 0 & \cdots & 0 & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 & 0 \\ & & & \ddots & & & \\ 0 & 0 & 0 & \dots & 2 & -1 & 0 \\ 0 & 0 & 0 & \dots & -1 & 2 & -2 \\ 0 & 0 & 0 & \dots & 0 & -1 & 2 \end{pmatrix} \quad \text{if } \ell \geq 2, \text{ and} \quad \begin{pmatrix} 2 & -4 \\ -1 & 2 \end{pmatrix} \quad \text{if } \ell = 1.$$

Let $\{\Lambda_i \mid i \in I\} \subset P$ denote fundamental dominant weights, so that $\langle h_i, \Lambda_j \rangle = \delta_{i,j}$, and let $P_+ \subset P$ denote the set of all dominant integral weights. Set

$$c = h_0 + \sum_{i=1}^{\ell} 2h_i, \qquad \delta = \sum_{i=0}^{\ell-1} 2\alpha_i + \alpha_\ell. \tag{4.1}$$

Then the $\Lambda_0, \dots, \Lambda_\ell, \delta$ form a $\mathbb{Z}$-basis for P, and $\langle c, \alpha_i \rangle = \langle h_i, \delta \rangle = 0$ for all $i \in I$.

In the case $h = \infty$, we make the following changes to these definitions. First, we let $\ell = \infty$, and $\mathfrak{g}$ denotes the Kac-Moody algebra of type B_∞, see [Kc, §7.11]. So $I = \{0, 1, 2, \dots\}$, corresponding to the nodes of the Dynkin diagram



Note certain notions, for example the element c from (4.1), only make sense if one passes to the completed algebra b_∞, see [Kc, §7.12], though the intended meaning whenever we make use of them should be obvious regardless.

Now, for either $h < \infty$ or $h = \infty$, we let $U_\mathbb{Q}$ denote the $\mathbb{Q}$-subalgebra of the universal enveloping algebra of $\mathfrak{g}$ generated by the Chevalley generators e_i, f_i, h_i $(i \in I)$. Recall these are subject only to the relations

$$[h_i, h_j] = 0, \qquad [e_i, f_j] = \delta_{i,j} h_i, \tag{4.2}$$

$$[h_i, e_j] = \langle h_i, \alpha_j\rangle e_j, \qquad [h_i, f_j] = -\langle h_i, \alpha_j\rangle f_j, \tag{4.3}$$

$$(\operatorname{ad} e_i)^{1-\langle h_i, \alpha_k\rangle} e_k = 0, \qquad (\operatorname{ad} f_i)^{1-\langle h_i, \alpha_k\rangle} f_k = 0 \tag{4.4}$$

for all $i, j, k \in I$ with $i \neq k$. We let $U_{\mathbb{Z}}$ denote the $\mathbb{Z}$-form of $U_{\mathbb{Q}}$ generated by the divided powers $e_i^{(n)} = e_i^n/n!$ and $f_i^{(n)} = f_i^n/n!$. Then, $U_{\mathbb{Z}}$ has the usual triangular decomposition

$$U_{\mathbb{Z}} = U_{\mathbb{Z}}^- U_{\mathbb{Z}}^0 U_{\mathbb{Z}}^+.$$

We are particularly concerned here with the plus part $U_{\mathbb{Z}}^+$, generated by all $e_i^{(n)}$. It is a graded Hopf algebra over $\mathbb{Z}$ via the *principal grading* $\deg(e_i^{(n)}) = n$ for all $i \in I, n \geq 0$.

§4-b. **Cyclotomic Hecke-Clifford superalgebras revisited.** Given $i \in I$, define

$$q(i) := 2\frac{q^{2i+1} + q^{-2i-1}}{q + q^{-1}} \in F. \tag{4.5}$$

Note in particular that $q(0) = 2$. For $\lambda \in P_+$, let $\mathcal{I}_\lambda$ denote the two-sided ideal of $\mathcal{H}_n$ generated by the element

$$(X_1 - 1)^{\langle h_0, \lambda\rangle} \prod_{i=1}^{\ell} (X_1 + X_1^{-1} - q(i))^{\langle h_i, \lambda\rangle} \tag{4.6}$$

Up to a power of the unit X_1, this is an element of the form (3.1), so the quotient superalgebra

$$\mathcal{H}_n^\lambda := \mathcal{H}_n / \mathcal{I}_\lambda$$

is a special case of the cyclotomic Hecke-Clifford superalgebras introduced in the previous section. This quotient of $\mathcal{H}_n$ defined for $\lambda \in P_+$ should not be confused with the parabolic subalgebra $\mathcal{H}_\mu$ defined earlier for μ a composition of n.

Theorem 3.6 immediately gives the following basis theorem for $\mathcal{H}_n^\lambda$:

Theorem 4.1. *For any $\lambda \in P_+$, the canonical images of the elements*

$$\{X^\alpha C^\beta T_w \mid \alpha \in \mathbb{Z}^n \text{ with } 0 \leq \alpha_1, \dots, \alpha_n < \langle c, \lambda\rangle, \beta \in \mathbb{Z}_2^n, w \in S_n\}$$

form a basis for $\mathcal{H}_n^\lambda$. In particular, $\dim \mathcal{H}_n^\lambda = (2\langle c, \lambda\rangle)^n (n!)$.

Remark 4.2. In the special case $\lambda = \Lambda_0$ is the first fundamental dominant weight, the cyclotomic Hecke-Clifford superalgebra $\mathcal{H}_n^\lambda$ can be identified with the superalgebra $\mathcal{H}_n^{\mathrm{fin}}$. This follows easily from Theorem 4.1: since $\langle c, \Lambda_0\rangle = 1$, $\mathcal{H}_n^{\Lambda_0}$ has basis given by the images of the elements $\{C^\beta T_w \mid \beta \in \mathbb{Z}_2^n, w \in S_n\}$ just as $\mathcal{H}_n^{\mathrm{fin}}$, and the multiplications are the same by construction. See also [JN, Prop. 3.5].

Introduce the functors

$$\mathrm{pr}^\lambda : \mathcal{H}_n\text{-mod} \to \mathcal{H}_n^\lambda\text{-mod}, \qquad \mathrm{infl}^\lambda : \mathcal{H}_n^\lambda\text{-mod} \to \mathcal{H}_n\text{-mod}\,. \tag{4.7}$$

Here, infl^λ is simply inflation along the canonical epimorphism $\mathcal{H}_n \to \mathcal{H}_n^\lambda$, while on a module M, $\mathrm{pr}^\lambda M = M/\mathcal{I}_\lambda M$ with the induced action of $\mathcal{H}_n^\lambda$. The functor infl^λ is right adjoint to pr^λ, i.e. there is a functorial isomorphism

$$\mathrm{Hom}_{\mathcal{H}_n^\lambda}(\mathrm{pr}^\lambda M, N) \simeq \mathrm{Hom}_{\mathcal{H}_n}(M, \mathrm{infl}^\lambda N). \tag{4.8}$$

Note we will generally be sloppy and omit the functor infl^λ in our notation. In other words, we generally identify $\mathcal{H}_n^\lambda$-mod with the full subcategory of $\mathcal{H}_n$-mod consisting of all modules M with $\mathcal{I}_\lambda M = 0$.

§4-c. **Elements $\tilde{\boldsymbol{\Phi}}_{\boldsymbol{j}}$.** We will need certain elements of $\mathcal{H}_n$ defined originally by Jones and Nazarov. Given $1 \le j < n$, define

$$z_j := X_j + X_j^{-1} - X_{j+1} - X_{j+1}^{-1} = X_j^{-1}(X_jX_{j+1} - 1)(X_jX_{j+1}^{-1} - 1), \tag{4.9}$$

$$\tilde{\Phi}_j := z_j^2 T_j + \xi \frac{z_j^2}{X_jX_{j+1}^{-1} - 1} - \xi \frac{z_j^2}{X_jX_{j+1} - 1} C_jC_{j+1}. \tag{4.10}$$

Then $\tilde{\Phi}_j$ is equal to $z_j^2\Phi_j$ where Φ_j is the element defined by Jones and Nazarov in [JN, (3.6)]. Note $\tilde{\Phi}_j$ really does make sense as an element of $\mathcal{H}_n$, unlike Φ_j which belongs to a certain localization. An easy calculation as in [JN, (3.7)] gives that

$$\tilde{\Phi}_j X_j^{\pm1} = X_{j+1}^{\pm1}\tilde{\Phi}_j, \qquad \tilde{\Phi}_j X_{j+1}^{\pm1} = X_j^{\pm1}\tilde{\Phi}_j, \qquad \tilde{\Phi}_j X_k^{\pm1} = X_k^{\pm1}\tilde{\Phi}_j, \tag{4.11}$$

$$\tilde{\Phi}_j C_j = C_{j+1}\tilde{\Phi}_j, \qquad \tilde{\Phi}_j C_{j+1} = C_j\tilde{\Phi}_j, \qquad \tilde{\Phi}_j C_k = C_k\tilde{\Phi}_j \tag{4.12}$$

for $k \neq j, j+1$. Moreover, [JN, Prop. 3.1] implies that

$$\begin{aligned}\tilde{\Phi}_j^2 = z_j^2\big(X_j^{-2}(X_jX_{j+1} - 1)^2(X_jX_{j+1}^{-1} - 1)^2 - \xi^2 X_j^{-1}X_{j+1}^{-1}(X_jX_{j+1} - 1)^2 \\ - \xi^2 X_j^{-1}X_{j+1}(X_jX_{j+1}^{-1} - 1)^2\big).\end{aligned} \tag{4.13}$$

In order to make use of this, we need the following technical lemma:

Lemma 4.3. *Suppose $a, b \in F^\times$ with $b + b^{-1} = q(i)$ for some $i \in I$. If*

$$\begin{aligned}a^{-2}(ab-1)^2(ab^{-1}-1)^2\big(a^{-2}(ab-1)^2(ab^{-1}-1)^2 - \xi^2 a^{-1}b^{-1}(ab-1)^2 \\ - \xi^2 a^{-1}b(ab^{-1}-1)^2\big) = 0.\end{aligned}$$

then $a + a^{-1} = q(j)$ for $j \in I$ with $|i - j| \le 1$.

Proof. Follow [JN, (4.1)–(4.4)]. □

A more lengthy calculation also as in [JN, Prop. 3.1] shows that the elements $\tilde{\Phi}_j$ satisfy the braid relations, i.e.

$$\tilde{\Phi}_i\tilde{\Phi}_j = \tilde{\Phi}_j\tilde{\Phi}_i, \qquad \tilde{\Phi}_i\tilde{\Phi}_{i+1}\tilde{\Phi}_i = \tilde{\Phi}_{i+1}\tilde{\Phi}_i\tilde{\Phi}_{i+1}, \tag{4.14}$$

for all admissible i, j with $|i - j| > 1$. This means that for any $w \in S_n$, we obtain well-defined elements $\tilde{\Phi}_w \in \mathcal{H}_n$, namely, $\tilde{\Phi}_w := \tilde{\Phi}_{i_1} \dots \tilde{\Phi}_{i_m}$ where $w = s_{i_1} \dots s_{i_m}$ is any reduced expression for w. According to (4.11),(4.12), these elements have the property that

$$\tilde{\Phi}_w X_i^{\pm1} = X_{wi}^{\pm1}\tilde{\Phi}_w, \qquad \tilde{\Phi}_w C_i = C_{wi}\tilde{\Phi}_w, \tag{4.15}$$

for all $w \in S_n, 1 \le i \le n$. Note we will not make essential use of (4.15) or the fact that the $\tilde{\Phi}_i$ satisfy the braid relations in what follows.

§4-d. **Integral representations.** Now call an $\mathcal{A}_n$-module M *integral* if it is finite dimensional and moreover all eigenvalues of $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$ on M are of the form $q(i)$ for $i \in I$, see (4.5). Call an $\mathcal{H}_n$-module, or more generally an $\mathcal{H}_\mu$-module for μ a composition of n, *integral* if it is integral on restriction to $\mathcal{A}_n$. In what follows we will restrict our attention to these modules, and write $\mathrm{Rep}_I\, \mathcal{H}_n$ (resp. $\mathrm{Rep}_I\, \mathcal{A}_n$, $\mathrm{Rep}_I\, \mathcal{H}_\mu$) for the full subcategory of $\mathcal{H}_n$-mod (resp. $\mathcal{A}_n$-mod, $\mathcal{H}_\mu$-mod) consisting of all integral modules.

Lemma 4.4. *Let M be a finite dimensional $\mathcal{H}_n$-module, and $1 \leq j \leq n$. Assume that the eigenvalues of $X_j + X_j^{-1}$ on M are of the form $q(i)$, $i \in I$. Then the same is true for the eigenvalues of all other $X_k + X_k^{-1}$, $k = 1, 2, \dots, n$.*

Proof. It suffices to show that the eigenvalues of $X_j + X_j^{-1}$ are of the form $q(i)$ if and only if the eigenvalues of $X_{j+1} + X_{j+1}^{-1}$ are of the same form, for $1 \leq j < n$. Actually, by an argument involving conjugation with the automorphism σ, it suffices just to prove the 'if' part. So assume that all eigenvalues of $X_{j+1} + X_{j+1}^{-1}$ on M are of the form $q(i)$ for various $i \in I$. Let $a \neq 0$ be an eigenvalue for the action of X_j on M. We have to prove that $a + a^{-1}$ is also of the form $q(i)$. Since X_j and X_{j+1} commute, we can pick v lying in the a-eigenspace of X_j so that v is also an eigenvector for X_{j+1}, of eigenvalue b say. By assumption, $b + b^{-1} = q(i)$ for some $i \in I$. Now let $\tilde{\Phi}_j$ be the element (4.10). By (4.11), $(X_{j+1} + X_{j+1}^{-1})\tilde{\Phi}_j = \tilde{\Phi}_j(X_j + X_j^{-1})$. So if $\tilde{\Phi}_j v \neq 0$, we get that

$$(a + a^{-1})\tilde{\Phi}_j v = \tilde{\Phi}_j(X_j + X_j^{-1})v = (X_{j+1} + X_{j+1}^{-1})\tilde{\Phi}_j v$$

so that $a + a^{-1} = q(i')$ for some $i' \in I$ by assumption. Else, $\tilde{\Phi}_j v = 0$ so $\tilde{\Phi}_j^2 v = 0$. So applying (4.13) and Lemma 4.3, we again get that $a + a^{-1} = q(i')$ for some $i' \in I$. $\square$

Corollary 4.5. *Let M be a finite dimensional $\mathcal{H}_n$-module. Then M is integral if and only if $\mathcal{I}_\lambda M = 0$ for some $\lambda \in P_+$.*

Proof. If $\mathcal{I}_\lambda M = 0$, then the eigenvalues of $X_1 + X_1^{-1}$ on M are of the form $q(i)$ for $i \in I$, by definition of $\mathcal{I}_\lambda$. Hence M is integral in view of Lemma 4.4. Conversely, suppose that M is integral. Then the minimal polynomial of $X_1 + X_1^{-1}$ on M is of the form $\prod_{i \in I}(t - q(i))^{\lambda_i}$ for some $\lambda_i \geq 0$. So if we set $\lambda = 2\lambda_0\Lambda_0 + \lambda_1\Lambda_1 + \cdots + \lambda_\ell\Lambda_\ell \in P_+$, we certainly have that the element (4.6) acts as zero on M. $\square$

Recall from §2-c that $\operatorname{Rep} \mathcal{H}_n^\lambda$ denotes the category of all finite dimensional $\mathcal{H}_n^\lambda$-modules. Corollary 4.5 implies that the functors pr^λ and infl^λ from (4.7) restrict to a well-defined adjoint pair of functors at the level of integral representations:

$$\mathrm{pr}^\lambda : \operatorname{Rep}_I \mathcal{H}_n \to \operatorname{Rep} \mathcal{H}_n^\lambda, \qquad \mathrm{infl}^\lambda : \operatorname{Rep} \mathcal{H}_n^\lambda \to \operatorname{Rep}_I \mathcal{H}_n. \tag{4.16}$$

Let us also check at this point that induction from a parabolic subalgebra of $\mathcal{H}_n$ preserves integral representations, the analogous fact for restriction being obvious.

Lemma 4.6. *Let μ be a composition of n and M be an integral $\mathcal{H}_\mu$-module. Then,* $\mathrm{ind}_\mu^n M$ *is an integral $\mathcal{H}_n$-module.*

Proof. By Theorem 2.2, $\mathrm{ind}_\mu^n M$ is spanned by elements $T_w \otimes m$ for $m \in M$, in particular it is finite dimensional. Let

$$Y_j = \prod_{i \in I}(X_j + X_j^{-1} - q(i)).$$

By Corollary 4.5, it suffices to show that Y_1^N annihilates $\mathrm{ind}_\lambda^n M$ for sufficiently large N. Consider $Y_1^N T_w \otimes m$ for $w \in S_n, m \in M$. We may write $T_w = T_u T_1 \dots T_k$ for $u \in S_{2\dots n} \cong S_{n-1}$ and $0 \leq k < n$. Then, Y_1^N commutes with T_u, so we just need to consider $Y_1^N T_1 \dots T_k \otimes m$. Now using the commutation relations, one checks that $Y_1^N T_1 \dots T_k \otimes m$ can be rewritten as an $\mathcal{H}_n$-linear combination of elements of the form $1 \otimes Y_j^{N'} m$ for $1 \leq j \leq n$ and $N - k \leq N' \leq N$. Since M is integral by assumption, we can choose N sufficiently large so that each such term is zero. $\square$

It follows that the functors $\mathrm{ind}_\mu^n, \mathrm{res}_\mu^n$ restrict to well-defined functors

$$\mathrm{ind}_\mu^n : \mathrm{Rep}_I \mathcal{H}_\mu \to \mathrm{Rep}_I \mathcal{H}_n, \qquad \mathrm{res}_\mu^n : \mathrm{Rep}_I \mathcal{H}_n \to \mathrm{Rep}_I \mathcal{H}_\mu \tag{4.17}$$

on integral representations. Similar remarks apply to more general induction and restriction between nested parabolic subalgebras of $\mathcal{H}_n$.

§4-e. **Modules over $\boldsymbol{\mathcal{A}_n}$.** Let $i \in I$ and define

$$b_\pm(i) = \frac{q(i)}{2} \pm \sqrt{\frac{q(i)^2}{4} - 1}, \tag{4.18}$$

i.e. the roots of the equation $x + x^{-1} = q(i)$. Let $L(i)$ denote the vector superspace on basis w, w', where w is even and w' is odd, made into an $\mathcal{A}_1$-module so that

$$C_1 w = w',\ C_1 w' = w,\ X_1^{\pm 1} w = b_\pm(i) w,\ X_1^{\pm 1} w' = b_\mp(i) w'.$$

One easily checks:

Lemma 4.7. *For each $i \in I$, $L(i)$ is an irreducible $\mathcal{A}_1$-module, of type* `M` *if $i \neq 0$ and of type* `Q` *if $i = 0$. Moreover, the modules $\{L(i) \mid i \in I\}$ form a complete set of pairwise non-isomorphic irreducibles in* $\mathrm{Rep}_I \mathcal{A}_1$.

Recall that $\mathcal{A}_n \cong \mathcal{A}_1 \otimes \cdots \otimes \mathcal{A}_1$ (n times) as superalgebras. Hence, for $\underline{i} = (i_1, \dots, i_n) \in I^n$, we can consider the irreducible $\mathcal{A}_n$-module $L(i_1) \circledast \cdots \circledast L(i_n)$. By Lemma 4.7 and the general theory of outer tensor products §2-b, one obtains:

Lemma 4.8. *The $\mathcal{A}_n$-modules $\{L(i_1) \circledast \cdots \circledast L(i_n) \mid \underline{i} \in I^n\}$ form a complete set of pairwise non-isomorphic irreducible $\mathcal{A}_n$-modules. Moreover, let γ_0 denote the number of $j = 1, \dots, n$ such that $i_j = 0$. Then, $L(i_1) \circledast \cdots \circledast L(i_n)$ is of type* `M` *if γ_0 is even and type* `Q` *if γ_0 is odd. Finally,* $\dim L(i_1) \circledast \cdots \circledast L(i_n) = 2^{n - \lfloor \gamma_0/2 \rfloor}$.

Now let M be any module in $\mathrm{Rep}_I \mathcal{A}_n$. For any $\underline{i} \in I^n$, let $M[\underline{i}]$ be the largest submodule of M all of whose composition factors are isomorphic to $L(i_1) \circledast \cdots \circledast L(i_n)$. Alternatively, since each $X_k + X_k^{-1}$ acts on $L(i_1) \circledast \cdots \circledast L(i_n)$ by the scalar $q(i_k)$ and all the scalars $q(i)$ for $i \in I$ are distinct, we can describe $M[\underline{i}]$ as the simultaneous generalized eigenspace for the commuting operators $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$ corresponding to eigenvalues $q(i_1), \dots, q(i_n)$, respectively. Hence:

Lemma 4.9. *For any $M \in \mathrm{Rep}_I \mathcal{A}_n$, $M = \bigoplus_{\underline{i} \in I^n} M[\underline{i}]$ as an $\mathcal{A}_n$-module.*

We write $K(\mathrm{Rep}_I \mathcal{A}_n)$, $K(\mathrm{Rep}_I \mathcal{H}_n)$, … for the Grothendieck groups of the categories $\mathrm{Rep}_I \mathcal{A}_n, \mathrm{Rep}_I \mathcal{H}_n$, … , defined as in §2-c. Note for an integral $\mathcal{A}_n$-module M, knowledge of the dimensions of the spaces $M[\underline{i}]$ for all $\underline{i}$ is equivalent to knowing the coefficients $a_{\underline{i}}$ when $[M] \in K(\mathrm{Rep}_I \mathcal{A}_n)$ is expanded as

$$[M] = \sum_{\underline{i} \in I^n} a_{\underline{i}} [L(i_1) \circledast \cdots \circledast L(i_n)]$$

in terms of the basis $\{[L(i_1) \circledast \cdots \circledast L(i_n)] \mid \underline{i} \in I^n\}$.

Now suppose instead that M is an integral $\mathcal{H}_n$-module, so that its restriction $\mathrm{res}^n_{1,\dots,1} M$ to $\mathcal{A}_n$ is in $\mathrm{Rep}_I \mathcal{A}_n$. We define the *formal character* of M by:

$$\mathrm{ch}\, M = [\mathrm{res}^n_{1,\dots,1} M] \in K(\mathrm{Rep}_I \mathcal{A}_n). \tag{4.19}$$

Since the functor $\mathrm{res}^n_{1,\dots,1}$ is exact, ch induces a homomorphism

$$\mathrm{ch}\ : K(\mathrm{Rep}_I\,\mathcal{H}_n) \to K(\mathrm{Rep}_I\,\mathcal{A}_n)$$

at the level of Grothendieck groups. We will later see that this map is actually injective (Theorem 5.12), justifying the terminology. Note we will occasionally consider characters of integral modules over parabolic subalgebras $\mathcal{H}_\mu$ for μ a composition of n, or over the cyclotomic algebras $\mathcal{H}^\lambda_n$ for $\lambda \in P_+$. The definitions are modified in these cases in obvious ways.

Lemma 4.10. *Let $\underline{i} = (i_1, \dots, i_n) \in I^n$. Then*

$$\mathrm{ch}\,\mathrm{ind}^n_{1,\dots,1} L(i_1) \circledast \cdots \circledast L(i_n) = \sum_{w \in S_n} [L(i_{w^{-1}1}) \circledast \cdots \circledast L(i_{w^{-1}n})]$$

Proof. This follows from Theorem 2.8 with $\mu = \nu = (1^n)$. □

Lemma 4.11. ("Shuffle Lemma") *Let $n = m+k$, and let $M \in \mathrm{Rep}_I\,\mathcal{H}_m$ and $K \in \mathrm{Rep}_I\,\mathcal{H}_k$ be irreducible. Assume*

$$\mathrm{ch}\,M = \sum_{\underline{i} \in I^m} a_{\underline{i}} [L(i_1) \circledast \cdots \circledast L(i_m)], \qquad \mathrm{ch}\,K = \sum_{\underline{j} \in I^k} b_{\underline{j}} [L(j_1) \circledast \cdots \circledast L(j_k))].$$

Then

$$\mathrm{ch}\,\mathrm{ind}^n_{m,k} M \circledast K = \sum_{\underline{i} \in I^m} \sum_{\underline{j} \in I^k} a_{\underline{i}} b_{\underline{j}} (\sum_{\underline{h}} L(h_1) \circledast \cdots \circledast L(h_n)),$$

where the last sum is over all $\underline{h} = (h_1, \dots, h_n) \in I^n$ which are obtained by shuffling $\underline{i}$ and $\underline{j}$, i.e. there exist $1 \le u_1 < \cdots < u_m \le n$ such that $(h_{u_1}, \dots, h_{u_m}) = (i_1, \dots, i_m)$, and $(h_1, \dots, \widehat{h}_{u_1}, \dots, \widehat{h}_{u_m}, \dots, h_n) = (j_1, \dots, j_k)$.

Proof. This follows from Theorem 2.8 with $\mu = (1^n)$ and $\nu = (m, k)$. □

§4-f. **Central characters.** Recall by Theorem 2.3 that every element z of the center $Z(\mathcal{H}_n)$ of $\mathcal{H}_n$ can be written as a symmetric polynomial $f(X_1 + X_1^{-1}, \dots, X_n + X_n^{-1})$ in the $X_k + X_k^{-1}$. Given $\underline{i} \in I^n$, we associate the *central character*

$$\chi_{\underline{i}} : Z(\mathcal{H}_n) \to F, \quad f(X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}) \mapsto f(q(i_1), \dots, q(i_n)).$$

Consider the natural left action of S_n on I^n by place permutation. We write $\underline{i} \sim \underline{j}$ if $\underline{i}, \underline{j}$ lie in the same orbit. The following lemma follows immediately from the fact that the $q(i)$ are distinct as i runs over the index set I.

Lemma 4.12. *For $\underline{i}, \underline{j} \in I^n$, $\chi_{\underline{i}} = \chi_{\underline{j}}$ if and only if $\underline{i} \sim \underline{j}$.*

Given $\underline{i} \in I^n$, we define its *weight* $\mathrm{wt}(\underline{i}) \in P$ by

$$\mathrm{wt}(\underline{i}) = \sum_{i \in I} \gamma_i \alpha_i \qquad \text{where} \qquad \gamma_i = \sharp\{j = 1, \dots, n \mid i_j = i\}. \tag{4.20}$$

So $\mathrm{wt}(\underline{i})$ is an element of the set Γ_n of non-negative integral linear combinations $\gamma = \sum_{i \in I} \gamma_i \alpha_i$ of the simple roots such that $\sum_{i \in I} \gamma_i = n$. Obviously, the S_n-orbit of $\underline{i}$ is uniquely determined by its weight, so we obtain a labelling of the orbits of S_n on I^n by the elements of Γ_n. We will also use the notation χ_γ for the central character $\chi_{\underline{i}}$ where $\underline{i}$ is any element of I^n with $\mathrm{wt}(\underline{i}) = \gamma$. So $\chi_{\underline{i}} = \chi_{\mathrm{wt}(\underline{i})}$.

Now let M be an integral $\mathcal{H}_n$-module and $\gamma = \sum_{i \in I} \gamma_i \alpha_i \in \Gamma_n$. We let $M[\gamma]$ denote the generalized eigenspace of M over $Z(\mathcal{H}_n)$ that corresponds to the central character χ_γ, i.e.

$$M[\gamma] = \{m \in M \mid (z - \chi_\gamma(z))^k m = 0 \text{ for all } z \in Z(\mathcal{H}_n) \text{ and } k \gg 0\}.$$

Observe this is an $\mathcal{H}_n$-submodule of M. Now, for any $\underline{i} \in I^n$ with $\mathrm{wt}(\underline{i}) = \gamma$, $Z(\mathcal{H}_n)$ acts on $L(i_1) \circledast \cdots \circledast L(i_n)$ via the central character χ_γ. So applying Lemma 4.12, we see that

$$M[\gamma] = \bigoplus_{\underline{i} \text{ with } \mathrm{wt}(\underline{i})=\gamma} M[\underline{i}],$$

recalling the decomposition of M as an $\mathcal{A}_n$-module from Lemma 4.9. Therefore:

Lemma 4.13. *Any integral $\mathcal{H}_n$-module M decomposes as*

$$M = \bigoplus_{\gamma \in \Gamma_n} M[\gamma]$$

as an $\mathcal{H}_n$-module.

Thus the $\{\chi_\gamma \,|\, \gamma \in \Gamma_n\}$ exhaust the possible central characters that can arise in an integral $\mathcal{H}_n$-module, while Lemma 4.10 shows that every such central character does arise in some integral $\mathcal{H}_n$-module.

Let us write $\mathrm{Rep}_\gamma \mathcal{H}_n$ for the full subcategory of $\mathrm{Rep}_I \mathcal{H}_n$ consisting of all modules M with $M[\gamma] = M$. Then, Lemma 4.13 implies that there is an equivalence of categories

$$\mathrm{Rep}_I \mathcal{H}_n \cong \bigoplus_{\gamma \in \Gamma_n} \mathrm{Rep}_\gamma \mathcal{H}_n. \tag{4.21}$$

We say that $\mathrm{Rep}_\gamma \mathcal{H}_n$ is the *block* of $\mathrm{Rep}_I \mathcal{H}_n$ corresponding to the central character χ_γ. In particular, if $M \neq 0$ is indecomposable then M belongs to $\mathrm{Rep}_\gamma \mathcal{H}_n$, i.e. $M = M[\gamma]$, for a unique $\gamma \in \Gamma_n$.

We can extend some of these notions to $\mathcal{H}_n^\lambda$-modules, for $\lambda \in P_+$. In particular, if $M \in \mathrm{Rep}\, \mathcal{H}_n^\lambda$, we also write $M[\gamma]$ for the summand $M[\gamma]$ of M defined by first viewing M as an $\mathcal{H}_n$-module by inflation. Also write $\mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$ for the full subcategory of $\mathrm{Rep}\, \mathcal{H}_n^\lambda$ consisting of the modules M with $M = M[\gamma]$. Thus we also have a decomposition

$$\mathrm{Rep}\, \mathcal{H}_n^\lambda \cong \bigoplus_{\gamma \in \Gamma_n} \mathrm{Rep}_\gamma \mathcal{H}_n^\lambda \tag{4.22}$$

induced by (4.21). Note though that we should not yet refer to $\mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$ as a block of $\mathrm{Rep}\, \mathcal{H}_n^\lambda$: the center of $\mathcal{H}_n^\lambda$ may be larger than the image of the center of $\mathcal{H}_n$, so we cannot yet assert that $Z(\mathcal{H}_n^\lambda)$ acts on $M[\gamma]$ by a single central character. Also we no longer know precisely which $\gamma \in \Gamma_n$ have the property that $\mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$ is non-trivial. These questions will be settled in §8-d.

§4-g. **Kato's theorem.** Let $i \in I$. Introduce the *principal series module*

$$L(i^n) := \mathrm{ind}_{1,\dots,1}^n L(i) \circledast \cdots \circledast L(i). \tag{4.23}$$

By Lemma 4.10, we know immediately that $\mathrm{ch}\, L(i^n) = n![L(i) \circledast \cdots \circledast L(i)]$, hence $L(i^n)$ belongs to the block $\mathrm{Rep}_{n\alpha_i} \mathcal{H}_n$. In particular, for each $k = 1, \dots, n$, the only eigenvalue of the element $X_k + X_k^{-1}$ on $L(i^n)$ is $q(i)$. In addition, if $i = 0$, then the only eigenvalue of each X_k is 1.

Lemma 4.14. *Let $n \geq 2$, $1 \leq j < n$, $i \in I - \{0\}$, and $v \in L(i) \boxtimes \cdots \boxtimes L(i)$ (n copies). Then $X_j^{-1}(1 - C_jC_{j+1})v \neq (1 - C_jC_{j+1})X_{j+1}v$.*

Proof. The elements of $\mathcal{A}_n$ which are involved in the inequality act only on the positions j and $j+1$ in the tensor product. So we may assume that $n = 2$ and $j = 1$. Let

$$v = aw \otimes w + bw \otimes w' + cw' \otimes w + dw' \otimes w'$$

for $a, b, c, d \in F$. Then

$$X_1^{-1}(1 - C_1C_2)v = (b_-(i)a + b_-(i)d)w \otimes w + (b_-(i)b + b_-(i)c)w \otimes w' \\ + (b_+(i)c - b_+(i)b)w' \otimes w + (b_+(i)d - b_+(i)a)w' \otimes w',$$

$$(1 - C_1C_2)X_2v = (b_+(i)a + b_-(i)d)w \otimes w + (b_-(i)b + b_+(i)c)w \otimes w' \\ + (b_+(i)c - b_-(i)b)w' \otimes w + (b_-(i)d - b_+(i)a)w' \otimes w'.$$

Now the lemma follows from the inequality $b_-(i) \neq b_+(i)$ for $i \neq 0$. $\square$

Lemma 4.15. *Let $i \in I$. Set $L = L(i) \circledast \cdots \circledast L(i)$, so $L(i^n) = \mathcal{H}_n \otimes_{\mathcal{A}_n} L$.*

(i) *If $i \neq 0$, the common $q(i)$-eigenspace of the operators $X_1 + X_1^{-1}, \dots, X_{n-1} + X_{n-1}^{-1}$ on $L(i^n)$ is precisely $1 \otimes L$, which is contained in the $q(i)$-eigenspace of $X_n + X_n^{-1}$ too. Moreover, all Jordan blocks of $X_1 + X_1^{-1}$ on $L(i^n)$ are of size n.*

(ii) *If $i = 0$, the common 1-eigenspace of the operators $X_1, \dots, X_{n-1}$ on $L(i^n)$ is precisely $1 \otimes L$, which is contained in the 1-eigenspace of X_n too. Moreover, all Jordan blocks of X_1 on $L(i^n)$ are of size n.*

Proof. We prove (i), (ii) being similar. Note $L(i^n) = \bigoplus_{x \in S_n} T_x \otimes L$, since by Theorem 2.2 we know that $\mathcal{H}_n$ is a free right $\mathcal{A}_n$-module on basis $\{T_x \mid x \in S_n\}$.

We first show that the eigenspace of $X_1 + X_1^{-1}$ is a sum of the subpaces of the form $T_y \otimes L$, where $y \in S_{2\dots n} \cong S_{n-1}$ is the subgroup of S_n generated by $s_2, \dots, s_{n-1}$. Well, any T_x can be written as $T_yT_1T_2\dots T_j$ for some $y \in S_{2\dots n}$ and $0 \leq j < n$. Note

$$(X_{j+1} + X_{j+1}^{-1} - q(i))v = 0$$

for any $v \in L$, by definition of L. Now the defining relations of $\mathcal{H}_n$ especially (2.21), (2.22) imply

$$(X_1 + X_1^{-1} - q(i))T_yT_1T_2\dots T_j \otimes v = \\ \xi T_yT_1\dots T_{j-1} \otimes (X_j^{-1}(1 - C_jC_{j+1}) - (1 - C_jC_{j+1})X_{j+1})v + (*),$$

where $(*)$ stands for a sum of terms which belong to subspaces of the form $T_{y'}T_1\dots T_k \otimes L$ for $y' \in S_{2\dots n}$ and $0 \leq k < j - 1$.

Now assume that a linear combination

$$z := \sum_{y \in S_{2\dots n}} \sum_{0 \leq j < n} \sum_{v \in L} c_{y,j,v} T_yT_1T_2\dots T_j \otimes v$$

is an eigenvector for $X_1 + X_1^{-1}$. Then it must be annihilated by $X_1 + X_1^{-1} - q(i)$. Choose the maximal j for which the coefficient $c_{y,j,v}$ is non-zero, and for this j choose the maximal (with respect to the Bruhat order) y such that $c_{y,j,v}$ is non-zero. Then the calculation above and Lemma 4.14 show that $(X_1 + X_1^{-1} - q(i))z \neq 0$ unless $j = 0$. This proves our claim on the eigenspace of $X_1 + X_1^{-1}$.

Now apply the same argument to see that the common eigenspace of $X_1 + X_1^{-1}$ and $X_2 + X_2^{-1}$ is spanned by $T_y \otimes L$ for $y \in S_{3\dots n}$, and so on, yielding the first claim in (i). Finally, define

$$V(m) := \{z \in L(i^n) \mid (X_1 + X_1^{-1} - q(i))^m z = 0\}.$$

It follows by induction from the calculation above and Lemma 4.14 that

$$V(m) = \mathrm{span}\{T_y T_1 T_2 \dots T_j \otimes v \mid y \in S_{2\dots n},\ j < m,\ v \in L\},$$

giving the second claim. □

Now we are ready to prove the main theorem giving the structure of the principal series module $L(i^n)$, compare [Kt].

Theorem 4.16. *Let $i \in I$ and $\mu = (\mu_1, \dots, \mu_u)$ be a composition of n.*

(i) *$L(i^n)$ is irreducible of the same type as $L(i) \circledast \cdots \circledast L(i)$, i.e. type* `M` *if either $i \neq 0$ or $i = 0$ and n is even, type* `Q` *otherwise, and it is the only irreducible module in its block.*

(ii) *All composition factors of $\mathrm{res}^n_\mu L(i^n)$ are isomorphic to $L(i^{\mu_1}) \circledast \cdots \circledast L(i^{\mu_u})$, and $\mathrm{soc}\,\mathrm{res}^n_\mu L(i^n)$ is irreducible.*

(iii) *$\mathrm{soc}\,\mathrm{res}^n_{n-1} L(i^n) \cong \mathrm{res}^{n-1,1}_{n-1} L(i^{n-1}) \circledast L(i)$.*

Proof. Denote $L(i) \circledast \cdots \circledast L(i)$ by L.

(i) Let M be a non-zero $\mathcal{H}_n$-submodule of $L(i^n)$. Then, $\mathrm{res}^n_{1,\dots,1} M$ must contain an $\mathcal{A}_1$-submodule N isomorphic to L. But the commuting operators $X_1 + X_1^{-1}, \dots, X_n + X_n^{-1}$ (or $X_1, \dots, X_n$ if $i = 0$) act on L as scalars, giving that N is contained in their common eigenspace on $L(i^n)$. But by Lemma 4.15, this implies that $N = 1 \otimes L$. This shows that M contains $1 \otimes L$, but this generates the whole of $L(i^n)$ over $\mathcal{H}_n$. So $M = L(i^n)$.

To see that the type of $L(i^n)$ is the same as the type of L, the functor $\mathrm{ind}^n_{1,\dots,1}$ determines a map

$$\mathrm{End}_{\mathcal{A}_n}(L) \to \mathrm{End}_{\mathcal{H}_n}(L(i^n)).$$

We just need to see that this is an isomorphism, which we do by constructing the inverse map. Let $f \in \mathrm{End}_{\mathcal{H}_n}(L(i^n))$. Then f leaves $1 \otimes L$ invariant by Lemma 4.15, so f restricts to an $\mathcal{A}_n$-endomorphism $\bar{f}$ of L.

Finally, to see that $L(i^n)$ is the only irreducible in its block, we have already observed using Lemma 4.10 that $\mathrm{ch}\, L(i^n) = n![L(i) \circledast \cdots \circledast L(i)]$. Hence all composition factors of $\mathrm{res}^n_{1,\dots,1} L(i^n)$ are isomorphic to $L(i) \circledast \cdots \circledast L(i)$. Now apply Frobenius reciprocity and the fact just proved that $L(i^n)$ is irreducible.

(ii) That all composition factors of $\mathrm{res}^n_\mu L(i^n)$ are isomorphic to $L(i^{\mu_1}) \circledast \cdots \circledast L(i^{\mu_u})$ follows from the parabolic analogue of (i). To see that $\mathrm{soc}\,\mathrm{res}^n_\mu L(i^n)$ is simple, note that the submodule $\mathcal{H}_\mu \otimes L$ of $\mathrm{res}_{\mathcal{H}_\mu} L(i^n)$ is isomorphic to $L(i^{\mu_1}) \circledast \cdots \circledast L(i^{\mu_l})$. This module is irreducible, and so it is contained in the socle. Conversely, let M be an irreducible $\mathcal{H}_\mu$-submodule of $L(i^n)$. Then using Lemma 4.15 as in the proof of (i), we see that M must contain $1 \otimes L$, hence $\mathcal{H}_\mu \otimes L$.

(iii) By (ii), $L(i^n)$ has a unique $\mathcal{H}_{n-1,1}$-submodule isomorphic to $L(i^{n-1}) \circledast L(i)$, namely, $\mathcal{H}_{n-1,1} \otimes L$. Since this is completely reducible on restriction to $\mathcal{H}_{n-1}$, it follows that $\mathcal{H}_{n-1,1} \otimes L \subseteq \mathrm{soc}\,\mathrm{res}^n_{n-1} L(i^n)$. Conversely, take any irreducible $\mathcal{H}_{n-1}$-submodule M of $L(i^n)$. The common eigenspace of $X_1 + X_1^{-1}, \dots, X_{n-1} + X_{n-1}^{-1}$ (resp. $X_1, \dots, X_{n-1}$ if $i = 0$) on M must lie in $1 \otimes L$ by Lemma 4.15. Hence, $M \subseteq \mathcal{H}_{n-1,1} \otimes L$ which completes the proof. □

§4-h. **Covering modules.** Fix $i \in I$ and $n \geq 1$ throughout the subsection. We will construct for each $m \geq 1$ an $\mathcal{H}_n$-module $L_m(i^n)$ with irreducible cosocle isomorphic to $L(i^n)$. Let $\mathcal{J}(i^n)$ denote the annihilator in $\mathcal{H}_n$ of $L(i^n)$. Introduce the quotient superalgebra

$$\mathcal{R}_m(i^n) := \mathcal{H}_n/\mathcal{J}(i^n)^m \tag{4.24}$$

for each $m \geq 1$. One checks that $\mathcal{J}(i^n)$ contains $(X_k + X_k^{-1} - q(i))^{n!}$ for each $k = 1, \dots, n$. It follows easily from this that each superalgebra $\mathcal{R}_m(i^n)$ is finite dimensional. Moreover, by Theorem 4.16, $L(i^n)$ is the unique irreducible $\mathcal{R}_m(i^n)$-module up to isomorphism.

Let $L_m(i^n)$ denote a projective cover of $L(i^n)$ in the category $\mathcal{R}_m(i^n)$-mod. For convenience, we also define $L_0(i^n) = \mathcal{R}_0(i^n) = 0$. Note we know the dimension of $L(i^n)$ from Lemma 4.8, and moreover $L(i^n)$ is of type $\mathtt{Q}$ if $i = 0$ and n is odd, type $\mathtt{M}$ otherwise. Using this and the general theory of finite dimensional superalgebras, one shows:

Lemma 4.17. *For each $m \geq 1$,*

$$\mathcal{R}_m(i^n) \simeq \begin{cases} (L_m(i^n) \oplus \Pi L_m(i^n))^{\oplus 2^{n-1}(n!)} & \text{if } i \neq 0, \\ (L_m(i^n) \oplus \Pi L_m(i^n))^{\oplus 2^{(n-2)/2}(n!)} & \text{if } i = 0 \text{ and } n \text{ is even,} \\ L_m(i^n)^{\oplus 2^{(n-1)/2}(n!)} & \text{if } i = 0 \text{ and } n \text{ is odd,} \end{cases}$$

as left $\mathcal{H}_n$-modules. Moreover, $L_m(i^n)$ admits an odd involution if and only if $i = 0$ and n is odd.

There are obvious surjections

$$\mathcal{R}_1(i^n) \twoheadleftarrow \mathcal{R}_2(i^n) \twoheadleftarrow \dots. \tag{4.25}$$

In the case $m = 1$, we certainly have that $L_1(i^n) \cong L(i^n)$; let us assume by the choice of $L_1(i^n)$ that in fact $L_1(i^n) = L(i^n)$. Then we can choose the $L_m(i^n)$ for $m > 1$ so that the the maps (4.25) induce even maps

$$L_1(i^n) \twoheadleftarrow L_2(i^n) \twoheadleftarrow \dots. \tag{4.26}$$

Moreover, in case $i = 0$ and n is odd, we can choose the odd involutions

$$\theta_m : L_m(i^n) \to L_m(i^n) \tag{4.27}$$

given by Lemma 4.17 in such a way that they are compatible with the maps in (4.26).

The significance of the $\mathcal{H}_n$-modules $\mathcal{R}_m(i^n)$ is explained by the following lemma:

Lemma 4.18. *Let M be an $\mathcal{H}_n$-module annihilated by $\mathcal{J}(i^n)^k$ for some k. Then, there is a natural isomorphism*

$$\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{R}_m(i^n), M) \xrightarrow{\sim} M$$

for all $m \geq k$.

Proof. The assumption implies that M is the inflation of an $\mathcal{R}_m(i^n)$-module. So

$$\mathrm{Hom}_{\mathcal{H}_n}(\mathcal{R}_m(i^n), M) \simeq \mathrm{Hom}_{\mathcal{R}_m(i^n)}(\mathcal{R}_m(i^n), M) \simeq M,$$

all isomorphisms being the natural ones. □

Let us finally consider the most important case $n = 1$ in more detail. In this case, one easily checks that the ideal $\mathcal{J}(i)^m$ is generated by $(X_1 + X_1^{-1} - q(i))^m$ if $i \neq 0$ or $(X_1 - 1)^m$ if $i = 0$. It follows that

$$\dim \mathcal{R}_m(i) = \begin{cases} 4m & \text{if } i \neq 0, \\ 2m & \text{if } i = 0. \end{cases} \tag{4.28}$$

Moreover, Lemma 4.17 shows in this case that

$$\mathcal{R}_m(i) \simeq \begin{cases} L_m(i) \oplus \Pi L_m(i) & \text{if } i \neq 0, \\ L_m(i) & \text{if } i = 0. \end{cases} \tag{4.29}$$

Hence, $\dim L_m(i) = 2m$ in either case. Using this, it follows easily that $L_m(i)$ can be described alternatively as the vector superspace on basis $w_1, \dots, w_m$, $w'_1, \dots, w'_m$, where each w_k is even and each w'_k is odd, with $\mathcal{H}_1$-module structure uniquely determined by

$$X_1 w_k = b_+(i) w_k + w_{k+1}, \qquad C_1 w_k = w'_k$$

for each $k = 1, \dots, m$, interpreting w_{m+1} as 0. Using this explicit description, one now checks routinely that $L_m(i)$ is uniserial with m composition factors all $\simeq L(i)$.

We can also describe the map $L_m(i) \twoheadleftarrow L_{m+1}(i)$ from (4.26) explicitly: it is the identity on $w_1, \dots, w_m, w'_1, \dots, w'_m$ but maps w_{m+1} and w'_{m+1} to zero. Also, the map θ_m from (4.27) can be chosen so that

$$w_k \mapsto \sqrt{-1} w'_k, \qquad w'_k \mapsto -\sqrt{-1} w_k$$

for each $k = 1, \dots, m$.

§4-i. **Modifications in the degenerate case.** For $i \in I$, the definition (4.5) of $q(i)$ becomes

$$q(i) = i(i+1) \in F \tag{4.30}$$

for each $i \in I$. For $\lambda \in P_+$, the quotient superalgebra $\mathcal{H}_n^\lambda$, is defined to be the quotient $\mathcal{H}_n / \mathcal{I}_\lambda$ of the affine Sergeev superalgebra $\mathcal{H}_n$ by the two-sided ideal $\mathcal{I}_\lambda$ generated by

$$x_1^{\langle h_0, \lambda \rangle} \prod_{i=1}^{\ell} (x_1^2 - q(i))^{\langle h_i, \lambda \rangle}. \tag{4.31}$$

The basis theorem for $\mathcal{H}_n^\lambda$ gives that $\mathcal{H}_n^\lambda$ has a basis given by the images of the elements

$$\{x^\alpha c^\beta w \mid \alpha \in \mathbb{Z}_{\geq 0}^n \text{ with } 0 \leq \alpha_i < \langle c, \lambda \rangle, \beta \in \mathbb{Z}_2^n, w \in S_n\}. \tag{4.32}$$

The definition of the category $\operatorname{Rep}_I \mathcal{H}_n$ of integral representations is modified in the appropriate way to ensure that the integral representations of $\mathcal{H}_n$ are precisely the inflations of finite dimensional representations of $\mathcal{H}_n^\lambda$ for $\lambda \in P_+$. To be precise, an integral $\mathcal{A}_n$-module now means a finite dimensional $\mathcal{A}_n$-module such that the eigenvalues of all $x_1^2, \dots, x_n^2$ are of the form $q(i)$ for $i \in I$. The appropriate analogue of Lemma 4.4, involving elements x_j^2 now of course, is proved using the elements

$$\Phi_j = s_j(x_j^2 - x_{j+1}^2) + (x_j + x_{j+1}) + c_j c_{j+1}(x_j - x_{j+1})$$

from [N, (3.4)]. Proofs of the basic properties of Φ_j, analogous to those in §4-c, can be found in [N, Prop. 3.2] and at the end of [N, §4].

The module $L(i)$ in §4-e is now defined to be the $\mathcal{A}_1$-module on basis w, w' with action

$$cw = w', \qquad cw' = w, \qquad xw = \sqrt{q(i)} w, \qquad xw' = -\sqrt{q(i)} w'.$$

The remaining definitions go through more or less unchanged: for example for an integral $\mathcal{H}_n$-module M, $M[\underline{i}]$ is the simultaneous generalized eigenspace of the operators $x_1^2, \dots, x_n^2$ corresponding to eigenvalues $q(i_1), \dots, q(i_n)$ respectively.

Kato's theorem (Theorem 4.16) is the same, as is Lemma 4.15 when $X_i + X_i^{-1}$ is replaced by x_i^2 and eigenvalue 1 for X_i is replaced by eigenvalue 0 for x_i in the usual way. Note the main technical fact needed in the proof of Lemma 4.15 is the following: for $i \neq 0$, and every $0 \neq v \in L(i) \boxtimes \cdots \boxtimes L(i)$ (n copies),

$$(x_j(1 - c_jc_{j+1}) + (1 - c_jc_{j+1})x_{j+1})v \neq 0$$

for each $j = 1, \dots, n-1$.

5. Crystal operators

§5-a. **Multiplicity-free socles.** The arguments in this subsection are based on [GV]. Let $M \in \mathrm{Rep}_I\, \mathcal{H}_n$ and $i \in I$. Define $\Delta_i M$ to be the generalized $q(i)$-eigenspace of $X_n + X_n^{-1}$ on M. Alternatively,

$$\Delta_i M = \bigoplus_{\underline{i} \in I^n,\ i_n = i} M[\underline{i}],$$

recalling the decomposition from Lemma 4.9. Note since $X_n + X_n^{-1}$ is central in the parabolic subalgebra $\mathcal{H}_{n-1,1}$ of $\mathcal{H}_n$, $\Delta_i M$ is invariant under this subalgebra. So in fact, Δ_i can be viewed as an exact functor

$$\Delta_i : \mathrm{Rep}_I\, \mathcal{H}_n \to \mathrm{Rep}_I\, \mathcal{H}_{n-1,1}, \tag{5.1}$$

being defined on morphisms simply as restriction. Clearly, there is an isomorphism of functors

$$\mathrm{res}^n_{n-1,1} \simeq \Delta_0 \oplus \Delta_1 \oplus \cdots \oplus \Delta_\ell.$$

Slightly more generally, given $m \geq 0$, define

$$\Delta_{i^m} : \mathrm{Rep}_I\, \mathcal{H}_n \to \mathrm{Rep}_I\, \mathcal{H}_{n-m,m} \tag{5.2}$$

so that $\Delta_{i^m} M$ is the simultaneous generalized $q(i)$-eigenspace of of the commuting operators $X_k + X_k^{-1}$ for $k = n-m+1, \dots, n$. In view of Theorem 4.16(i), $\Delta_{i^m} M$ can also be characterized as the largest submodule of $\mathrm{res}^n_{n-m,m} M$ all of whose composition factors are of the form $N \circledast L(i^m)$ for irreducible $N \in \mathrm{Rep}_I\, \mathcal{H}_{n-m}$.

The definition of Δ_{i^m} implies functorial isomorphisms

$$\mathrm{Hom}_{\mathcal{H}_{n-m,m}}(N \boxtimes L(i^m), \Delta_{i^m} M) \simeq \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}^n_{n-m,m} N \boxtimes L(i^m), M)$$

for $N \in \mathrm{Rep}_I\, \mathcal{H}_{n-m}$, $M \in \mathrm{Rep}_I\, \mathcal{H}_n$. For irreducible N this immediately imples:

$$\mathrm{Hom}_{\mathcal{H}_{n-m,m}}(N \circledast L(i^m), \Delta_{i^m} M) \cong \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}^n_{n-m,m} N \circledast L(i^m), M). \tag{5.3}$$

Also from definitions we get:

Lemma 5.1. *Let $M \in \mathrm{Rep}_I\, \mathcal{H}_n$ with $\mathrm{ch}\, M = \sum_{\underline{i} \in I^n} a_{\underline{i}} [L(i_1) \circledast \cdots \circledast L(i_n)]$. Then we have that $\mathrm{ch}\, \Delta_{i^m} M = \sum_{\underline{j}} a_{\underline{j}} [L(j_1) \circledast \cdots \circledast L(j_n)]$, summing over all $\underline{j} \in I^n$ with $j_{n-m+1} = \cdots = j_n = i$.*

Now for $i \in I$ and $M \in \mathrm{Rep}_I\, \mathcal{H}_n$, define

$$\varepsilon_i(M) = \max\{m \geq 0 \mid \Delta_{i^m} M \neq 0\}. \tag{5.4}$$

Note Lemma 5.1 shows that $\varepsilon_i(M)$ can be worked out just from knowledge of the character $\mathrm{ch}\, M$.

Lemma 5.2. *Let $M \in \mathrm{Rep}_I\, \mathcal{H}_n$ be irreducible, $i \in I$, $\varepsilon = \varepsilon_i(M)$. If $N \circledast L(i^m)$ is an irreducible submodule of $\Delta_{i^m} M$ for some $0 \leq m \leq \varepsilon$, then $\varepsilon_i(N) = \varepsilon - m$.*

Proof. The definitions imply immediately that $\varepsilon_i(N) \leq \varepsilon - m$. For the reverse inequality, (5.3) and the irreducibility of M gives that M is a quotient of $\mathrm{ind}_{n-m,m}^n N \circledast L(i^m)$. So applying the exact functor Δ_{i^ε}, $\Delta_{i^\varepsilon} M \neq 0$ is a quotient of $\Delta_{i^\varepsilon}(\mathrm{ind}_{n-m,m}^n N \circledast L(i^m))$. In particular, $\Delta_{i^\varepsilon}(\mathrm{ind}_{n-m,m}^n N \circledast L(i^m)) \neq 0$. Now one gets that $\varepsilon_i(N) \geq \varepsilon - m$ applying the Shuffle Lemma (Lemma 4.11) and Lemma 5.1. □

Lemma 5.3. *Let $m \geq 0$, $i \in I$ and N be an irreducible module in $\mathrm{Rep}_I \mathcal{H}_n$ with $\varepsilon_i(N) = 0$. Set $M := \mathrm{ind}_{n,m}^{n+m} N \circledast L(i^m)$. Then:*

(i) *$\Delta_{i^m} M \cong N \circledast L(i^m)$;*
(ii) *$\mathrm{cosoc}\, M$ is irreducible with $\varepsilon_i(\mathrm{cosoc}\, M) = m$;*
(iii) *all other composition factors L of M have $\varepsilon_i(L) < m$.*

Proof. (i) Clearly a copy of $N \circledast L(i^m)$ appears in $\Delta_{i^m} M$. But by the Shuffle Lemma and Lemma 5.1, $\dim \Delta_{i^m} M = \dim N \circledast L(i^m)$, hence $\Delta_{i^m} M \cong N \circledast L(i^m)$.

(ii) By (5.3), a copy of $N \circledast L(i^m)$ appears in $\Delta_{i^m} Q$ for any non-zero quotient Q of M, in particular for any constituent of $\mathrm{cosoc}\, M$. But by (i), $N \circledast L(i^m)$ only appears once in $\Delta_{i^m} M$, hence $\mathrm{cosoc}\, M$ must be irreducible.

(iii) We have shown that $\Delta_{i^m} M = \Delta_{i^m}(\mathrm{cosoc}\, M)$. Hence, $\Delta_{i^m} L = 0$ for any other composition factor of M by exactness of Δ_{i^m}. □

Lemma 5.4. *Let $M \in \mathrm{Rep}_I \mathcal{H}_n$ be irreducible, $i \in I$, $\varepsilon = \varepsilon_i(M)$. Then, $\Delta_{i^\varepsilon} M$ is isomorphic to $N \circledast L(i^\varepsilon)$ for some irreducible $\mathcal{H}_{n-\varepsilon}$-module N with $\varepsilon_i(N) = 0$.*

Proof. Pick an irreducible submodule of $\Delta_{i^\varepsilon} M$. In view of Theorem 4.16(i), it must be of the form $N \circledast L(i^\varepsilon)$ for some irreducible $\mathcal{H}_{n-\varepsilon}$-module N. Moreover, $\varepsilon_i(N) = 0$ by Lemma 5.2. By (5.3) and the irreducibility of M, M is a quotient of $\mathrm{ind}_{n-\varepsilon,\varepsilon}^n N \circledast L(i^\varepsilon)$. Hence, $\Delta_{i^\varepsilon} M$ is a quotient of $\Delta_{i^\varepsilon} \mathrm{ind}_{n-\varepsilon,\varepsilon}^n N \circledast L(i^\varepsilon)$. But this is isomorphic to $N \circledast L(i^\varepsilon)$ by Lemma 5.3(i). This shows that $\Delta_{i^\varepsilon} M \cong N \circledast L(i^\varepsilon)$. □

Lemma 5.5. *Let $m \geq 0$, $i \in I$ and N be an irreducible module in $\mathrm{Rep}_I \mathcal{H}_n$. Set $M := \mathrm{ind}_{n,m}^{n+m}(N \circledast L(i^m))$. Then, $\mathrm{cosoc}\, M$ is irreducible with $\varepsilon_i(\mathrm{cosoc}\, M) = \varepsilon_i(N) + m$, and all other composition factors L of M have $\varepsilon_i(L) < \varepsilon_i(N) + m$.*

Proof. Let $\varepsilon = \varepsilon_i(N)$. By Lemma 5.4, we have that $\Delta_{i^\varepsilon} N = K \circledast L(i^\varepsilon)$ for an irreducible $K \in \mathrm{Rep}_I \mathcal{H}_{n-\varepsilon}$ with $\varepsilon_i(K) = 0$. By (5.3) and the irreducibility of N, N is a quotient of $\mathrm{ind}_{n-\varepsilon,\varepsilon}^n K \circledast L(i^\varepsilon)$. So the transitivity of induction implies that $\mathrm{ind}_{n,m}^{n+m} N \circledast L(i^m)$ is a quotient of $\mathrm{ind}_{n-\varepsilon,\varepsilon+m}^{n+m} K \circledast L(i^{\varepsilon+m})$. Now everything follows from Lemma 5.3. □

Theorem 5.6. *Let M be an irreducible module in $\mathrm{Rep}_I \mathcal{H}_n$ and $i \in I$. Then, for any $0 \leq m \leq \varepsilon_i(M)$, $\mathrm{soc}\, \Delta_{i^m} M$ is an irreducible $\mathcal{H}_{n-m,m}$-module of the same type as M, and is isomorphic to $L \circledast L(i^m)$ for some irreducible $\mathcal{H}_{n-m}$-module L with $\varepsilon_i(L) = \varepsilon_i(M) - m$.*

Proof. Let $\varepsilon = \varepsilon_i(M)$. Suppose $K_1 \circledast L(i^m) \oplus K_2 \circledast L(i^m)$ is a submodule of $\Delta_{i^m} M$, for irreducibles K_1, K_2. By Lemma 5.2, we have $\varepsilon_i(K_j) = \varepsilon - m$, hence $\Delta_{i^{m-\varepsilon}} K_j \cong K_j' \circledast L(i^{m-\varepsilon})$ for some irreducible K_j', for each $j = 1, 2$. This shows that $\mathrm{res}_{n-\varepsilon,1,\dots,1}^{n-\varepsilon,\varepsilon} \Delta_{i^\varepsilon} M$ contains $K_1' \circledast L(i)^{\circledast\varepsilon} \oplus K_2' \circledast L(i)^{\circledast\varepsilon}$ as a submodule. But according to Lemma 5.4 and Theorem 4.16, $\mathrm{res}_{n-\varepsilon,1,\dots,1}^{n-\varepsilon,\varepsilon} \Delta_{i^\varepsilon} M$ has irreducible socle, so this is a contradiction. Hence, $\mathrm{soc}\, \Delta_{i^m} M$ is irreducible. Now certainly $\mathrm{soc}\, \Delta_{i^m} M \cong L \circledast L(i^m)$ for some irreducible $\mathcal{H}_{n-m}$-module L, by Theorem 4.16(i). It remains to show that $L \circledast L(i^m)$ has the same type as M. Note by Lemma 5.5, $\mathrm{ind}_{n-m,m}^n L \circledast L(i^m)$ has irreducible cosocle, necessarily isomorphic to

M by Frobenius reciprocity. So applying (5.3) we have that

$$\begin{aligned}\mathrm{End}_{\mathcal{H}_{n-m,m}}(L \circledast L(i^m)) &\simeq \mathrm{Hom}_{\mathcal{H}_{n-m,m}}(L \circledast L(i^m), \Delta_{i^m} M)\\ &\simeq \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}_{n-m,m}^n L \circledast L(i^m), M) \simeq \mathrm{End}_{\mathcal{H}_n}(M),\end{aligned}$$

which implies the statement concerning types. □

The theorem has the following consequence:

Corollary 5.7. *For irreducible $M \in \mathrm{Rep}_I\, \mathcal{H}_n$, the socle of $\mathrm{res}^n_{n-1,1} M$ is multiplicity-free.*

We can also apply the theorem to study $\mathrm{res}^n_{n-1} M$, meaning the restriction of M to the subalgebra $\mathcal{H}_{n-1} \subset \mathcal{H}_n$, see (2.27).

Corollary 5.8. *For an irreducible $M \in \mathrm{Rep}_I\, \mathcal{H}_n$ with $\varepsilon_i(M) > 0$,*

$$\mathrm{soc}\, \mathrm{res}^{n-1,1}_{n-1} \circ \Delta_i(M) \simeq \begin{cases} L \oplus \Pi L & \text{if } M \text{ is of type } \mathtt{Q} \text{ or } i \neq 0,\\ L & \text{if } M \text{ is of type } \mathtt{M} \text{ and } i = 0,\end{cases}$$

for some irreducible $\mathcal{H}_{n-1}$-module L of the same type as M if $i \neq 0$ and of the opposite type to M if $i = 0$.

Proof. Let $\delta := 1$ if M is of type $\mathtt{M}$ and $i = 0$, $\delta := 2$ otherwise. By Theorem 5.6, the socle of $\Delta_i M$ is isomorphic to $L \circledast L(i)$ for some irreducible $\mathcal{H}_{n-1}$-module L, and

$$\mathrm{res}^{n-1,1}_{n-1} L \circledast L(i) \cong L^{\oplus \delta},$$

indeed it is exactly as in the statement of the corollary. Now take any irreducible submodule K of $\mathrm{res}^{n-1,1}_{n-1} \circ \Delta_i(M)$. Consider the $\mathcal{H}_{n-1,1}$-submodule $\mathcal{H}_1' K$, where $\mathcal{H}_1'$ is the subalgebra generated by $C_n, X_n^{\pm 1}$. All composition factors of $\mathcal{H}_1' K$ are isomorphic to $K \circledast L(i)$. In particular, the socle of $\mathcal{H}_1' K$ is isomorphic to $K \circledast L(i)$ which implies $K \cong L$. We have now shown that $\mathrm{soc}\, \mathrm{res}^{n-1,1}_{n-1} \Delta_i(M) \cong L^{\oplus \delta'}$ for some $\delta' \geq \delta$.

By Lemma 5.2, $\varepsilon_i(L) = \varepsilon - 1$ where $\varepsilon = \varepsilon_i(M)$, and $\Delta_{i^{\varepsilon-1}} L$ is irreducible by Lemma 5.4. So at least δ' copies of $\Delta_{i^{\varepsilon-1}} L$ appear in $\mathrm{soc}\, \mathrm{res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-1} \Delta_{i^\varepsilon}(M)$. But $\Delta_{i^\varepsilon} M \cong N \circledast L(i^\varepsilon)$ for some irreducible N, so applying Theorem 4.16(iii) and the facts about type in Theorem 5.6,

$$\begin{aligned}\mathrm{soc}\, \mathrm{res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-1} \Delta_{i^\varepsilon}(M) &\cong \mathrm{soc}\, \mathrm{res}^{n-\varepsilon,\varepsilon}_{n-\varepsilon,\varepsilon-1} N \circledast L(i^\varepsilon)\\ &\cong \mathrm{res}^{n-\varepsilon,\varepsilon-1,1}_{n-\varepsilon,\varepsilon-1} N \circledast L(i^{\varepsilon-1}) \circledast L(i) \cong (N \circledast L(i^{\varepsilon-1}))^{\oplus \delta}.\end{aligned}$$

Hence $\delta' \leq \delta$ too. □

§5-b. **Operators $\tilde{e}_i$ and $\tilde{f}_i$.** Let $\mathrm{Irr}_I \mathcal{H}_n$ denote the set of isomorphism classes of irreducible modules in $\mathrm{Rep}_I\, \mathcal{H}_n$. We define

$$B(\infty) = \bigcup_{n \geq 0} \mathrm{Irr}_I \mathcal{H}_n, \tag{5.5}$$

For each $i \in I$, we define the *affine crystal operators*

$$\tilde{e}_i : B(\infty) \cup \{0\} \to B(\infty) \cup \{0\}, \tag{5.6}$$

$$\tilde{f}_i : B(\infty) \cup \{0\} \to B(\infty) \cup \{0\} \tag{5.7}$$

as follows. First, we set $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$. Now let M be an irreducible module in $\mathrm{Rep}_I\, \mathcal{H}_n$. Then, $\tilde{f}_i M$ is defined by

$$\tilde{f}_i M := \mathrm{cosoc}\, \mathrm{ind}^{n+1}_{n,1} M \circledast L(i), \tag{5.8}$$

which is irreducible by Lemma 5.5. To define $\tilde{e}_i M$, Theorem 5.6 shows that either $\Delta_i M = 0$ or $\operatorname{soc} \Delta_i M \cong N \circledast L(i)$ for an irreducible module $N \in \operatorname{Rep}_I \mathcal{H}_{n-1}$. In the former case, we define $\tilde{e}_i M = 0$; in the latter case, we define $\tilde{e}_i M = N$. Thus:

$$\operatorname{soc} \Delta_i M \cong (\tilde{e}_i M) \circledast L(i). \tag{5.9}$$

Note right away from Lemma 5.2 that

$$\varepsilon_i(M) = \max\{m \geq 0 \mid \tilde{e}_i^m M \neq 0\}, \tag{5.10}$$

while a special case of Lemma 5.5 shows that

$$\varepsilon_i(\tilde{f}_i M) = \varepsilon_i(M) + 1. \tag{5.11}$$

Lemma 5.9. *Let M be an irreducible in $\operatorname{Rep}_I \mathcal{H}_n$, $i \in I$ and $m \geq 0$.*

(i) $\operatorname{soc} \Delta_{i^m} M \cong (\tilde{e}_i^m M) \circledast L(i^m)$.

(ii) $\operatorname{cosoc} \operatorname{ind}_{n,m}^{n+m} M \circledast L(i^m) \cong \tilde{f}_i^m M$.

Proof. (i) If $m > \varepsilon_i(M)$, then both parts in the equality above are zero. Let $m \leq \varepsilon_i(M)$. Clearly, $(\tilde{e}_i^m M)$ is a submodule of $\operatorname{res}_{n-m}^{n-m,m} \Delta_{i^m} M$. Hence, $(\tilde{e}_i^m M) \circledast L(i)^{\circledast m}$ is a submodule of $\operatorname{res}_{n-m,1,\dots,1}^{n-m,m} \Delta_{i^m} M$. So Frobenius reciprocity gives that $(\tilde{e}_i^m M) \circledast L(i^m)$ is a submodule of $\Delta_{i^m} M$. Now we are done by Theorem 5.6.

(ii) By exactness of induction, $\tilde{f}_i^m M$ is a quotient of $\operatorname{ind}_{n,m}^{n+m} M \circledast L(i^m)$. Now the result follows from the simplicity of the cosocle, see Lemma 5.5. □

Lemma 5.10. *Let $M \in \operatorname{Rep}_I \mathcal{H}_n$ and $N \in \operatorname{Rep}_I \mathcal{H}_{n+1}$ be irreducible, and $i \in I$. Then, $\tilde{f}_i M \cong N$ if and only if $\tilde{e}_i N \cong M$.*

Proof. Suppose $\tilde{f}_i M \cong N$. Then by (5.3), $\operatorname{Hom}_{\mathcal{H}_{n,1}}(M \circledast L(i), \Delta_i N) \neq 0$, so $M \circledast L(i)$ appears in the socle of $\Delta_i N$. This means that $M \circledast L(i) \cong (\tilde{e}_i N) \circledast L(i)$, whence $M \cong \tilde{e}_i N$. The converse is similar. □

Corollary 5.11. *Let M, N be irreducible modules in $\operatorname{Rep}_I \mathcal{H}_n$. Then, $\tilde{f}_i M \cong \tilde{f}_i N$ if and only if $M \cong N$. Similarly, providing $\varepsilon_i(M), \varepsilon_i(N) > 0$, $\tilde{e}_i M \cong \tilde{e}_i N$ if and only if $M \cong N$.*

We can also define cyclotomic analogues of the crystal operators. So now suppose that $\lambda \in P_+$, and let $\operatorname{Irr} \mathcal{H}_n^\lambda$ denote the set of isomorphism classes of irreducible $\mathcal{H}_n^\lambda$-modules. Define

$$B(\lambda) = \bigcup_{n \geq 0} \operatorname{Irr} \mathcal{H}_n^\lambda. \tag{5.12}$$

The functors $\operatorname{infl}^\lambda$ and $\operatorname{pr}^\lambda$ induce maps

$$\operatorname{infl}^\lambda : B(\lambda) \cup \{0\} \to B(\infty) \cup \{0\}, \qquad \operatorname{pr}^\lambda : B(\infty) \cup \{0\} \to B(\lambda) \cup \{0\}, \tag{5.13}$$

with $\operatorname{pr}^\lambda \circ \operatorname{infl}^\lambda(L) = L$ for each $L \in B(\lambda)$. In other words, we can view $B(\lambda)$ as a subset of $B(\infty)$ via the embedding $\operatorname{infl}^\lambda$. Now by restricting $\tilde{e}_i$ and $\tilde{f}_i$ to $B(\lambda) \subset B(\infty)$, we obtain the *cyclotomic crystal operators*, namely,

$$\tilde{e}_i^\lambda = \operatorname{pr}^\lambda \circ \tilde{e}_i \circ \operatorname{infl}^\lambda : B(\lambda) \cup \{0\} \to B(\lambda) \cup \{0\}, \tag{5.14}$$

$$\tilde{f}_i^\lambda = \operatorname{pr}^\lambda \circ \tilde{f}_i \circ \operatorname{infl}^\lambda : B(\lambda) \cup \{0\} \to B(\lambda) \cup \{0\} \tag{5.15}$$

for each $i \in I$ and $\lambda \in P_+$. Note that $\tilde{e}_i$ already maps $B(\lambda)$ into $B(\lambda) \cup \{0\}$, so we always have that $\tilde{e}_i M = \tilde{e}_i^\lambda M$ for $M \in B(\lambda)$. This is certainly not the case for $\tilde{f}_i$: for $M \in B(\lambda)$, it will often be the case that $\tilde{f}_i^\lambda(M) = 0$ even though $\tilde{f}_i(M)$ is never zero.

§5-c. **Independence of irreducible characters.** We can now prove an important theorem, compare [V$_1$, §5.5]:

Theorem 5.12. *The map* ch $: K(\mathrm{Rep}_I\,\mathcal{H}_n) \to K(\mathrm{Rep}_I\,\mathcal{A}_n)$ *is injective.*

Proof. We need to show that $\{\mathrm{ch}\,L \mid [L] \in \mathrm{Irr}_I\mathcal{H}_n\}$ is a linearly independent set in $K(\mathrm{Rep}_I\,\mathcal{A}_n)$. Proceed by induction on n, the case $n = 0$ being trivial. Suppose $n > 0$ and

$$\sum_{L \in \mathrm{Irr}_I \mathcal{H}_n} a_L \mathrm{ch}\,L = 0$$

for some $a_L \in \mathbb{Z}$. Choose any $i \in I$. We will show by downward induction on $k = n, \dots, 1$ that $a_L = 0$ for all L with $\varepsilon_i(L) = k$. Since every irreducible L has $\varepsilon_i(L) > 0$ for at least one $i \in I$, this is enough to complete the proof.

Consider first the case that $k = n$. Then, $\Delta_{i^n} L = 0$ except if $L \cong L(i^n)$, by Theorem 4.16(i). Since ch $\Delta_{i^n} L$ can be worked out just from knowledge of ch L using Lemma 5.1, we deduce on applying Δ_{i^n} to the equation that the coefficient of ch $L(i^n)$ is zero. Thus the induction starts. Now suppose $1 \leq k < n$ and that we have shown $a_L = 0$ for all L with $\varepsilon_i(L) > k$. Apply Δ_{i^k} to the equation to deduce that

$$\sum_{L \text{ with } \varepsilon_i(\mathrm{L})=\mathrm{k}} a_L \mathrm{ch}\,\Delta_{i^k} L = 0.$$

Now each such $\Delta_{i^k} L$ is irreducible, hence isomorphic to $(\tilde{e}_i^k L) \circledast L(i^k)$, according to Lemmas 5.4 and 5.9(i). Moreover, for $L \ncong L'$, $\tilde{e}_i^k L \ncong \tilde{e}_i^k L'$ by Corollary 5.11. So now the induction hypothesis on n gives that all such coefficients a_L are zero, as required. □

Corollary 5.13. *If L is an irreducible module in* $\mathrm{Rep}_I\,\mathcal{H}_n$*, then $L \cong L^\tau$.*

Proof. Since $\tau(X_i) = X_i$, τ leaves characters invariant. Hence it leaves irreducibles invariant since they are determined up to isomorphism by their character according to the theorem. □

We can also show at this point that the type of an irreducible module L is determined by the type of its central character:

Lemma 5.14. *Suppose $L \in \mathrm{Rep}_I\,\mathcal{H}_n$ is irreducible with central character χ_γ where $\gamma = \sum_{i \in I} \gamma_i \alpha_i \in \Gamma_n$. Then, L is of type* `Q` *is γ_0 is odd, type* `M` *if γ_0 is even.*

Proof. Proceed by induction on n, the case $n = 0$ being trivial. If $n > 1$, let L be an irreducible $\mathcal{H}_n$-module with central character χ_γ. Choose $i \in I$ so that $\tilde{e}_i L \neq 0$. By definition, $\tilde{e}_i L$ has central character $\gamma - \alpha_i$. So by the induction hypothesis, $\tilde{e}_i L$ is of type `Q` if $\gamma_0 - \delta_{i,0}$ is odd, type `M` otherwise. But by general theory §2-b and Lemma 4.7, $(\tilde{e}_i L) \circledast L(i)$ is of the opposite type to $\tilde{e}_i L$ if $i = 0$, of the same type if $i \neq 0$. Hence, $(\tilde{e}_i L) \circledast L(i)$ is of type `Q` if γ_0 is odd, type `M` otherwise. Finally, the proof is completed by Theorem 5.6, since this shows that L has the same type as soc $\Delta_i L = (\tilde{e}_i L) \circledast L(i)$. □

§5-d. **Crystal graphs.** We can view the datum $(B(\infty), \tilde{e}_i, \tilde{f}_i)$ as a combinatorial structure: the *crystal graph*. This is the directed graph with vertices the set $B(\infty)$ and an edge

$$[M] \xrightarrow{i} [N]$$

whenever $[M], [N] \in B(\infty)$ satisfy $\tilde{f}_i M \cong N$, or equivalently, by Lemma 5.10, $M \cong \tilde{e}_i N$. Similarly, $(B(\lambda), \tilde{e}_i^\lambda, \tilde{f}_i^\lambda)$ can be viewed as a crystal graph.

Motivated by this, we introduce some notation to label the isomorphism classes of irreducible representations, or equivalently the vertices of the crystal graph. Write $\mathbf{1}$ for the (trivial) irreducible module of $\mathcal{H}_0$. If L is an irreducible module in $\mathrm{Rep}_I\,\mathcal{H}_n$, one easily shows using Lemma 5.10 repeatedly that

$$L \cong \tilde{f}_{i_n} \dots \tilde{f}_{i_2}\tilde{f}_{i_1}\mathbf{1}$$

for at least one tuple $\underline{i} = (i_1, i_2, \dots, i_n) \in I^n$. So if we define

$$L(\underline{i}) = L(i_1, \dots, i_n) := \tilde{f}_{i_n} \dots \tilde{f}_{i_2}\tilde{f}_{i_1}\mathbf{1},$$

we obtain a labelling of all irreducibles in $\mathrm{Rep}_I\,\mathcal{H}_n$ by tuples in I^n. For example, $L(i, i, \dots, i)$ (n times) is precisely the principal series module $L(i^n)$ introduced in (4.23). Similarly, any irreducible $\mathcal{H}_n^\lambda$-module can be represented as

$$L^\lambda(\underline{i}) = L^\lambda(i_1, \dots, i_n) := \tilde{f}^\lambda_{i_n}\tilde{f}^\lambda_{i_{n-1}} \dots \tilde{f}^\lambda_{i_1}\mathbf{1}_{\boldsymbol{\lambda}},$$

where $\mathbf{1}_{\boldsymbol{\lambda}}$ is the irreducible $\mathcal{H}_0^\lambda$-module. Of course, $L^\lambda(\underline{i}) \cong \mathrm{pr}^\lambda L(\underline{i})$.

Thus, our labelling of the irreducible modules in $\mathrm{Rep}_I\,\mathcal{H}_n$ is by *paths* in the crystal graph starting from $\mathbf{1}$, and similarly for $\mathrm{Rep}\,\mathcal{H}_n^\lambda$. Of course, the problem with this labelling is that a given irreducible L will in general be parametrized by several *different* tuples $\underline{i} \in I^n$, corresponding to different paths from $\mathbf{1}$ to L. But basic properties of $L(\underline{i})$ are easy to read off from the notation: for instance the central character of $L(\underline{i})$ is $\chi_{\underline{i}}$, so by Lemma 5.14, $L(\underline{i})$ is of type $\mathtt{Q}$ if an odd number of the i_j are zero, type $\mathtt{M}$ otherwise.

§5-e. **Boring central characters.** Given $\underline{i} = (i_1, \dots, i_n)$, let

$$\mathrm{ind}(\underline{i}) = \mathrm{ind}(i_1, \dots, i_n) := \mathrm{ind}^n_{1,\dots,1} L(i_1) \circledast \dots \circledast L(i_n).$$

Note the character of $\mathrm{ind}(\underline{i})$ is $\sum_{w \in S_n} L(i_{w^{-1}1}) \circledast \dots \circledast L(i_{w^{-1}n})$. So every irreducible constituent of $\mathrm{ind}(\underline{i})$ belongs to the block $\mathrm{Rep}_\gamma\,\mathcal{H}_n$, where $\gamma = \mathrm{wt}(\underline{i})$.

Lemma 5.15. *Let $\gamma \in \Gamma_n$ and pick any $\underline{i} \in I^n$ with $\mathrm{wt}(\underline{i}) = \gamma$. Then:*

(i) $\mathrm{res}^n_{1,\dots,1} L(\underline{i})$ *has a submodule isomorphic to* $L(i_1) \circledast \dots \circledast L(i_n)$*;*

(ii) $\mathrm{ind}(\underline{i})$ *contains a copy of* $L(\underline{i})$ *in its cosocle;*

(iii) *every irreducible module in the block* $\mathrm{Rep}_\gamma\,\mathcal{H}_n$ *appears at least once as a constituent of* $\mathrm{ind}(\underline{i})$.

Proof. (i) Proceed by induction on n. For the induction step, let $\underline{j} = (i_1, \dots, i_{n-1})$. By Frobenius reciprocity, there is a non-zero (hence necessarily injective) $\mathcal{H}_{n-1,1}$-module homomorphism from $L(\underline{j}) \circledast L(i_n)$ to $\mathrm{res}^n_{n-1,1} L(\underline{i})$. Hence by induction we get a copy of $L(i_1) \circledast \dots \circledast L(i_{n-1}) \circledast L(i_n)$ in $\mathrm{res}^{\mathcal{H}_n}_{\mathcal{A}_n} L(\underline{i})$.

(ii) Use (i) and Frobenius reciprocity.

(iii) We have just shown that $L(\underline{i})$ appears in $\mathrm{ind}(\underline{i})$. But for any other $\underline{j}$ with the same weight as $\underline{i}$, $\mathrm{ind}(\underline{j})$ has the same character as $\mathrm{ind}(\underline{i})$, hence they have the same set of composition factors thanks to Theorem 5.12. Hence, $L(\underline{i})$ appears in $\mathrm{ind}(\underline{j})$. □

We also need the following criterion for irreducibility, compare [G$_1$, Lemma 5.9]:

Lemma 5.16. *Let M, N be irreducibles in $\mathrm{Rep}_I\,\mathcal{H}_m, \mathrm{Rep}_I\,\mathcal{H}_n$ respectively. Suppose*

(i) $\mathrm{ind}^{m+n}_{m,n} M \circledast N \cong \mathrm{ind}^{n+m}_{n,m} N \circledast M$*;*

(ii) *$M \circledast N$ appears in $\mathrm{res}^{m+n}_{m,n}\mathrm{ind}^{m+n}_{m,n} M \circledast N$ with multiplicity one.*

Then, $\mathrm{ind}^{m+n}_{m,n} M \circledast N$ *is irreducible.*

Proof. Suppose for a contradiction that $K = \mathrm{ind}_{m,n}^{m+n} M \circledast N$ is reducible. Then we can find a proper irreducible submodule S, and set $Q = K/S$. By Frobenius reciprocity, $M \circledast N$ appears in $\mathrm{res}^{m+n}_{m,n} Q$ with non-zero multiplicity. Hence, it cannot appear in $\mathrm{res}^{m+n}_{m,n} S$ by assumption (ii). But assumption (i), Corollary 5.13 and Theorem 2.14 show that $K \cong K^\tau$. Hence, K also has a quotient isomorphic to $S^\tau \cong S$, and the Frobenius reciprocity argument implies that $M \circledast N$ appears in $\mathrm{res}^{m+n}_{m,n} S$. □

Lemma 5.17. *Let $\underline{i} \in I^m, \underline{j} \in I^n$ be tuples such that $|i_a - j_b| > 1$ for all $1 \leq a \leq m, 1 \leq b \leq n$. Then, $\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j}) \cong \mathrm{ind}_{n,m}^{m+n} L(\underline{j}) \circledast L(\underline{i})$ is irreducible.*

Proof. By Lemma 5.16 and the Shuffle Lemma, it suffices to show that

$$\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j}) \cong \mathrm{ind}_{n,m}^{m+n} L(\underline{j}) \circledast L(\underline{i}).$$

By the Mackey Theorem, $\mathrm{res}^{m+n}_{m,n} \mathrm{ind}^{m+n}_{n,m} L(\underline{j}) \circledast L(\underline{i})$ contains $L(\underline{i}) \circledast L(\underline{j})$ as a summand with multiplicity one, all other constituents lying in different blocks. Hence by Frobenius reciprocity, there exists a non-zero homomorphism

$$f : \mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j}) \to \mathrm{ind}_{n,m}^{m+n} L(\underline{j}) \circledast L(\underline{i}).$$

Every homomorphic image of $\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j})$ contains an $\mathcal{H}_{m,n}$-submodule isomorphic to $L(\underline{i}) \circledast L(\underline{j})$. So, by Lemma 5.15(i), we see that the image of f contains an $\mathcal{A}_{m+n}$-submodule V isomorphic to $L(i_1) \circledast \cdots \circledast L(i_m) \circledast L(j_1) \circledast \cdots \circledast L(j_n)$.

Now we claim that the image of f also contains an $\mathcal{A}_{m+n}$-submodule isomorphic to $L(j_1) \circledast \cdots \circledast L(j_n) \circledast L(i_1) \circledast \cdots \circledast L(i_m)$. To see this, consider $\tilde{\Phi}_m V$. Pick a common eigenvector $v \in V$ for the operators $X_m^{\pm 1}$ and $X_{m+1}^{\pm 1}$; then $(X_m + X_m^{-1})v = q(i_m)v$ and $(X_{m+1} + X_{m+1}^{-1})v = q(j_1)v$. So according to (4.13), $\tilde{\Phi}_m^2$ acts on v by a scalar, and the assumption that $|i_m - j_1| > 1$ combined with Lemma 4.3 shows that this scalar is necessarily non-zero. Thus, $\tilde{\Phi}_m V \neq 0$, so by (4.15) it is an irreducible $\mathcal{A}_{m+n}$-module, namely,

$$\tilde{\Phi}_m V \cong L(i_1) \circledast \cdots \circledast L(i_{m-1}) \circledast L(j_1) \circledast L(i_m) \circledast L(j_2) \circledast \cdots \circledast L(j_n).$$

Next apply $\tilde{\Phi}_{m-1}, \dots, \tilde{\Phi}_1$ to move $L(j_1)$ to the first position, and continue in this way to complete the proof of the claim.

We have now shown that the image of f contains $L(j_1) \circledast \cdots \circledast L(j_n) \circledast L(i_1) \circledast \cdots \circledast L(i_m)$. But by the Shuffle Lemma, all such composition factors of $\mathrm{res}^{m+n}_{1,\dots,1} \mathrm{ind}^{m+n}_{n,m} L(\underline{j}) \circledast L(\underline{i})$ necessarily lie in the irreducible $\mathcal{H}_{n,m}$-submodule $1 \otimes L(\underline{j}) \circledast L(\underline{i})$ of the induced module. Since this generates all of $\mathrm{ind}^{m+n}_{n,m} L(\underline{j}) \circledast L(\underline{i})$ as an $\mathcal{H}_{m+n}$-module, this shows that f is surjective. Hence f is an isomorphism by dimension, which completes the proof. □

Theorem 5.18. *Let $\underline{i} \in I^m, \underline{j} \in I^n$ be tuples such that $|i_a - j_b| \neq 1$ for all $1 \leq a \leq m, 1 \leq b \leq n$. Then, $\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j}) \cong \mathrm{ind}_{n,m}^{m+n} L(\underline{j}) \circledast L(\underline{i})$ is irreducible. Moreover, every other irreducible module lying in the same block as $\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j})$ is of the form $\mathrm{ind}_{m,n}^{m+n} L(\underline{i}') \circledast L(\underline{j}')$ for permutations $\underline{i}'$ of $\underline{i}$ and $\underline{j}'$ of $\underline{j}$.*

Proof. The second statement of the theorem is an easy consequence of the first and Lemma 5.15(iii). For the first statement, proceed by induction on $m+n$, the case $m+n = 1$ being trivial. For $m + n > 1$, we may assume by Lemma 5.17 that there exists $k \in I$ that appears in both the tuples $\underline{i}$ and $\underline{j}$. Note then that for every $a = 1, \dots, m$, either $i_a = k$ or $|i_a - k| > 1$, and similarly for every $b = 1, \dots, n$, either $j_b = k$ or $|j_b - k| > 1$. So by the induction hypothesis, we have that

$$L(\underline{i}) \cong \mathrm{ind}^n_{n-r,r} L(\underline{i}') \circledast L(k^r), \qquad L(\underline{j}) \cong \mathrm{ind}^m_{m-s,s} L(\underline{j}') \circledast L(k^s)$$

for some $r, s \geq 1$, where $\underline{i}', \underline{j}'$ are tuples with no entries equal to k. By Theorem 2.14, Corollary 5.13 and Lemma 5.17,

$$\mathrm{ind}_{r,m-s}^{r+m-s} L(k^r) \circledast L(\underline{j}') \cong \mathrm{ind}_{m-s,r}^{m-s+r} L(\underline{j}') \circledast L(k^r).$$

So using Theorem 4.16(i) and transitivity of induction,

$$\begin{aligned}\mathrm{ind}_{m,n}^{m+n} L(\underline{i}) \circledast L(\underline{j}) &\cong \mathrm{ind}_{n-r,r,m-s,s}^{m+n} L(\underline{i}') \circledast L(k^r) \circledast L(\underline{j}') \circledast L(k^s) \\ &\cong \mathrm{ind}_{n-r,m-s,r+s}^{m+n} L(\underline{i}') \circledast L(\underline{j}') \circledast L(k^{r+s}).\end{aligned}$$

Finally this is irreducible by the induction hypothesis and Lemma 5.17. □

§5-f. **Some character calculations.** At this point we need to compute the characters of certain very special $\mathcal{H}_n$-modules explicitly.

Lemma 5.19. *Let $i, j \in I$ with $|i - j| = 1$. Then, for all $a, b \geq 0$ with $a + b < -\langle h_i, \alpha_j \rangle$, there is a non-split short exact sequence*

$$0 \longrightarrow L(i^{a+1} j i^b) \longrightarrow \mathrm{ind}_{a+b+1,1}^{a+b+2} L(i^a j i^b) \circledast L(i) \longrightarrow L(i^a j i^{b+1}) \longrightarrow 0.$$

Moreover, for every $a, b \geq 0$ with $a + b \leq -\langle h_i, \alpha_j \rangle$,

$$\mathrm{ch}\, L(i^a j i^b) = (a!)(b!)[L(i)^{\circledast a} \circledast L(j) \circledast L(i)^{\circledast b}].$$

Proof. We proceed by induction on $n = 0, 1, \dots, -\langle h_i, \alpha_j \rangle$ to show that

$$\mathrm{ch}\, L(i^n j) = n![L(i)^{\circledast n} \circledast L(j)],$$

this being immediate in case $n = 0$. For $n > 0$, let $M := \mathrm{ind}_{n,1}^{n+1} L(i^{n-1} j) \circledast L(i)$. We know by the inductive hypothesis and the Shuffle Lemma that

$$\mathrm{ch}\, M = n![L(i)^{\circledast n} \circledast L(j)] + (n-1)![L(i)^{\circledast (n-1)} \circledast L(j) \circledast L(i)].$$

Now consider the $\mathcal{H}_{n,1}$-submodule

$$N := (X_{n+1} + X_{n+1}^{-1} - q(i))M \cong L(i^n) \circledast L(j)$$

of M. The key point is that N is stable under the action of T_n, hence all of $\mathcal{H}_{n+1}$. Although this is in principle an elementary calculation, it turns out to be extremely lengthy. It was carried out by hand for $n \leq 2$ but for the cases $n = 3, 4$ (when ℓ necessarily equals 1), we had to resort to a computer calculation using the GAP computer algebra package. This proves the existence of an irreducible $\mathcal{H}_{n+1}$-module N with character $n![L(i)^{\circledast n} \circledast L(j)]$. This must be $L(i^n j)$, by Lemma 5.15(i), completing the proof of the induction step.

Now we explain how to deduce the characters of the remaining irreducibles in the block. In the argument just given, the quotient module M/N has character $(n-1)![L(i)^{\circledast(n-1)} \circledast L(j) \circledast L(i)]$, so must be $L(i^{n-1} j i)$. Twisting with the automorphism σ proves that there exist irreducibles with characters $n![L(j) \circledast L(i)^{\circledast n}]$ and $(n-1)![L(i) \circledast L(j) \circledast L(i)^{\circledast (n-1)}]$, which must be $L(j i^n)$ and $L(i j i^{n-1})$ respectively by Lemma 5.15(i) once more. This covers everything unless $n = 4$, when we necessarily have that $i = 0, j = 1$ and $\ell = 1$. In this case, we have shown already that there exist four irreducibles with characters

$$\begin{aligned}&\mathrm{ch}\, L(00001) = 24[L(0)^{\circledast 4} \circledast L(1)], \quad \mathrm{ch}\, L(00010) = 6[L(0)^{\circledast 3} \circledast L(1) \circledast L(0)],\\ &\mathrm{ch}\, L(10000) = 24[L(1) \circledast L(0)^{\circledast 4}], \quad \mathrm{ch}\, L(01000) = 6[L(0) \circledast L(1) \circledast L(0)^{\circledast 3}].\end{aligned}$$

So by Lemma 5.15, there must be exactly one more irreducible module in the block, namely $L(00100)$, since none of the above involve the character $[L(0)^{\circledast 2} \circledast L(1) \circledast L(0)^{\circledast 2}]$. Considering the character of $\mathrm{ind}_{n,1}^{n+1} L(0010) \circledast L(0)$ shows that $\mathrm{ch}\, L(00100)$ is either $4[L(0)^{\circledast 2} \circledast L(1) \circledast$

$L(0)^{\circledast 2}]$ or $4[L(0)^{\circledast 2} \circledast L(1) \circledast L(0)^{\circledast 2}] + 6[L(0)^{\circledast 3} \circledast L(1) \circledast L(0)]$. But the latter is not σ-invariant so cannot occur as there would then be too many irreducibles.

Now that the characters are known, it is finally a routine matter using the Shuffle Lemma and Lemma 5.5 to prove the existence of the required non-split sequence. $\square$

Lemma 5.20. *Let $i, j \in I$ with $|i-j| = 1$ and set $n = 1 - \langle h_i, \alpha_j \rangle$. Then*

$$L(i^n j) \cong L(i^{n-1} j i).$$

Moreover, for every $a, b \geq 0$ with $a + b = -\langle h_i, \alpha_j \rangle$,

$$L(i^a j i^{b+1}) \cong \mathrm{ind}_{n,1}^{n+1} L(i^a j i^b) \circledast L(i) \cong \mathrm{ind}_{1,n}^{n+1} L(i) \circledast L(i^a j i^b)$$

with character $a!(b+1)![L(i)^{\circledast a} \circledast L(j) \circledast L(i)^{\circledast (b+1)}] + (a+1)!b![L(i)^{\circledast (a+1)} \circledast L(j) \circledast L(i)^{\circledast b}]$.

Proof. Let $M = \mathrm{ind}_{n,1}^{n+1} L(i^{n-1} j) \circledast L(i)$. We first claim that M is irreducible. To prove this, arguing in the same way as the proof of Lemma 5.19, it suffices to show that the $\mathcal{H}_{n,1}$-submodule

$$(X_{n+1} + X_{n+1}^{-1} - q(i))M \cong L(i^n) \circledast L(j)$$

of M is *not* invariant under T_n. Again this was checked by an explicit computer calculation. Hence, there is an irreducible $\mathcal{H}_n$-module M with character

$$n![L(i)^{\circledast n} \circledast L(j)] + (n-1)![L(i)^{\circledast (n-1)} \circledast L(j) \circledast L(i)].$$

Hence $\tilde{e}_i M \cong L(i^{n-1} j)$ and $\tilde{e}_j M \cong L(i^n)$ by Theorem 5.6. So we deduce that $M \cong L(i^{n-1} j i) \cong L(i^n j)$ thanks to Lemma 5.10.

Now consider the remaining irreducibles in the block. There are at most $(n-1)$ remaining, namely $L(i^a j i^{b+1})$ for $a \geq 0, b \geq 1$ with $a+b = -\langle h_i, \alpha_j \rangle$. Considering the known characters of $\mathrm{ind}_{n,1}^{n+1} L(i^a j i^b) \circledast L(i)$ and arguing in a similar way to the second paragraph of the proof of the preceeding lemma, the remainder of the lemma follows without further calculation. $\square$

Remark 5.21. It is worth pointing out at this point that by further computer calculations, we have calculated the characters of all irreducibles in $\mathrm{Rep}_I \mathcal{H}_n$ for $n \leq 4$, or $n \leq 6$ in case $\ell = 1$. The results are listed in the appendix. Note we make no use of these calculations other than in the cases treated in Lemmas 5.19 and 5.20 above.

§5-g. **Higher crystal operators.** In this subsection we will introduce certain generalizations of the crystal operators $\tilde{f}_j$, following the ideas of [G$_1$, §10]. The results of this subsection are only needed in §6-e below. To simplify notation, we will write simply ind in place of ind_μ^n throughout the subsection.

Lemma 5.22. *Let $i, j \in I$ with $i \neq j$. For any $a, b \geq 0$ with $a + b = -\langle h_i, \alpha_j \rangle$,*

$$\mathrm{ind}\, L(i^a j i^b) \circledast L(i^m) \cong \mathrm{ind}\, L(i^m) \circledast L(i^a j i^b)$$

is irreducible.

Proof. We first claim that

$$\mathrm{ind}\, L(i^a j i^b) \circledast L(i^m) \cong \mathrm{ind}\, L(i^m) \circledast L(i^a j i^b).$$

This is immediate from Lemma 5.17 in case $|i-j|>1$. If $|i-j|=1$, then transitivity of induction and Lemma 5.20 give that

$$\begin{aligned}\operatorname{ind} L(i^aji^b)\circledast L(i^m) &\cong \operatorname{ind} L(i^aji^b)\circledast L(i)\circledast L(i^{m-1})\\ &\cong \operatorname{ind} L(i)\circledast L(i^aji^b)\circledast L(i^{m-1}),\end{aligned}$$

and now repeating this argument $(m-1)$ more times gives the claim.

Hence, by Corollary 5.13 and Theorem 2.14, $K := \operatorname{ind} L(i^aji^b)\circledast L(i^m)$ is self-dual. Now suppose for a contradiction that K is reducible. Then we can pick a proper irreducible submodule S of K, and set $Q := K/S$. Applying Lemmas 5.19 and 4.11,

$$\operatorname{ch} K = \sum_{k=0}^{m}\binom{m}{k}(a+k)!(b+m-k)![L(i)^{\circledast(a+k)}\circledast L(j)\circledast L(i)^{\circledast(b+m-k)}].$$

By Frobenius reciprocity, Q contains an $\mathcal{H}_{a+b+1,m}$-submodule isomorphic to $L(i^aji^b)\circledast L(i^m)$. So by Lemma 5.15(i), the irreducible $\mathcal{A}_{a+b+m+1}$-module $L(i)^{\circledast a}\circledast L(j)\circledast L(i)^{\circledast(b+m)}$ appears in Q with non-zero multiplicity, hence in fact by Theorem 4.16(i) it must appear with multiplicity $a!(b+m)!$ (viewing Q as a module over $\mathcal{H}_{a,1,b+m}$). It follows that $L(i)^{\circledast a}\circledast L(j)\circledast L(i)^{\circledast(b+m)}$ is not a composition factor of S. But this is a contradiction, since as K is self-dual, $S\cong S^\tau$ is a quotient module of K hence must contain $L(i)^{\circledast a}\circledast L(j)\circledast L(i)^{\circledast(b+m)}$ by the Frobenius reciprocity argument again. □

Lemma 5.23. *Let $i,j\in I$ with $i\neq j$. For any $a\geq 1$ and $b\geq 0$ with $a+b=-\langle h_i,\alpha_j\rangle$, any irreducible module M in $\operatorname{Rep}_I \mathcal{H}_n$ and any $m\geq 0$,*

$$\operatorname{cosoc}\operatorname{ind} M\circledast L(i^m)\circledast L(i^aji^b)$$

is irreducible.

Proof. By the argument in the proof of Lemma 5.5, it suffices to prove this in the special case that $\varepsilon_i(M)=0$. Let $k=m+a+b+1$. Recall from the previous lemma that

$$N := \operatorname{ind} L(i^m)\circledast L(i^aji^b)$$

is an irreducible $\mathcal{H}_k$-module. Moreover by Lemma 5.19, $\operatorname{ch} L(i^aji^b) = (a!)(b!)[L(i)^{\circledast a}\circledast L(j)\circledast L(i)^{\circledast b}]$. So since $\varepsilon_i(M)=0$ and $a>0$, the Mackey Theorem and a block argument shows that

$$\operatorname{res}^{n+k}_{n,k}(\operatorname{ind} M\circledast L(i^m)\circledast L(i^aji^b))\cong (M\circledast N)\oplus U$$

for some $\mathcal{H}_{n,k}$-module U all of whose composition factors lie in different blocks to those of $M\circledast N$. Now let $H := \operatorname{cosoc}\operatorname{ind} M\circledast L(i^m)\circledast L(i^aji^b)$. It follows from above that $\operatorname{res}^{n+k}_{n,k} H\cong (M\circledast N)\oplus\tilde{U}$ where $\tilde{U}$ is some quotient module of U. Then:

$$\begin{aligned}\operatorname{Hom}_{\mathcal{H}_{n+k}}(H,H) &\cong \operatorname{Hom}_{\mathcal{H}_{n+k}}(\operatorname{ind} M\circledast L(i^m)\circledast L(i^aji^b), H)\\ &\cong \operatorname{Hom}_{\mathcal{H}_{n,k}}(M\circledast N, \operatorname{res}^{n+k}_{n,k} H)\\ &\cong \operatorname{Hom}_{\mathcal{H}_{n,k}}(M\circledast N, M\circledast N\oplus\tilde{U})\cong \operatorname{Hom}_{\mathcal{H}_{n,k}}(M\circledast N, M\circledast N).\end{aligned}$$

Since H is completely reducible and $M\circledast N$ is irreducible, this implies that H is irreducible too, as required. □

Now we can define the higher crystal operators. Let $i,j\in I$ with $i\neq j$, and $a\geq 1, b\geq 0$ with $a+b=-\langle h_i,\alpha_j\rangle$. Then the special case $m=0$ of the theorem shows that

$$\tilde{f}_{i^aji^b}M := \operatorname{cosoc}\operatorname{ind} M\circledast L(i^aji^b)$$

is irreducible for every irreducible M in $\mathrm{Rep}_I\,\mathcal{H}_n$. Thus we have defined an operator

$$\tilde{f}_{i^aji^b} : B(\infty) \to B(\infty). \tag{5.16}$$

Lemma 5.24. *Take $i, j \in I$ with $i \neq j$ and set $k = -\langle h_i, \alpha_j\rangle$. Let M be an irreducible module in $\mathrm{Rep}_I\,\mathcal{H}_n$.*

(i) *There exists a unique integer a with $0 \le a \le k$ such that for every $m \ge 0$ we have*

$$\varepsilon_i(\tilde{f}_i^m \tilde{f}_j M) = m + \varepsilon_i(M) - a.$$

(ii) *Assume $m \ge k$. Then a copy of $\tilde{f}_i^m \tilde{f}_j M$ appears in the cosocle of*

$$\mathrm{ind}\, \tilde{f}_i^{m-k} M \circledast L(i^a j i^{k-a}).$$

In particular, if $a \ge 1$, then $\tilde{f}_i^m \tilde{f}_j M \cong \tilde{f}_{i^aji^{k-a}} \tilde{f}_i^{m-k} M$.

Proof. Let $\varepsilon = \varepsilon_i(M)$ and write $M = \tilde{f}_i^\varepsilon N$ for irreducible $N \in \mathrm{Rep}_I\,\mathcal{H}_{n-\varepsilon}$ with $\varepsilon_i(N) = 0$. It suffices to prove (i) for any fixed choice of m, the conclusion for all other $m \ge 0$ then following immediately by (5.11). So take $m \ge k$. Note that $\tilde{f}_i^m \tilde{f}_j M = \tilde{f}_i^m \tilde{f}_j \tilde{f}_i^\varepsilon N$ is a quotient of

$$\mathrm{ind}\, N \circledast L(i^\varepsilon) \circledast L(j) \circledast L(i)^{\circledast k} \circledast L(i^{m-k}),$$

which by Lemma 5.19 has a filtration with factors isomorphic to

$$F_a := \mathrm{ind}\, N \circledast L(i^\varepsilon) \circledast L(i^a j i^{k-a}) \circledast L(i^{m-k}), \quad 0 \le a \le k.$$

So $\tilde{f}_i^m \tilde{f}_j M$ is a quotient of some such factor, and to prove (i) it remains to show that $\varepsilon_i(L) = \varepsilon + m - a$ for any irreducible quotient L of F_a. The inequality $\varepsilon_i(L) \le \varepsilon + m - a$ is clear from the Shuffle Lemma. On the other hand, by transitivity of induction and Lemma 5.22, $F_a \cong \mathrm{ind}\, N \circledast (\mathrm{ind}\, L(i^a j i^{k-a}) \circledast L(i^{\varepsilon+m-k}))$. So by Frobenius Reciprocity, the irreducible module $N \circledast (\mathrm{ind}\, L(i^a j i^{k-a}) \circledast L(i^{\varepsilon+m-k}))$ is contained in $\mathrm{res}_{n-\varepsilon,m+1+\varepsilon} L$. Hence $\varepsilon_i(L) \ge \varepsilon + m - a$.

For (ii), by Lemma 5.22, we also have $F_a \cong \mathrm{ind}\, N \circledast L(i^{m-k+\varepsilon}) \circledast L(i^a j i^{k-a})$, and by the Shuffle Lemma, the only irreducible factors K of F_a with $\varepsilon_i(K) = \varepsilon + m - a$ come from its quotient

$$\mathrm{ind}\, \tilde{f}_i^{m-k+\varepsilon} N \circledast L(i^a j i^{k-a}) \cong \mathrm{ind} \tilde{f}_i^{m-k} M \circledast L(i^a j i^{k-a}).$$

Finally, in case $a \ge 1$, the cosocle of the last module is precisely $\tilde{f}_{i^aji^{k-a}} \tilde{f}_i^{m-k} M$. $\square$

§5-h. **Modifications in the degenerate case.** Other than replacing $X_i + X_i^{-1}$ by x_i^2 everywhere, everything in this section goes through in the degenerate case in exactly the same way. Note the computer calculations in §5-f were checked separately in the degenerate case.

6. Induction and restriction

§6-a. ***i*-induction and *i*-restriction.** Fix $\lambda \in P_+$ throughout the subsection. Recall the definition of the functors Δ_i for $i \in I$, see §5-a. Let us denote the composite functor $\mathrm{res}^{n-1,1}_{n-1} \circ \Delta_i$ instead by

$$\mathrm{res}_i : \mathrm{Rep}_I\,\mathcal{H}_n \to \mathrm{Rep}_I\,\mathcal{H}_{n-1}, \tag{6.1}$$

for any n. Note if M is an $\mathcal{H}_n^\lambda$-module, then $\mathrm{res}_i M$ is automatically an $\mathcal{H}_{n-1}^\lambda$-module. So the restriction of the functor res_i gives a functor which we also denote

$$\mathrm{res}_i : \mathrm{Rep}\,\mathcal{H}_n^\lambda \to \mathrm{Rep}\,\mathcal{H}_{n-1}^\lambda. \tag{6.2}$$

We now focus on this cyclotomic case.

There is an alternative definition of res_i: if M is a module in $\mathrm{Rep}_\gamma\,\mathcal{H}_n^\lambda$ for some fixed $\gamma = \sum_{j\in I}\gamma_j\alpha_j \in \Gamma_n$ then

$$\mathrm{res}_i M = \begin{cases} (\mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M)[\gamma - \alpha_i] & \text{if } \gamma_i > 0,\\ 0 & \text{if } \gamma_i = 0.\end{cases} \tag{6.3}$$

This description makes it clear how to define an analogous (additive) functor

$$\mathrm{ind}_i : \mathrm{Rep}\,\mathcal{H}_n^\lambda \to \mathrm{Rep}\,\mathcal{H}_{n+1}^\lambda. \tag{6.4}$$

Using (4.22) and additivity, it suffices to define this on an object M belonging to $\mathrm{Rep}_\gamma\,\mathcal{H}_n^\lambda$ for fixed $\gamma = \sum_{j\in I}\gamma_j\alpha_j \in \Gamma_n$. Then, we set

$$\mathrm{ind}_i M = (\mathrm{ind}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} M)[\gamma + \alpha_i]. \tag{6.5}$$

By the definitions (6.3) and (6.5) and Lemma 4.9, we have that

$$\mathrm{ind}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} M = \bigoplus_{i\in I}\mathrm{ind}_i M, \qquad \mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M = \bigoplus_{i\in I}\mathrm{res}_i M. \tag{6.6}$$

To complete the definition of the functor ind_i, it is defined on a morphism f simply by restriction of the corresponding morphism $\mathrm{ind}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} f$. We stress that the functor ind_i depends fundamentally on the fixed choice of λ, unlike res_i which is just the restriction of its affine counterpart.

Lemma 6.1. *For $\lambda \in P_+$ and each $i \in I$,*

(i) ind_i *and* res_i *are both left and right adjoint to each other, hence they are exact and send projectives to projectives;*
(ii) ind_i *and* res_i *commute with duality, i.e. there are natural isomorphisms*

$$\mathrm{ind}_i(M^\tau) \simeq (\mathrm{ind}_i M)^\tau, \qquad \mathrm{res}_i(M^\tau) \simeq (\mathrm{res}_i M)^\tau$$

for each finite dimensional $\mathcal{H}_n^\lambda$-module M.

Proof. We know that $\mathrm{ind}_i M$ and $\mathrm{res}_i M$ are summands of $\mathrm{ind}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} M$ and $\mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M$ respectively. Moreover, τ-duality leaves central characters invariant because $\tau(X_j) = X_j$ for each j. Now everything follows easily applying Corollary 3.15. □

In order to refine the definitions of ind_i and res_i in the next subsection, we need to give an alternative definition, due to Grojnowski [G$_1$, §8] in the untwisted case. Recall the definition of the left $\mathcal{H}_1$-modules $\mathcal{R}_m(i)$ for $i \in I, m \geq 0$ from (4.24). The limits in the next lemma are taken with respect to the systems induced by the maps (4.25).

Lemma 6.2. *For every finite dimensional $\mathcal{H}_n^\lambda$-module M and $i \in I$, there are natural isomorphisms*

$$\mathrm{ind}_i M \simeq \varprojlim \mathrm{pr}^\lambda \mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_m(i),$$
$$\mathrm{res}_i M \simeq \varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}_1'}(\mathcal{R}_m(i), M)$$

(in the second case, $\mathcal{H}'_1$ denotes the subalgebra of $\mathcal{H}_{n-1,1}$ generated by $C_n, X_n^{\pm 1}$ and the $\mathcal{H}_{n-1}$-module structure is defined by $(hf)(r) = h(f(r))$ for $f \in \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M)$ and $r \in \mathcal{R}_m(i)$).

Proof. For res_i, it suffices to consider the effect on $M \in \mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$ for $\gamma = \sum_{j\in I} \gamma_j \alpha_j \in \Gamma_n$ with $\gamma_i > 0$, both sides of what we are trying to prove clearly being zero if $\gamma_i = 0$. Then, for all sufficiently large m, Lemma 4.18 (in the special case $n = 1$) implies that

$$\mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M) \simeq (\mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M)[\gamma - \alpha_i].$$

Hence,

$$\varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M) \simeq (\mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M)[\gamma - \alpha_i] = \mathrm{res}_i M.$$

This proves the lemma for res_i.

To deduce the statement about induction, it now suffices by uniqueness of adjoint functors to show that $\varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} ? \boxtimes \mathcal{R}_m(i)$ is left adjoint to $\varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), ?)$. Let $N \in \mathrm{Rep}\,\mathcal{H}_{n-1}^\lambda$ and $M \in \mathrm{Rep}\,\mathcal{H}_n^\lambda$. First observe as explained in the previous paragraph that the direct system

$$\mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_1(i), M) \hookrightarrow \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_2(i), M) \hookrightarrow \dots$$

stabilizes after finitely many terms. We claim that the inverse system

$$\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} N \boxtimes \mathcal{R}_1(i) \twoheadleftarrow \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} N \boxtimes \mathcal{R}_2(i) \twoheadleftarrow \dots$$

also stabilizes after finitely many terms. To see this, it suffices to show that the dimension of $\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} N \boxtimes \mathcal{R}_m(i)$ is bounded above independently of m. Well, each $\mathcal{R}_m(i)$ is generated as an $\mathcal{H}'_1$-module by a subspace W isomorphic (as a vector space) to the cosocle $\mathcal{R}_1(i)$ of $\mathcal{R}_m(i)$. Then $\mathrm{ind}_{n,1}^{n+1} N \boxtimes \mathcal{R}_m(i)$ is generated as an $\mathcal{H}_{n+1}$-module by the subspace $W' = 1 \otimes (N \otimes W)$, also of dimension independent of m. Finally, $\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} N \boxtimes \mathcal{R}_m(i)$ is a quotient of the vector space $\mathcal{H}_{n+1}^\lambda \otimes_F W'$, whose dimension is independent of m.

Now we can complete the proof of adjointness. Using the fact from the previous paragraph that the direct and inverse systems stabilize after finitely many terms, we have natural isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{H}_n^\lambda}(\varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n-1,1}^n N \boxtimes \mathcal{R}_m(i), M) &\simeq \varinjlim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(\mathrm{pr}^\lambda \mathrm{ind}_{n-1,1}^n N \boxtimes \mathcal{R}_m(i), M)\\
&\simeq \varinjlim \mathrm{Hom}_{\mathcal{H}_n}(\mathrm{ind}_{n-1,1}^n N \boxtimes \mathcal{R}_m(i), M)\\
&\simeq \varinjlim \mathrm{Hom}_{\mathcal{H}_{n-1,1}}(N \boxtimes \mathcal{R}_m(i), \mathrm{res}_{n-1,1}^n M)\\
&\simeq \varinjlim \mathrm{Hom}_{\mathcal{H}_{n-1}}(N, \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M))\\
&\simeq \varinjlim \mathrm{Hom}_{\mathcal{H}_{n-1}^\lambda}(N, \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M))\\
&\simeq \mathrm{Hom}_{\mathcal{H}_{n-1}^\lambda}(N, \varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_1}(\mathcal{R}_m(i), M)).
\end{aligned}$$

This completes the argument. □

§6-b. **Operators e_i and f_i.** Continue with $\lambda \in P_+$ being fixed. We wish to refine the definitions of the functors res_i and ind_i to give operators, denoted e_i and f_i respectively, from *irreducible* $\mathcal{H}_n^\lambda$-modules to *isomorphism classes* of $\mathcal{H}_{n-1}^\lambda$- (resp. $\mathcal{H}_{n+1}^\lambda$-) modules.

Actually, e_i is simply the restriction to $\mathrm{Irr}\,\mathcal{H}_n^\lambda$ of an operator also denoted e_i on the irreducible modules in $\mathrm{Rep}_I\,\mathcal{H}_n$. We define this first; recall the definition of the module

$L_m(i)$ from §4-h. Let M be an irreducible in $\mathrm{Rep}_I\,\mathcal{H}_n$. Let $\mathcal{H}'_1$ denote the subalgebra of $\mathcal{H}_n$ generated by $C_n, X_n^{\pm 1}$. For each $m \geq 1$, we define an $\mathcal{H}_{n-1}$-module

$$\overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_m(i), M) \tag{6.7}$$

as follows. If M is of type $\mathtt{M}$ or $i \neq 0$, this is simply the space $\mathrm{Hom}_{\mathcal{H}'_1}(L_m(i), M)$ viewed as an $\mathcal{H}_{n-1}$-module in the same way as in Lemma 6.2. But if M is of type $\mathtt{Q}$ and $i = 0$, we can pick an odd involution $\theta_M : M \to M$ and also have the odd involutions $\theta_m : L_m(i) \to L_m(i)$ from (4.27). Let

$$\theta_M \otimes \theta_m : \mathrm{Hom}_{\mathcal{H}'_1}(L_m(i), M) \to \mathrm{Hom}_{\mathcal{H}'_1}(L_m(i), M)$$

denote the map defined by $((\theta_M \otimes \theta_m)f)(v) = (-1)^{\bar{f}}\theta_M(f(\theta_m v))$. One checks that $(\theta_M \otimes \theta_m)^2 = 1$, hence the ± 1-eigenspaces of $\theta_M \otimes \theta_m$ split $\mathrm{Hom}_{\mathcal{H}'_1}(L_m(i), M)$ into a direct sum of two isomorphic $\mathcal{H}_{n-1}$-modules (because there is an obvious odd automorphism swapping the two eigenspaces). Now in this case, we define $\overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_m(i), M)$ to be the 1-eigenspace (say).

In either case, we have a direct system

$$\overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_1(i), M) \hookrightarrow \overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_2(i), M) \hookrightarrow \dots$$

induced by the inverse system (4.26). Now define

$$e_i M = \varinjlim \overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_m(i), M), \tag{6.8}$$

giving us the affine version of the operator e_i. Note if M is an $\mathcal{H}_n^\lambda$-module then each $\overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_m(i), M)$ is an $\mathcal{H}_{n-1}^\lambda$-module, so

$$e_i M = \varinjlim \mathrm{pr}^\lambda \overline{\mathrm{Hom}}_{\mathcal{H}'_1}(L_m(i), M). \tag{6.9}$$

We take (6.9) as our definition of the operator e_i in the cyclotomic case. Comparing (6.9) with Lemma 6.2 and using (4.29), one sees at once that:

Lemma 6.3. *Let $i \in I$ and M be an irreducible module in $\mathrm{Rep}_I\,\mathcal{H}_n$, or an irreducible $\mathcal{H}_n^\lambda$-module. Then,*

$$\mathrm{res}_i M \simeq \begin{cases} e_i M & \text{if } i = 0 \text{ and } M \text{ is of type } \mathtt{M}, \\ e_i M \oplus \Pi e_i M & \text{otherwise.} \end{cases}$$

Now we turn to the definition of $f_i M$ which, just like $\mathrm{ind}_i M$, only makes sense in the cyclotomic case. So, let M an irreducible $\mathcal{H}_n^\lambda$-module. We need to extend the definition of the operation $\circledast$ to give meaning to the notation $M \circledast L_m(i)$, for each $m \geq 1$. If either M is of type $\mathtt{M}$ or $i \neq 0$, then $M \circledast L_m(i) := M \boxtimes L_m(i)$. But if M is of type $\mathtt{Q}$ and $i = 0$, pick an odd involution $\theta_M : M \to M$. Then, the $\pm\sqrt{-1}$-eigenspaces of $\theta_M \otimes \theta_m$ acting on the left on $M \boxtimes L_m(i)$ split it into a direct sum of two isomorphic $\mathcal{H}_{n,1}$-modules. Let $M \circledast L_m(i)$ denote the $\sqrt{-1}$-eigenspace (say) for each m.

We then have an inverse system $M \circledast L_1(i) \twoheadleftarrow M \circledast L_2(i) \twoheadleftarrow \dots$ of $\mathcal{H}_{n,1}$-modules induced by the maps from (4.26). Now we can define

$$f_i M = \varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i). \tag{6.10}$$

Comparing the definition with the proof of Lemma 6.2, one sees that the inverse limit stabilizes after finitely many terms, hence that $f_i M$ really is a well-defined finite dimensional $\mathcal{H}_{n+1}^\lambda$-module. Indeed:

Lemma 6.4. *Let $i \in I$ and M be an irreducible $\mathcal{H}_n^\lambda$-module. Then,*

$$\mathrm{ind}_i M \simeq \begin{cases} f_i M & \text{if } i = 0 \text{ and } M \text{ is of type } \mathtt{M}, \\ f_i M \oplus \Pi f_i M & \text{otherwise.} \end{cases}$$

Lemma 6.5. *Let $i \in I$ and M be an irreducible module in $\mathrm{Rep}_I\, \mathcal{H}_n$. Then, $e_i M$ is non-zero if and only if $\tilde{e}_i M \neq 0$, in which case it is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\tilde{e}_i M$.*

Proof. To see that $e_i M$ has irreducible socle $\tilde{e}_i M$ whenever it is non-zero, combine Lemma 6.3 with Corollary 5.8. The remaining facts follow since M is self-dual by Lemma 5.13, and res_i commutes with duality by Lemma 6.1(ii). □

Before the next theorem, we recall again that the operator f_i depends critically on the fixed choice of λ.

Theorem 6.6. *Let $\lambda \in P_+$ and $i \in I$. Then, for any irreducible $\mathcal{H}_n^\lambda$-module M,*

(i) *$e_i M$ is non-zero if and only if $\tilde{e}_i^\lambda M \neq 0$, in which case it is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\tilde{e}_i^\lambda M$;*

(ii) *$f_i M$ is non-zero if and only if $\tilde{f}_i^\lambda M \neq 0$, in which case it is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $\tilde{f}_i^\lambda M$.*

Proof. (i) This is immediate from Lemma 6.5.

(ii) We deduce this from (i) by an adjointness argument. Let M be an irreducible $\mathcal{H}_n^\lambda$-module, and N be an irreducible $\mathcal{H}_{n+1}^\lambda$-module. Let δ_M equal 1 if $i = 0$ and M is of type $\mathtt{M}$, 2 otherwise, and define δ_N similarly. Then, by Lemmas 6.1(i), 6.3 and 6.4,

$$\begin{aligned} \dim \mathrm{Hom}_{\mathcal{H}_{n+1}^\lambda}(f_i M, N) &= \frac{1}{\delta_M} \dim \mathrm{Hom}_{\mathcal{H}_{n+1}^\lambda}(\mathrm{ind}_i M, N) \\ &= \frac{1}{\delta_M} \dim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(M, \mathrm{res}_i N) = \frac{\delta_N}{\delta_M} \dim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(M, e_i N). \end{aligned}$$

By (i), the latter is zero unless $M = \tilde{e}_i N$, or equivalently $N = \tilde{f}_i M$ by Lemma 5.10. Taking into account the superalgebra analogue of Schur's lemma using Lemma 5.14, one deduces that $\mathrm{cosoc}\, f_i M \cong \tilde{f}_i M$. Finally, note $f_i M$ is self-dual by Lemma 6.1(ii) so everything else follows. □

Remark 6.7. Let us also point out, as follows easily from the definitions, that $e_i M$ and $f_i M$ admit odd involutions if either $i \neq 0$ and M is of type $\mathtt{Q}$, or $i = 0$ and M is of type $\mathtt{M}$.

§6-c. **Divided powers.** Continue with $\lambda \in P_+$ and fix $i \in I$. We can generalize the definitions of e_i, f_i to define operators denoted $e_i^{(r)}, f_i^{(r)}$ on irreducible $\mathcal{H}_n^\lambda$-modules, for each $r \geq 1$. It will be the case that $e_i^{(1)} = e_i, f_i^{(1)} = f_i$. For the definitions, we make use of the covering modules $L_m(i^r)$ from §4-h.

Let M be an irreducible module in $\mathrm{Rep}_I\, \mathcal{H}_n$. If $r > n$, we set $e_i^{(r)} M = 0$. Otherwise, let $\mathcal{H}'_r$ denote the subalgeba of $\mathcal{H}_n$ generated by $X_{n-r+1}^{\pm 1}, \dots, X_n^{\pm 1}$, $C_{n-r+1}, \dots, C_n$, $T_{n-r+1}, \dots, T_{n-1}$. We have a direct system

$$\overline{\mathrm{Hom}}_{\mathcal{H}'_r}(L_1(i^r), M) \hookrightarrow \overline{\mathrm{Hom}}_{\mathcal{H}'_r}(L_2(i^r), M) \hookrightarrow \dots$$

induced by the inverse system (4.26), where $\overline{\mathrm{Hom}}$ is interpreted in exactly the same way as in §6-b using the generalized maps θ_m from (4.27) in case $i = 0$. Now define

$$e_i^{(r)} M = \varinjlim \overline{\mathrm{Hom}}_{\mathcal{H}'_r}(L_m(i^r), M). \tag{6.11}$$

As in §6-b, if M is an $\mathcal{H}_n^\lambda$-module then $e_i^{(r)} M$ is too, so that

$$e_i^{(r)} M = \varinjlim \mathrm{pr}^\lambda \overline{\mathrm{Hom}}_{\mathcal{H}'_r}(L_m(i^r), M) \tag{6.12}$$

in the cyclotomic case.

To define $f_i^{(r)}$, which as usual only makes sense in the cyclotomic case, let M be an irreducible $\mathcal{H}_n^\lambda$-module. We have an inverse system $M \circledast L_1(i^r) \twoheadleftarrow M \circledast L_2(i^r) \twoheadleftarrow \dots$ of $\mathcal{H}_{n,r}$-modules induced by the maps from (4.26), again interpreting $\circledast$ as in §6-b. Now we can define

$$f_i^{(r)} M = \varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,r}^{n+r} M \circledast L_m(i^r). \tag{6.13}$$

Lemma 6.8. *Let $i \in I, r \geq 1$ and M be an irreducible $\mathcal{H}_n^\lambda$-module. Then:*

$$(\mathrm{res}_i)^r M \simeq \begin{cases} (e_i^{(r)} M \oplus \Pi e_i^{(r)} M)^{\oplus 2^{r-1}(r!)} & \text{if } i \neq 0, \\ e_i^{(r)} M^{\oplus 2^{(r-1)/2}(r!)} & \text{if } i = 0,\ r \text{ is odd, } M \text{ is of type } \mathtt{M}, \\ (e_i^{(r)} M \oplus \Pi e_i^{(r)} M)^{\oplus 2^{\lfloor (r-1)/2 \rfloor}(r!)} & \text{otherwise;} \end{cases}$$

$$(\mathrm{ind}_i)^r M \simeq \begin{cases} (f_i^{(r)} M \oplus \Pi f_i^{(r)} M)^{\oplus 2^{r-1}(r!)} & \text{if } i \neq 0, \\ f_i^{(r)} M^{\oplus 2^{(r-1)/2}(r!)} & \text{if } i = 0,\ r \text{ is odd, } M \text{ is of type } \mathtt{M}, \\ (f_i^{(r)} M \oplus \Pi f_i^{(r)} M)^{\oplus 2^{\lfloor (r-1)/2 \rfloor}(r!)} & \text{otherwise.} \end{cases}$$

Proof. Using Lemma 4.17 and the definitions, it suffices to show that

$$(\mathrm{res}_i)^r M \simeq \varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_r}(\mathcal{R}_m(i^r), M),$$
$$(\mathrm{ind}_i)^r M \simeq \varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,r}^{n+r} M \boxtimes \mathcal{R}_m(i^r).$$

For $(\mathrm{res}_i)^r$, this follows from Lemma 4.18 in exactly the same way as in the proof of Lemma 6.2. Now $(\mathrm{ind}_i)^r$ is left adjoint to $(\mathrm{res}_i)^r$, so the statement for induction follows from uniqueness of adjoint functors on checking that the functor $\varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,r}^{n+r} ? \boxtimes \mathcal{R}_m(i^r)$ is left adjoint to $\varinjlim \mathrm{pr}^\lambda \mathrm{Hom}_{\mathcal{H}'_r}(\mathcal{R}_m(i^r), ?)$. The latter follows as in the proof of Lemma 6.2. □

Since we have defined the operator $e_i^{(r)}$ on irreducible modules we get induced operators also denoted $e_i^{(r)}$ at the level of Grothendieck groups, namely,

$$e_i^{(r)} : K(\mathrm{Rep}_I \mathcal{H}_n) \to K(\mathrm{Rep}_I \mathcal{H}_{n-r}), \qquad e_i^{(r)} : K(\mathrm{Rep}\, \mathcal{H}_n^\lambda) \to K(\mathrm{Rep}\, \mathcal{H}_{n-r}^\lambda),$$

in the affine and cyclotomic cases respectively. Similarly $f_i^{(r)}$ induces an operator

$$f_i^{(r)} : K(\mathrm{Rep}\, \mathcal{H}_n^\lambda) \to K(\mathrm{Rep}\, \mathcal{H}_{n+r}^\lambda)$$

on Grothendieck groups. We record:

Lemma 6.9. *As operators on the Grothendieck group $K(\mathrm{Rep}\, \mathcal{H}_n^\lambda)$ (or on $K(\mathrm{Rep}_I \mathcal{H}_n)$ in the case of e_i), we have that $e_i^r = (r!) e_i^{(r)}$ and $f_i^r = (r!) f_i^{(r)}$.*

Proof. Let us prove in the case $i \neq 0$. The proof for $i = 0$ is the same idea, though the details are more delicate (one needs to use Lemma 5.14 too). By Lemmas 6.3 and 6.4, we have that

$$(\mathrm{res}_i)^r = 2^r e_i^r, \qquad (\mathrm{ind}_i)^r = 2^r f_i^r$$

as operators on the Grothendieck group. By Lemma 6.8, we have that

$$(\mathrm{res}_i)^r = 2^r (r!) e_i^{(r)}, \qquad (\mathrm{ind}_i)^r = 2^r (r!) f_i^{(r)}.$$

This is enough. □

Let us finally note that we have only defined the operators $e_i^{(r)}$ and $f_i^{(r)}$ on irreducible modules. However, the definitions could be made more generally on pairs (M, θ_M), where M is an $\mathcal{H}_n^\lambda$-module (or an integral $\mathcal{H}_n$-module in the case of e_i) and $\theta_M : M \to M$ is either the identity map or else an odd involution of M. In case $\theta_M = \mathrm{id}_M$, the definitions of $e_i^{(r)} M$ and $f_i^{(r)} M$ are exactly the same as the case M is irreducible of type $\mathtt{M}$ above. In case θ_M is an odd involution, the definitions of $e_i^{(r)} M$ and $f_i^{(r)} M$ are exactly the same as the case M is irreducible of type $\mathtt{Q}$ above, substiting the given map θ_M for the canonical odd involution of M in the situation above.

This remark applies especially to give us modules $e_i^{(r)} P_M, f_i^{(r)} P_M$, where P_M is the projective cover of an irreducible $\mathcal{H}_n^\lambda$-module: in this case, if M is of type $\mathtt{Q}$, the odd involution θ_M of M lifts to a unique odd involution also denoted θ_M of the projective cover. On doing this, we have that

$$[e_i^{(r)} P_M] = e_i^{(r)} [P_M], \qquad [f_i^{(r)} P_M] = f_i^{(r)} [P_M] \tag{6.14}$$

where the equalities are written in $K(\mathrm{Rep}\, \mathcal{H}_{n-r}^\lambda)$ and $K(\mathrm{Rep}\, \mathcal{H}_{n+r}^\lambda)$ respectively. To prove this, one needs to observe that all composition factors of P_M are of the same type as M by Lemma 5.14.

Note Lemma 6.8 is also true if M is replaced by its projective cover P_M, the proof being the same as above. In particular, this shows that $e_i^{(r)} P_M$ is a summand of $(\mathrm{res}_i)^r P_M$, and similarly for f_i. So Lemma 6.1(i) gives that $e_i^{(r)} P_M$ and $f_i^{(r)} P_M$ are also projective modules. Hence $e_i^{(r)}$ and $f_i^{(r)}$ induce operators with the same names on the Grothendieck groups of *projective* modules too:

$$e_i^{(r)} : K(\mathrm{Proj}\, \mathcal{H}_n^\lambda) \to K(\mathrm{Proj}\, \mathcal{H}_{n-r}^\lambda), \qquad f_i^{(r)} : K(\mathrm{Proj}\, \mathcal{H}_n^\lambda) \to K(\mathrm{Proj}\, \mathcal{H}_{n+r}^\lambda).$$

Moreover, by the same argument as in the proof of Lemma 6.9, we have:

Lemma 6.10. *As operators on the Grothendieck group $K(\mathrm{Proj}\, \mathcal{H}_n^\lambda)$, we have that*

$$e_i^r [P_M] = (r!) [e_i^{(r)} P_M], \qquad f_i^r [P_M] = (r!) [f_i^{(r)} P_M],$$

for all irreducible $\mathcal{H}_n^\lambda$-modules M.

§6-d. **Alternative descriptions of ε_i.** In this subsection we give three new interpretations of the functions ε_i, precisely as in [G_1, Theorem 9.13].

Theorem 6.11. *Let $i \in I$ and M be an irreducible module in $\mathrm{Rep}_I\, \mathcal{H}_n$. Then*

(i) *$[e_i M] = \varepsilon_i(M)[\tilde{e}_i M] + \sum c_a [N_a]$ where the N_a are irreducibles with $\varepsilon_i(N_a) < \varepsilon_i(\tilde{e}_i M)$;*
(ii) *$\varepsilon_i(M)$ is the maximal size of a Jordan block of $X_n + X_n^{-1}$ (resp. X_n if $i = 0$) on M with eigenvalue $q(i)$ (resp. eigenvalue 1 if $i = 0$);*
(iii) *$\mathrm{End}_{\mathcal{H}_{n-1}}(e_i M) \simeq \mathrm{End}_{\mathcal{H}_{n-1}}(\tilde{e}_i M)^{\oplus \varepsilon_i(M)}$ as vector superspaces.*

Proof. Let $\varepsilon = \varepsilon_i(M)$ and $N = \tilde{e}_i^{\varepsilon} M$.

(i) By Lemma 5.4 and Frobenius reciprocity, there is a short exact sequence

$$0 \longrightarrow R \longrightarrow \mathrm{ind}_{n-\varepsilon,\varepsilon}^{n} N \circledast L(i^{\varepsilon}) \longrightarrow M \longrightarrow 0.$$

Moreover, all composition factors L of R have $\varepsilon_i(L) < \varepsilon$ by Lemma 5.3(iii). Applying the exact functor Δ_i, we obtain an exact sequence

$$0 \longrightarrow \Delta_i R \longrightarrow \Delta_i \mathrm{ind}_{n-\varepsilon,\varepsilon} N \circledast L(i^{\varepsilon}) \longrightarrow \Delta_i M \longrightarrow 0.$$

By the Mackey Theorem, $\Delta_i \mathrm{ind}_{n-\varepsilon,\varepsilon}^{n} N \circledast L(i^{\varepsilon}) \cong \mathrm{ind}_{n-\varepsilon,\varepsilon-1,1}^{n-1,1} N \circledast \Delta_i L(i^{\varepsilon})$. By considering characters $[\Delta_i L(i^{\varepsilon})] = \varepsilon [L(i^{\varepsilon-1}) \circledast L(i)]$. Hence,

$$[\Delta_i \mathrm{ind}_{n-\varepsilon,\varepsilon}^{n} N \circledast L(i^{\varepsilon})] = \varepsilon [\mathrm{ind}_{n-\varepsilon,\varepsilon-1,1}^{n-1,1} N \circledast L(i^{\varepsilon-1}) \circledast L(i)]. \tag{6.15}$$

Using Lemma 5.3 again, the cosocle of $\mathrm{ind}_{n-\varepsilon,\varepsilon-1,1}^{n-1,1} N \circledast L(i^{\varepsilon-1}) \circledast L(i)$ is $(\tilde{e}_i M) \circledast L(i)$, and all other composition factors of this module are of the form $L \circledast L(i)$ with $\varepsilon_i(L) < \varepsilon - 1$. Moreover, all composition factors of $\Delta_i R$ are of the form $L \circledast L(i)$ with $\varepsilon_i(L) < \varepsilon - 1$. So we have now shown that

$$[\Delta_i M] = \varepsilon [\tilde{e}_i M \circledast L(i)] + \sum c_a [N_a \circledast L(i)]$$

for irreducibles N_a with $\varepsilon_i(N_a) < \varepsilon_i(\tilde{e}_i M)$. The conclusion follows on applying Lemma 6.3.

(ii) We give the argument for the case $i \neq 0$, the case $i = 0$ being similar but using Lemma 4.15(ii) instead of Lemma 4.15(i). We know that $\Delta_{i^{\varepsilon}} M \cong N \circledast L(i^{\varepsilon})$. So, applying the automorphism σ to Lemma 4.15(i), we deduce that the maximal size of a Jordan block of $X_n + X_n^{-1}$ on $\Delta_{i^{\varepsilon}} M$ is ε. Hence the maximal size of a Jordan block of $X_n + X_n^{-1}$ on $\Delta_i M$ is at least ε.

On the other hand, the argument given above in deriving (6.15) shows that the module $\Delta_i \mathrm{ind}_{n-\varepsilon,\varepsilon}^{n} N \circledast L(i^{\varepsilon})$ has a filtration with ε factors, each of which is isomorphic to $\mathrm{ind}_{n-\varepsilon,\varepsilon-1,1}^{n-1,1} N \circledast L(i^{\varepsilon-1}) \circledast L(i)$. Since $(X_n + X_n^{-1} - q(i))$ annihilates $\mathrm{ind}_{n-\varepsilon,\varepsilon-1,1}^{n-1,1} N \circledast L(i^{\varepsilon-1}) \circledast L(i)$, it follows that $(X_n + X_n^{-1} - q(i))^{\varepsilon}$ annihilates $\Delta_i \mathrm{ind}_{n-\varepsilon,\varepsilon}^{n} N \circledast L(i^{\varepsilon})$. So certainly $(X_n + X_n^{-1} - q(i))^{\varepsilon}$ annihilates its quotient $\Delta_i M$. So the maximal size of a Jordan block of $X_n + X_n^{-1}$ on $\Delta_i M$ is at most ε.

(iii) Let $z = (X_n + X_n^{-1} - q(i))$ if $i \neq 0$ and $(X_n - 1)$ if $i = 0$. Consider the effect of left multiplication by z on the $\mathcal{H}_{n-1}$-module $R = \mathrm{res}_{n-1}^{n-1,1} \circ \Delta_i(M)$. Note R is equal to either $e_i M$ or $e_i M \oplus \Pi e_i M$, by Lemma 6.3. In the latter case, $(X_n + X_n^{-1})$ (resp. X_n) acts as a scalar on $\mathrm{soc}\, R \simeq \tilde{e}_i M \oplus \Pi \tilde{e}_i M$, hence it leaves the two indecomposable summands invariant. This shows that in any case left multiplication by z (which centralizes the subalgebra $\mathcal{H}_{n-1}$ of $\mathcal{H}_n$) induces an $\mathcal{H}_{n-1}$-endomorphism $\theta : e_i M \to e_i M$. But by (ii), $\theta^{\varepsilon-1} \neq 0$ and $\theta^{\varepsilon} = 0$. Hence, $1, \theta, \dots, \theta^{\varepsilon-1}$ give ε linearly independent even $\mathcal{H}_{n-1}$-endomorphisms of $e_i M$. In view of Remark 6.7, we automatically get from these ε linearly independent odd endomorphisms in case $\tilde{e}_i M$ is of type $\mathtt{Q}$, so we have now shown that

$$\dim \mathrm{End}_{\mathcal{H}_{n-1}}(e_i M) \geq \varepsilon \dim \mathrm{End}_{\mathcal{H}_{n-1}}(\tilde{e}_i M).$$

On the other hand, $e_i M$ has irreducible cosocle $\tilde{e}_i M$, and this appears in $e_i M$ with multiplicity ε by (i), so the reverse inequality also holds. $\square$

Corollary 6.12. *Let M, N be irreducible $\mathcal{H}_n$-modules with $M \not\cong N$. Then, for every $i \in I$, $\mathrm{Hom}_{\mathcal{H}_{n-1}}(e_i M, e_i N) = 0$.*

Proof. Suppose there is a non-zero homomorphism $\theta : e_i M \to e_i N$. Then, since $e_i M$ has simple head $\tilde{e}_i M$, we see that $e_i N$ has $\tilde{e}_i M$ as a composition factor. Hence, by Theorem 6.11(i), $\varepsilon_i(\tilde{e}_i N) \geq \varepsilon_i(\tilde{e}_i M)$. Dualizing and applying the same argument gives the inequality the other way round, hence $\varepsilon_i(\tilde{e}_i N) = \varepsilon_i(\tilde{e}_i M)$. But then, $\tilde{e}_i M$ is a composition factor of $e_i N$ with $\varepsilon_i(\tilde{e}_i M) = \varepsilon_i(\tilde{e}_i N)$, hence by Theorem 6.11(i) again, $\tilde{e}_i M \cong \tilde{e}_i N$. But this contradicts Corollary 5.11. □

To state the next corollary, we first need to introduce the $*$-operation on the crystal graph. This will play a fundamental role later on. Suppose M is an irreducible module in $\operatorname{Rep}_I \mathcal{H}_n$ and $0 \leq m \leq n$. Using Lemma 2.9 for the second equality in (6.17), define

$$\tilde{e}_i^* M = (\tilde{e}_i(M^\sigma))^\sigma, \tag{6.16}$$

$$\tilde{f}_i^* M = (\tilde{f}_i(M^\sigma))^\sigma = \operatorname{cosoc} \operatorname{ind}_{1,n}^{n+1} L(i) \circledast M, \tag{6.17}$$

$$\varepsilon_i^*(M) = \varepsilon_i(M^\sigma) = \max\{m \geq 0 \mid (\tilde{e}_i^*)^m M \neq 0\}. \tag{6.18}$$

Note $\varepsilon_i^*(M)$ can be worked out just from knowledge of the character M: $\varepsilon_i^*(M)$ is the maximum k such that $[L(i)^{\circledast k} \circledast \dots]$ appears in $\operatorname{ch} M$.

Recalling the definition of the ideal $\mathcal{I}_\lambda$ generated by the element (4.6), Theorem 6.11(ii) has the following important corollary:

Corollary 6.13. *Let $\lambda \in P_+$ and M be an irreducible in $\operatorname{Rep}_I \mathcal{H}_n$. Then $\mathcal{I}_\lambda M = 0$ if and only if $\varepsilon_i^*(M) \leq \langle h_i, \lambda \rangle$ for all $i \in I$.*

§6-e. **Functions $\boldsymbol{\varphi_i}$.** Fix $\lambda \in P_+$ throughout this subsection. Let M be an irreducible $\mathcal{H}_n^\lambda$-module. Recall from (5.10) that for $i \in I$,

$$\varepsilon_i(M) = \max\{m \geq 0 \mid (\tilde{e}_i^\lambda)^m M \neq 0\}, \tag{6.19}$$

since $\tilde{e}_i^\lambda$ is simply the restriction of $\tilde{e}_i$. Analogously, we define

$$\varphi_i(M) = \max\{m \geq 0 \mid (\tilde{f}_i^\lambda)^m M \neq 0\}. \tag{6.20}$$

We will see shortly (Corollary 6.17) that $\varphi_i(M) < \infty$ always so that the definition makes sense. Note unlike $\varepsilon_i(M)$, the integer $\varphi_i(M)$ depends on the fixed choice of λ.

As in §5-d, $\mathbf{1}_{\boldsymbol{\lambda}}$ denotes the irreducible $\mathcal{H}_0^\lambda$-module.

Lemma 6.14. $\varepsilon_i(\mathbf{1}_{\boldsymbol{\lambda}}) = 0$ *and* $\varphi_i(\mathbf{1}_{\boldsymbol{\lambda}}) = \langle h_i, \lambda \rangle$.

Proof. The statement involving ε_i is obvious. For $\varphi_i(\mathbf{1}_{\boldsymbol{\lambda}})$, note that $\tilde{f}_i^m \mathbf{1}_{\boldsymbol{\lambda}} = L(i^m)$ and

$$\varepsilon_i^*(L(i^m)) = m, \qquad \varepsilon_j^*(L(i^m)) = 0$$

for every $j \neq i$. Hence by Corollary 6.13, $\operatorname{pr}^\lambda L(i^m) \neq 0$ if and only if $m \leq \langle h_i, \lambda \rangle$. This implies that $\varphi_i(\mathbf{1}_{\boldsymbol{\lambda}}) = \langle h_i, \lambda \rangle$. □

Lemma 6.15. *Let $i, j \in I$ with $i \neq j$ and M be an irreducible module in $\operatorname{Rep}_I \mathcal{H}_n$. Then, $\varepsilon_j^*(\tilde{f}_i^m M) \leq \varepsilon_j^*(M)$ for every $m \geq 0$.*

Proof. Follows from the Shuffle Lemma. □

Lemma 6.16. *Let $i, j \in I$ with $i \neq j$. Let M be an irreducible module in $\operatorname{Rep} \mathcal{H}_n^\lambda$ such that $\varphi_j(M) > 0$. Then, $\varphi_i(\tilde{f}_j M) - \varepsilon_i(\tilde{f}_j M) \leq \varphi_i(M) - \varepsilon_i(M) - \langle h_i, \alpha_j \rangle$.*

Proof. Let $\varepsilon = \varepsilon_i(M), \varphi = \varphi_i(M)$ and $k = -\langle h_i, \alpha_j\rangle$. By Lemma 5.24, there exist unique $a, b \geq 0$ with $a + b = k$ such that $\varepsilon_i(\tilde{f}_j M) = \varepsilon - a$. We need to show that $\varphi_i(\tilde{f}_j M) \leq \varphi + b$, which follows if we can show that $\mathrm{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ for all $m > \varphi + b$. We claim that

$$\varepsilon_i^*(\tilde{f}_i^{m+b} \tilde{f}_j M) \geq \varepsilon_i^*(\tilde{f}_i^m M)$$

for all $m \geq 0$. Given the claim, we know by the definition of φ, Corollary 6.13 and Lemma 6.15 that $\varepsilon_i^*(\tilde{f}_i^m M) > \langle h_i, \lambda\rangle$ for all $m > \varphi$. So the claim implies that $\varepsilon_i^*(\tilde{f}_i^m \tilde{f}_j M) > \langle h_i, \lambda\rangle$ for all $m > \varphi + b$, hence by Corollary 6.13 once more, $\mathrm{pr}^\lambda \tilde{f}_i^m \tilde{f}_j M = 0$ as required.

To prove the claim, note that $a \leq \varepsilon$, so $b + \varepsilon \geq k$. Hence, Lemma 5.24(ii) shows that there is a surjection

$$\mathrm{ind}_{n,m-a,a+b+1}^{n+m+b+1} M \circledast L(i^{m-a}) \circledast L(i^a j i^b) \twoheadrightarrow \tilde{f}_i^{m+b} \tilde{f}_j M.$$

By Lemma 5.19, $\mathrm{res}^{a+b+1}_{a,b+1} L(i^a j i^b) \cong L(i^a) \circledast L(j i^b)$. Hence by Frobenius reciprocity, there is a surjection

$$\mathrm{ind}_{a,b+1}^{a+b+1} L(i^a) \circledast L(j i^b) \twoheadrightarrow L(i^a j i^b).$$

Combining, we have proved existence of a surjection

$$\mathrm{ind}_{n,m,b+1}^{n+m+b+1} M \circledast L(i^m) \circledast L(j i^b) \twoheadrightarrow \tilde{f}_i^{m+b} \tilde{f}_j M.$$

Hence by Frobenius reciprocity there is a non-zero map

$$(\mathrm{ind}_{n,m}^{n+m} M \circledast L(i^m)) \circledast L(j i^b) \to \mathrm{res}^{n+m+b+1}_{n+m,b+1} \tilde{f}_i^{m+b} \tilde{f}_j M.$$

Since the left hand module has irreducible cosocle $\tilde{f}_i^m M \circledast L(j i^b)$, we deduce that $\tilde{f}_i^{m+b} \tilde{f}_j M$ has a constituent isomorphic to $\tilde{f}_i^m M$ on restriction to the subalgebra $\mathcal{H}_{n+m} \subseteq \mathcal{H}_{n+m+b+1}$. This implies the claim. $\square$

Corollary 6.17. *Let $\lambda \in P_+$ and M be an irreducible $\mathcal{H}_n^\lambda$-module with central character χ_γ for some $\gamma \in \Gamma_n$. Then, $\varphi_i(M) - \varepsilon_i(M) \leq \langle h_i, \lambda - \gamma\rangle$.*

Proof. Proceed by induction on n, the case $n = 0$ being immediate by Lemma 6.14. For $n > 0$, we may write $M = \tilde{f}_j N$ for some irreducible $\mathcal{H}_{n-1}^\lambda$-module N with $\varphi_j(N) > 0$. By induction, $\varphi_i(N) - \varepsilon_i(N) \leq \langle h_i, \lambda - \gamma + \alpha_j\rangle$. The conclusion follows from Lemma 6.16. $\square$

§6-f. **Alternative descriptions of $\boldsymbol{\varphi_i}$.** Keep $\lambda \in P_+$ fixed. Now we wish to prove the analogue of Theorem 6.11 for the function φ_i. This is considerably more difficult to do. Let M be an irreducible $\mathcal{H}_n^\lambda$-module. Recall that

$$f_i M = \varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i), \qquad \mathrm{ind}_i M = \varprojlim \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \boxtimes \mathcal{R}_m(i),$$

and that the inverse limits stabilize after finitely many terms. Define $\tilde{\varphi}_i(M)$ to be the stabilization point of the limit, i.e. the least $m \geq 0$ such that $f_i M = \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)$, or equivalently $\mathrm{ind}_i M = \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \boxtimes \mathcal{R}_m(i)$. The first lemma follows [G$_1$, Theorem 9.15].

Lemma 6.18. *Let M be an irreducible $\mathcal{H}_n^\lambda$-module and $i \in I$. Then:*

(i) $[f_i M] = \tilde{\varphi}_i(M)[\tilde{f}_i M] + \sum c_a [N_a]$ *where the N_a are irreducible $\mathcal{H}_{n+1}^\lambda$-modules with $\varepsilon_i(N_a) < \varepsilon_i(\tilde{f}_i M)$;*

(ii) $\mathrm{End}_{\mathcal{H}_{n+1}^\lambda}(f_i M) \simeq \mathrm{End}_{\mathcal{H}_{n+1}^\lambda}(\tilde{f}_i M)^{\oplus \tilde{\varphi}_i(M)}$ *as vector superspaces.*

Proof. (i) Take any $m \geq 1$. Since pr^λ is right exact, the natural surjection $L_m(i) \twoheadrightarrow L_{m-1}(i)$ and the obvious embedding $L_m(i) \hookrightarrow L_{m+1}(i)$ (see §4-h) induce a commutative diagram

$$\begin{array}{ccccccc}
\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L(i) & \xrightarrow{\alpha_m} & \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i) & \xrightarrow{\beta_m} & \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_{m-1}(i) & \rightarrow & 0 \\
\| & & \downarrow & & & & \\
\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L(i) & \xrightarrow{\alpha_{m+1}} & \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_{m+1}(i) & \xrightarrow{\beta_{m+1}} & \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i) & \rightarrow & 0
\end{array}$$

where the rows are exact. Note if $\alpha_m = 0$ then $\alpha_{m+1} = 0$. It follows that if β_m is an isomorphism so is $\beta_{m'}$ for every $m' \geq m$. So by definition of $\tilde{\varphi}_i(M)$, the maps $\beta_1, \beta_2, \dots, \beta_{\tilde{\varphi}_i(M)}$ are not isomorphisms but all other $\beta_{m'}, m' > \tilde{\varphi}_i(M)$ are isomorphisms.

Now to prove (i), we show by induction on $m = 0, 1, \dots, \tilde{\varphi}_i(M)$ that

$$[\mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)] = m[\tilde{f}_i M] + \text{lower terms},$$

where the lower terms are irreducible $\mathcal{H}_{n+1}^\lambda$-modules N with $\varepsilon_i(N) < \varepsilon_i(\tilde{f}_i M)$. This is vacuous if $m = 0$. For $m > 0$, β_m is not an isomorphism, so $\alpha_m \neq 0$. Hence, by Lemma 5.5, the image of α_m contains a copy of $\tilde{f}_i M$ plus lower terms. Now the induction step is immediate.

(ii) Take $m = \tilde{\varphi}_i(M)$. One easily shows using the explicit construction of $L_m(i)$ in §4-h that there is an even endomorphism $\theta : L_m(i) \to L_m(i)$ of $\mathcal{H}_1$-modules, such that the image of θ^k is $\simeq L_{m-k}(i)$ for each $0 \leq k \leq m$. Frobenius reciprocity induces superalgebra homomorphisms

$$\mathrm{End}_{\mathcal{H}_1}(L_m(i)) \hookrightarrow \mathrm{End}_{\mathcal{H}_{n,1}}(M \circledast L_m(i)) \hookrightarrow \mathrm{End}_{\mathcal{H}_{n+1}}(\mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)).$$

So θ induces an even $\mathcal{H}_{n+1}$-endomorphism $\tilde{\theta}$ of $\mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)$, such that the image of $\tilde{\theta}^k$ is $\simeq \mathrm{ind}_{n,1}^{n+1} M \circledast L_{m-k}(i)$ for $0 \leq k \leq m$. Now apply the right exact functor pr^λ to get an even $\mathcal{H}_{n+1}^\lambda$-endomorphism

$$\hat{\theta} : \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i) \to \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)$$

induced by $\tilde{\theta}$. Note $\hat{\theta}^m = 0$ and $\hat{\theta}^{m-1} \neq 0$ because its image coincides with the image of the non-zero map α_m in the proof of (i). Hence, $1, \hat{\theta}, \dots, \hat{\theta}^{m-1}$ are linearly independent, even endomorphisms of $f_i M$. Now the proof of (ii) is completed in the same way as in the proof of Theorem 6.11(iii). □

Corollary 6.19. *Let M, N be irreducible $\mathcal{H}_n^\lambda$-modules with $M \not\cong N$. Then, for every $i \in I$, $\mathrm{Hom}_{\mathcal{H}_{n+1}^\lambda}(f_i M, f_i N) = 0$.*

Proof. Repeat the argument in the proof of Corollary 6.12, but using $\tilde{\varphi}_i$ and Lemma 6.18(i) in place of ε_i and Theorem 6.11(i). □

The main thing now is to prove that $\tilde{\varphi}_i(M) = \varphi_i(M)$. Note right away from the definitions that $\tilde{\varphi}_i(M) = 0$ if and only if $\varphi_i(M) = 0$.

Lemma 6.20. *If M is an irreducible $\mathcal{H}_n^\lambda$-module then*

$$(\tilde{\varphi}_0(M) - \varepsilon_0(M)) + 2\sum_{i=1}^{\ell}(\tilde{\varphi}_i(M) - \varepsilon_i(M)) = \langle c, \lambda \rangle.$$

Proof. By Frobenius reciprocity and Theorem 3.9, we have that

$$\begin{aligned}\mathrm{End}_{\mathcal{H}^\lambda_{n+1}}(\mathrm{ind}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n} M) &\simeq \mathrm{Hom}_{\mathcal{H}^\lambda_n}(M, \mathrm{res}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n}\mathrm{ind}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n} M)\\ &\simeq \mathrm{Hom}_{\mathcal{H}^\lambda_n}(M, M\oplus \Pi M)^{\oplus\langle c,\lambda\rangle} \oplus \mathrm{Hom}_{\mathcal{H}^\lambda_n}(M, \mathrm{ind}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}}\mathrm{res}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}} M)\\ &\simeq \mathrm{Hom}_{\mathcal{H}^\lambda_n}(M, M\oplus \Pi M)^{\oplus\langle c,\lambda\rangle} \oplus \mathrm{End}_{\mathcal{H}^\lambda_{n-1}}(\mathrm{res}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}} M).\end{aligned}$$

Hence, by Schur's lemma,

$$\dim \mathrm{End}_{\mathcal{H}^\lambda_{n+1}}(\mathrm{ind}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n} M) - \dim \mathrm{End}_{\mathcal{H}^\lambda_{n-1}}(\mathrm{res}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}} M) = \begin{cases} 2\langle c,\lambda\rangle & \text{if } M \text{ is of type } \mathtt{M},\\ 4\langle c,\lambda\rangle & \text{if } M \text{ is of type } \mathtt{Q}.\end{cases}$$

Now if M is of type $\mathtt{M}$,

$$\mathrm{ind}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n} M \simeq f_0 M \oplus \bigoplus_{i=1}^{\ell} (f_i M \oplus \Pi f_i M), \qquad \mathrm{res}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}} M \simeq e_0 M \oplus \bigoplus_{i=1}^{\ell}(e_i M \oplus \Pi e_i M).$$

by Lemmas 6.3, 6.4 and (6.6). Hence by Lemma 6.18(ii) and Theorem 6.11(iii),

$$\dim \mathrm{End}_{\mathcal{H}^\lambda_{n+1}}(\mathrm{ind}^{\mathcal{H}^\lambda_{n+1}}_{\mathcal{H}^\lambda_n} M) = 2\tilde{\varphi}_0(M) + 4\sum_{i=1}^{\ell}\tilde{\varphi}_i(M),$$
$$\dim \mathrm{End}_{\mathcal{H}^\lambda_{n-1}}(\mathrm{res}^{\mathcal{H}^\lambda_n}_{\mathcal{H}^\lambda_{n-1}} M) = 2\varepsilon_0(M) + 4\sum_{i=1}^{\ell}\varepsilon_i(M),$$

and the conclusion follows in this case. The argument for M of type $\mathtt{Q}$ is similar. □

Lemma 6.21. *Let M be an irreducible $\mathcal{H}^\lambda_n$-module and $i \in I$. Let $c_i = 1$ if $i = 0$, $c_i = 2$ otherwise. Then,*

$$[\mathrm{res}_i\mathrm{ind}_i M : M] = 2c_i\varepsilon_i(\tilde{f}_i M)\tilde{\varphi}_i(M), \qquad [\mathrm{ind}_i\mathrm{res}_i M : M] = 2c_i\varepsilon_i(M)\tilde{\varphi}_i(\tilde{e}_i M),$$
$$\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M \simeq (M\oplus\Pi M)^{\oplus c_i\tilde{\varphi}_i(M)}, \qquad \mathrm{soc}\,\mathrm{ind}_i\mathrm{res}_i M \simeq (M\oplus\Pi M)^{\oplus c_i\varepsilon_i(M)}.$$

Proof. The statement about composition multiplicities follows from Theorem 6.11(i) and Lemma 6.18(i), taking into account how res_i and ind_i are related to e_i and f_i as explained in Lemmas 6.3 and 6.4. Now consider the statement about socles. We consider only $\mathrm{res}_i\mathrm{ind}_i M$, the other case being entirely similar but using results from §6-d instead. By adjointness, it suffices to be able to compute $\mathrm{Hom}_{\mathcal{H}^\lambda_{n+1}}(\mathrm{ind}_i N, \mathrm{ind}_i M)$ for any irreducible $\mathcal{H}^\lambda_n$-module N. But in view of Lemma 6.4, this can be computed from knowledge of $\mathrm{Hom}_{\mathcal{H}^\lambda_{n+1}}(f_i N, f_i M)$ which is known by Corollary 6.19 (if $M \not\cong N$) and Lemma 6.18(ii) (if $M \cong N$). The details are similar to those in the proof of Lemma 6.20, so we omit them. □

The proof of the next lemma is based on [V_2, Lemma 6.1].

Lemma 6.22. *Let M be an irreducible $\mathcal{H}^\lambda_n$-module and $i \in I$. There are maps*

$$\mathrm{ind}_i\mathrm{res}_i M \xrightarrow{\psi} \mathrm{res}_i\mathrm{ind}_i M \xrightarrow{\mathrm{can}} \mathrm{res}_i\mathrm{ind}_i M/\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M,$$

whose composite is surjective.

Proof. Let $k = \tilde{\varphi}_i(M)$ and

$$\pi : \mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i) \twoheadrightarrow \mathrm{pr}^\lambda \mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i) = \mathrm{ind}_i M$$

be the quotient map. Let $\mathcal{H}'_1$ denote the subalgebra of $\mathcal{H}_n$ generated by $X_n^{\pm 1}, C_n$, and set $z = X_n + X_n^{-1} - q(i)$ (resp. $X_n - 1$ if $i = 0$). Recall from §4-h that viewed as an $\mathcal{H}'_1$-module, we have that $\mathcal{R}_k(i) \simeq \mathcal{H}'_1/(z^k)$. In particular, $\mathcal{R}_k(i)$ is a cyclic module generated by the image $\tilde{1}$ of $1 \in \mathcal{H}'_1$.

We first observe that for any $m \geq \varepsilon_i(M) + \tilde{\varphi}_i(M)$, z^m annihilates the vector

$$\pi[T_n \otimes (u \otimes v)] \in \mathrm{ind}_i M$$

for any $u \in M, v \in \mathcal{R}_k(i)$. This follows from the relations in $\mathcal{H}_{n+1}$, e.g. in case $i \neq 0$ one ultimately appeals to the facts that $(X_n + X_n^{-1} - q(i))^{\varepsilon_i(M)}$ annihilates u (see Theorem 6.11(ii)) and $(X_{n+1} + X_{n+1}^{-1} - q(i))^{\tilde{\varphi}_i(M)}$ annihilates v. It follows that the unique $\mathcal{H}_{n-1,1}$-homomorphism $(\mathrm{res}_i M) \boxtimes \mathcal{H}'_1 \to \mathrm{res}^n_{n-1,1}\mathrm{res}_i\mathrm{ind}_i M$ such that $u \otimes 1 \mapsto \pi[T_n \otimes (u \otimes \tilde{1})]$ for each $u \in \mathrm{res}_i M \subseteq M$ factors to induce a well-defined $\mathcal{H}_{n-1,1}$-module homorphism $(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(i) \to \mathrm{res}^n_{n-1,1}\mathrm{res}_i\mathrm{ind}_i M$. We then get from Frobenius reciprocity an induced map

$$\psi_m : \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(i) \to \mathrm{res}_i\mathrm{ind}_i M \tag{6.21}$$

for each $m \geq \varepsilon_i(M) + \tilde{\varphi}_i(M)$. Each ψ_m factors through the quotient $\mathrm{pr}^\lambda\mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(i)$, so we get an induced map

$$\psi : \mathrm{ind}_i\mathrm{res}_i M = \varprojlim \mathrm{pr}^\lambda\mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(i) \to \mathrm{res}_i\mathrm{ind}_i M.$$

It remains to show that the composite of ψ with the canonical epimorphism from $\mathrm{res}_i\mathrm{ind}_i M$ to $\mathrm{res}_i\mathrm{ind}_i M/\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M$ is surjective.

Let $x = (n, n+1) \in S_{n+1}$. By Mackey Theorem there exists an exact sequence

$$0 \to M \boxtimes \mathcal{R}_k(i) \to \mathrm{res}^{n+1}_{n,1}(\mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i)) \to \mathrm{ind}^{n,1}_{n-1,1,1}{}^x((\mathrm{res}^n_{n-1,1}M) \boxtimes \mathcal{R}_k(i)) \to 0.$$

In other words, there is an $\mathcal{H}_{n,1}$-isomorphism

$$\begin{array}{rcl} \mathrm{ind}^{n,1}_{n-1,1,1}{}^x((\mathrm{res}^n_{n-1,1}M) \boxtimes \mathcal{R}_k(i)) & \stackrel{\sim}{\longrightarrow} & \mathrm{res}^{n+1}_{n,1}(\mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i))/(M \boxtimes \mathcal{R}_k(i)), \\ h \otimes (u \otimes v) & \mapsto & hT_n \otimes u \otimes v + M \boxtimes \mathcal{R}_k(i) \end{array}$$

for $h \in \mathcal{H}_n, u \in M, v \in \mathcal{R}_k(i)$, where $M \boxtimes \mathcal{R}_k(i)$ is embedded into $\mathrm{res}^{n+1}_{n,1}(\mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i))$ as $1 \otimes M \otimes \mathcal{R}_k(i)$. Recall from (4.28) that $\dim \mathcal{R}_k(i) = 2kc_i$ where c_i is as in Lemma 6.21. Hence, applying Lemma 6.21,

$$\mathrm{res}^{n,1}_n M \boxtimes \mathcal{R}_k(i) \simeq (M \oplus \Pi M)^{kc_i} \simeq \mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M.$$

So applying the exact functor $\mathrm{res}_i = \mathrm{res}^{n,1}_n \circ \Delta_i$ to the isomorphism above we get an isomorphism

$$\begin{array}{rcl} \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_k(i) & \stackrel{\sim}{\longrightarrow} & \mathrm{res}_i(\mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_k(i))/\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M, \\ h \otimes u \otimes v & \mapsto & hT_n \otimes u \otimes v + \mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M. \end{array}$$

It follows that there is a surjection

$$\theta : \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_k(i) \twoheadrightarrow \mathrm{res}_i\mathrm{ind}_i M/\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M$$

such that the diagram

$$\begin{array}{ccc} \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(i) & \xrightarrow{\psi_m} & \mathrm{res}_i\mathrm{ind}_i M \\ \downarrow & & \downarrow{\scriptstyle \mathrm{can}} \\ \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_k(i) & \xrightarrow{\theta} & \mathrm{res}_i\mathrm{ind}_i M/\mathrm{soc}\,\mathrm{res}_i\mathrm{ind}_i M \end{array}$$

commutes for all $m \geq \varepsilon_i(M) + \tilde{\varphi}_i(M)$, where ψ_m is the map from (6.21) and the left hand arrow is the natural surjection. Now surjectivity of θ immediately implies the desired surjectivity of the composite. $\square$

Lemma 6.23. *Let M be an irreducible $\mathcal{H}_n^\lambda$-module with $\varepsilon_i(M) > 0$. Then,*

$$\tilde{\varphi}_i(\tilde{e}_i M) = \tilde{\varphi}_i(M) + 1.$$

Proof. Let us first show that

$$\tilde{\varphi}_i(\tilde{e}_i M) \geq \tilde{\varphi}_i(M) + 1. \tag{6.22}$$

Recall $\varphi_i(M) = 0$ if and only if $\tilde{\varphi}_i(M) = 0$. Suppose first that $\varphi_i(M) = 0$, when $\varphi_i(\tilde{e}_i M) \neq 0$. Then, $\tilde{\varphi}_i(M) = 0$ and $\tilde{\varphi}_i(\tilde{e}_i M) \neq 0$, so the conclusion certainly holds in this case. So we may assume that $\varphi_i(M) > 0$, hence $\tilde{\varphi}_i(M) \neq 0$. Note by Lemma 6.21,

$$[\mathrm{res}_i \mathrm{ind}_i M / \mathrm{soc}\, \mathrm{res}_i \mathrm{ind}_i M : M] = 2c_i \varepsilon_i(\tilde{f}_i M)\tilde{\varphi}_i(M) - 2c_i \tilde{\varphi}_i(M) = 2c_i \varepsilon_i(M)\tilde{\varphi}_i(M) \neq 0.$$

In particular, the map ψ in Lemma 6.22 is non-zero. Now Lemma 6.22 implies that the multiplicity of M in $\mathrm{im}\, \psi$ is *strictly* greater than $2c_i\varepsilon_i(M)\tilde{\varphi}_i(M)$, since at least one composition factor of $\mathrm{soc}\, \mathrm{im}\, \psi \subseteq \mathrm{soc}\, \mathrm{res}_i \mathrm{ind}_i M$ must be sent to zero on composing with the second map can. Using another part of Lemma 6.21, this shows that

$$2c_i\varepsilon_i(M)\tilde{\varphi}_i(\tilde{e}_i M) > 2c_i\varepsilon_i(M)\tilde{\varphi}_i(M)$$

and (6.22) follows.

Now using (6.22) and Lemma 6.21, we see that in the Grothendieck group,

$$[\mathrm{res}_i \mathrm{ind}_i M - \mathrm{ind}_i \mathrm{res}_i M : M] \leq 2c_i(\tilde{\varphi}_i(M) - \varepsilon_i(M)),$$

with equality if and only if equality holds in (6.22). By central character considerations, for $i \neq j$, $[\mathrm{res}_i \mathrm{ind}_j M : M] = [\mathrm{ind}_j \mathrm{res}_i M : M] = 0$. So using (6.6) we deduce that

$$[\mathrm{res}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} \mathrm{ind}^{\mathcal{H}_{n+1}^\lambda}_{\mathcal{H}_n^\lambda} M - \mathrm{ind}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} \mathrm{res}^{\mathcal{H}_n^\lambda}_{\mathcal{H}_{n-1}^\lambda} M : M] \leq 2(\tilde{\varphi}_0(M) - \varepsilon_0(M)) + 4\sum_{i=1}^{\ell}(\tilde{\varphi}_i(M) - \varepsilon_i(M))$$

with equality if and only if equality holds in (6.22) for all $i \in I$. Now Lemma 6.20 shows that the right hand side equals $2\langle c, \lambda \rangle$, which does indeed equal the left hand side thanks to Theorem 3.9. $\square$

Corollary 6.24. *For any irreducible $\mathcal{H}_n^\lambda$-module M, $\varphi_i(M) = \tilde{\varphi}_i(M)$.*

Proof. We proceed by induction on $\varphi_i(M)$, the conclusion being known already in case $\varphi_i(M) = 0$. For the induction step, take an irreducible $\mathcal{H}_n^\lambda$-module N with $\varphi_i(N) > 0$, so $N = \tilde{e}_i M$ where $M = \tilde{f}_i N$ is an irreducible $\mathcal{H}_{n+1}^\lambda$-module with $\varepsilon_i(M) > 0$, $\varphi_i(M) < \varphi_i(N)$. Then by Lemma 6.23 and the induction hypothesis,

$$\tilde{\varphi}_i(N) = \tilde{\varphi}_i(\tilde{e}_i M) = \tilde{\varphi}_i(M) + 1 = \varphi_i(M) + 1 = \varphi_i(\tilde{e}_i M) = \varphi_i(N).$$

This completes the induction step. $\square$

As a first consequence, we can improve Corollary 6.17:

Lemma 6.25. *Let M be an irreducible $\mathcal{H}_n^\lambda$-module of central character χ_γ for $\gamma \in \Gamma_n$. Then, $\varphi_i(M) - \varepsilon_i(M) = \langle h_i, \lambda - \gamma \rangle$.*

Proof. In view of Corollary 6.17, it suffices to show that

$$(\varphi_0(M) - \varepsilon_0(M)) + 2\sum_{i=1}^{\ell}(\varphi_i(M) - \varepsilon_i(M)) = \langle c, \lambda \rangle.$$

But this is immediate from Lemma 6.20 and Corollary 6.24. □

We are finally ready to assemble the results of the subsection to obtain the full analogue of Theorem 6.11 for φ_i:

Theorem 6.26. *Let $i \in I$ and M be an irreducible $\mathcal{H}_n^\lambda$-module. Then,*

(i) $[f_i M] = \varphi_i(M)[\tilde{f}_i M] + \sum c_a[N_a]$ *where the N_a are irreducibles with* $\varphi_i(N_a) < \varphi_i(\tilde{f}_i M)$*;*

(ii) $\varphi_i(M)$ *is the least* $m \geq 0$ *such that* $f_i M = \mathrm{pr}^\lambda \mathrm{ind}_{n,1}^{n+1} M \circledast L_m(i)$*;*

(iii) $\mathrm{End}_{\mathcal{H}_{n+1}}(f_i M) \simeq \mathrm{End}_{\mathcal{H}_{n+1}}(\tilde{f}_i M)^{\oplus \varphi_i(M)}$ *as vector superspaces.*

Proof. (i) Since $\varphi_i(M) = \tilde{\varphi}_i(M)$ by Corollary 6.24, we know by Lemma 6.18(i) that

$$[f_i M] = \varphi_i(M)[\tilde{f}_i M] + \sum c_a[N_a]$$

where the N_a are irreducibles with $\varepsilon_i(N_a) < \varepsilon_i(\tilde{f}_i M)$. Suppose that $M \in \mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$, for $\gamma \in \Gamma_n$. Then $\tilde{f}_i M$ and each N_a have central character $\chi_{\gamma+\alpha_i}$, since they are all composition factors of $\mathrm{ind}_i M$. So by Lemma 6.25,

$$\varphi_i(N_a) = \langle h_i, \lambda - \gamma - \alpha_i \rangle + \varepsilon_i(N_a), \qquad \varphi_i(\tilde{f}_i M) = \langle h_i, \lambda - \gamma - \alpha_i \rangle + \varepsilon_i(\tilde{f}_i M).$$

It follows that $\varphi_i(N_a) < \varphi_i(\tilde{f}_i N_a)$ too.

(ii) This is just the definition of $\tilde{\varphi}_i(M)$ combined with Corollary 6.24.

(iii) This follows from Lemma 6.18(ii). □

7. Construction of $U_{\mathbb{Z}}^+$

§7-a. **Grothendieck groups revisited.** Let us now write

$$K(\infty) = \bigoplus_{n \geq 0} K(\mathrm{Rep}_I \mathcal{H}_n) \tag{7.1}$$

for the sum over all n of the Grothendieck groups of the categories $\mathrm{Rep}_I \mathcal{H}_n$. Also, write

$$K(\infty)_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} K(\infty), \tag{7.2}$$

extending scalars. Thus $K(\infty)$ is a free $\mathbb{Z}$-module with canonical basis given by $B(\infty)$, the isomorphism classes of irreducible modules, and $K(\infty)_{\mathbb{Q}}$ is the $\mathbb{Q}$-vector space on basis $B(\infty)$. We will always view $K(\infty)$ as a lattice in $K(\infty)_{\mathbb{Q}}$.

We let $K(\infty)^*$ denote the *restricted dual* of $K(\infty)$, namely, the set of functions $f : K(\infty) \to \mathbb{Z}$ such that f vanishes on all but finitely many elements of $B(\infty)$. Thus $K(\infty)^*$ is also a free $\mathbb{Z}$-module, with canonical basis

$$\{\delta_M \mid [M] \in B(\infty)\}$$

dual to the basis $B(\infty)$ of $K(\infty)$, i.e. $\delta_M([M]) = 1$, $\delta_M([N]) = 0$ for $[N] \in B(\infty)$ with $N \not\cong M$. Note for an arbitrary $N \in \mathrm{Rep}_I \mathcal{H}_n$, $\delta_M([N])$ simply computes the *composition multiplicity* $[N : M]$ of the irreducible module M as a composition factor of N. Finally, we write $B(\infty)_{\mathbb{Q}}^* := \mathbb{Q} \otimes_{\mathbb{Z}} B(\infty)^*$, which can be identified with the restricted dual of $B(\infty)_{\mathbb{Q}}$.

Entirely similar definitions can be made for each $\lambda \in P_+$:

$$K(\lambda) = \bigoplus_{n \geq 0} K(\mathrm{Rep}_I \mathcal{H}_n^\lambda) \tag{7.3}$$

denotes the Grothendieck groups of the categories $\mathrm{Rep}\,\mathcal{H}_n^\lambda$ for all n. Again, $K(\lambda)$ is a free $\mathbb{Z}$-module on the basis $B(\lambda)$ of isomorphism classes of irreducible modules. Moreover, infl^λ induces a canonical embedding

$$\mathrm{infl}^\lambda : K(\lambda) \hookrightarrow K(\infty).$$

We will generally identify $K(\lambda)$ with its image under this embedding. We also define $K(\lambda)^*$ and $K(\lambda)_{\mathbb{Q}} = \mathbb{Q} \otimes_{\mathbb{Z}} K(\lambda)$ as above.

Recall the operators e_i and more generally the divided power operators $e_i^{(r)}$ for $r \geq 1$, defined on irreducible modules in $\mathrm{res}_I \mathcal{H}_n$ in (6.8) and (6.11) respectively. These induce linear maps

$$e_i^{(r)} : K(\infty) \to K(\infty) \tag{7.4}$$

for each $r \geq 1$. Similarly, the operators $e_i^{(r)}$ and $f_i^{(r)}$ from (6.12) and (6.13) respectively induce maps

$$e_i^{(r)}, f_i^{(r)} : K(\lambda) \to K(\lambda). \tag{7.5}$$

Recall by Lemma 6.9 that

$$e_i^r = (r!)e_i^{(r)}, \qquad f_i^r = (r!)f_i^{(r)}. \tag{7.6}$$

Extending scalars, the maps $e_i^{(r)}, f_i^{(r)}$ induce linear maps on $K(\infty)_{\mathbb{Q}}$ and $K(\lambda)_{\mathbb{Q}}$ too.

§7-b. **Hopf algebra structure.** Now we wish to give $K(\infty)$ the structure of a graded Hopf algebra over $\mathbb{Z}$. To do this, recall the canonical isomorphism

$$K(\mathrm{Rep}_I \mathcal{H}_m) \otimes_{\mathbb{Z}} K(\mathrm{Rep}_I \mathcal{H}_n) \to K(\mathrm{Rep}_I \mathcal{H}_{m,n}) \tag{7.7}$$

from (2.4), for each $m, n \geq 0$. The exact functor $\mathrm{ind}_{m,n}^{m+n}$ induces a well-defined map

$$\mathrm{ind}_{m,n}^{m+n} : K(\mathrm{Rep}_I \mathcal{H}_{m,n}) \to K(\mathrm{Rep}_I \mathcal{H}_{m+n}).$$

Composing with the isomorphism (7.7) and taking the direct sum over all $m, n \geq 0$, we obtain a homogeneous map

$$\diamond : K(\infty) \otimes_{\mathbb{Z}} K(\infty) \to K(\infty). \tag{7.8}$$

By transitivity of induction, this makes $K(\infty)$ into an associative graded $\mathbb{Z}$-algebra. Moreover, there is a unit

$$\iota : \mathbb{Z} \to K(\infty) \tag{7.9}$$

mapping 1 to the identity module $\mathbf{1}$ of $\mathcal{H}_0$. Finally, since the duality τ induces the identity map at the level of Grothendieck groups, Theorem 2.14 implies that the multiplication $\diamond$ is commutative (in the usual unsigned sense).

Now consider how to define the comultiplication. The exact functor res_{n_1,n_2}^n induces a map

$$\mathrm{res}_{n_1,n_2}^n : K(\mathrm{Rep}_I \mathcal{H}_n) \to K(\mathrm{Rep}_I \mathcal{H}_{n_1,n_2}).$$

On composing with the isomorphism (7.7), we obtain maps

$$\Delta^n_{n_1,n_2} : K(\mathrm{Rep}_I\,\mathcal{H}_n) \to K(\mathrm{Rep}_I\,\mathcal{H}_{n_1}) \otimes_{\mathbb{Z}} K(\mathrm{Rep}_I\,H_{n_2}), \tag{7.10}$$

$$\Delta^n = \sum_{n_1+n_2=n} \Delta^n_{n_1,n_2} : K(\mathrm{Rep}_I\,\mathcal{H}_n) \to \bigoplus_{n_1+n_2=n} K(\mathrm{Rep}_I\,\mathcal{H}_{n_1}) \otimes_{\mathbb{Z}} K(\mathrm{Rep}_I\,H_{n_2}). \tag{7.11}$$

Now taking the direct sum over all $n \geq 0$ gives a homomogeneous map

$$\Delta : K(\infty) \to K(\infty) \otimes_{\mathbb{Z}} K(\infty). \tag{7.12}$$

Transitivity of restriction implies that Δ is coassociative, while the homogeneous projection onto $K(\mathrm{Rep}_I\,\mathcal{H}_0) \cong \mathbb{Z}$ gives a counit

$$\varepsilon : K(\infty) \to \mathbb{Z}. \tag{7.13}$$

Thus $K(\infty)$ is also a graded coalgebra over $\mathbb{Z}$. Now finally:

Theorem 7.1. *$(K(\infty), \diamond, \Delta, \iota, \varepsilon)$ is a commutative, graded Hopf algebra over $\mathbb{Z}$.*

Proof. It just remains to check that Δ is an algebra homomorphism, which follows using the Mackey Theorem. Note in checking the details, one needs to use Lemma 5.14 to take the definition of $\circledast$ into account correctly. □

We record the following lemma explaining how to compute the action of e_i on $K(\infty)$ explicitly in terms of Δ:

Lemma 7.2. *Let M be a module in $\mathrm{Rep}_I\,\mathcal{H}_n$. Write*

$$\Delta^n_{n-1,1}[M] = \sum_a [M_a] \otimes [N_a]$$

for irreducible $\mathcal{H}_{n-1}$-modules M_a and irreducible $\mathcal{H}_1$-modules N_a. Then,

$$e_i[M] = \sum_{a \text{ with } N_a \cong L(i)} [M_a].$$

Proof. This is immediate from Lemma 6.3. □

Lemma 7.3. *The operators $e_i : K(\infty) \to K(\infty)$ satisfy the Serre relations, i.e.*

$$\begin{aligned}
e_i e_j &= e_j e_i && \text{if } |i-j| > 1;\\
e_i^2 e_j + e_j e_i^2 &= 2 e_i e_j e_i && \text{if } |i-j| = 1, i \neq 0, j \neq \ell;\\
e_0^3 e_1 + 3 e_0 e_1 e_0^2 &= 3 e_0^2 e_1 e_0 + e_1 e_0^3 && \text{if } \ell \neq 1;\\
e_{\ell-1}^3 e_\ell + 3 e_{\ell-1} e_\ell e_{\ell-1}^2 &= 3 e_{\ell-1}^2 e_\ell e_{\ell-1} + e_\ell e_{\ell-1}^3 && \text{if } \ell \neq 1;\\
e_0^5 e_1 + 5 e_0 e_1 e_0^4 + 10 e_0^3 e_1 e_0^2 &= 10 e_0^2 e_1 e_0^3 + 5 e_0^4 e_1 e_0 + e_1 e_0^5 && \text{if } \ell = 1,
\end{aligned}$$

for $i, j \in I$.

Proof. In view of Lemma 7.2 and coassociativity of Δ, this reduces to checking it on irreducible $\mathcal{H}_n$-modules for $n = 2, 3, 4, 4, 6$ respectively. For this, the character information in Lemmas 5.19 and 5.20 is sufficient. □

Now consider $K(\lambda)$ for $\lambda \in P_+$. This has a natural structure as $K(\infty)$-comodule: viewing $K(\lambda)$ as a subset of $K(\infty)$, the comodule structure map is the restriction

$$\Delta^\lambda : K(\lambda) \to K(\lambda) \otimes_{\mathbb{Z}} K(\infty)$$

of Δ. In other words, each $K(\lambda)$ is a subcomodule of the right regular $K(\infty)$-comodule.

One checks from the definitions that the dual maps to $\diamond, \Delta, \iota, \varepsilon$ induce on $K(\infty)^*$ the structure of a cocommutative graded Hopf algebra. From this point of view, each $K(\lambda)$ is a left $K(\infty)^*$-module in the natural way: $f \in K(\infty)^*$ acts on the left on $K(\lambda)$ as the map $(\mathrm{id}\,\bar{\otimes} f) \circ \Delta^\lambda$. Similarly, $K(\infty)$ is itself a left $K(\infty)^*$-module, indeed in this case the action is even *faithful*.

Lemma 7.4. *The operator $e_i^{(r)}$ acts on $K(\infty)$ (resp. $K(\lambda)$ for any $\lambda \in P_+$) in the same way as the basis element $\delta_{L(i^r)}$ of $K(\infty)^*$.*

Proof. Let M be an irreducible module in $\mathrm{Rep}_I \mathcal{H}_n$ or $\mathrm{Rep}\,\mathcal{H}_n^\lambda$. Let

$$\Delta_{n-r,r}[M] = \sum_a [M_a] \otimes [N_a]$$

for irreducible $\mathcal{H}_{n-r}^\lambda$-modules M_a and irreducible $\mathcal{H}_r$-modules N_a. By the definition of the action of $K(\infty)^*$ on $K(\lambda)$, it follows that

$$\delta_{L(i^r)}[M] = \sum_{a \text{ with } N_a \cong L(i^r)} [M_a].$$

Hence, since $[\mathrm{res}^r_{1,\dots,1} L(i^r)] = (r!)[L(i)^{\circledast r}]$, we get that $\delta^r_{L(i)}$ acts in the same way as $(r!)\delta_{L(i^r)}$. So in view of (7.6), it just remains to check that $\delta_{L(i)}$ acts in the same way as e_i, which follows by Lemma 7.2. $\square$

Lemma 7.5. *There is a unique homomorphism $\pi : U_\mathbb{Z}^+ \to K(\infty)^*$ of graded Hopf algebras such that $e_i^{(r)} \mapsto \delta_{L(i^r)}$ for each $i \in I$ and $r \geq 1$.*

Proof. Since $K(\infty)$ is a faithful $K(\infty)^*$-module, (7.6) and Lemmas 7.3 and 7.4 imply that the operators $\delta_{L(i^r)}$ satisfy the same relations as the generators $e_i^{(r)}$ of $U_\mathbb{Z}^+$. This implies existence of a unique such algebra homomorphism. The fact that π is a coalgebra map follows because

$$\Delta(\delta_{L(i)}) = \delta_{L(i)} \otimes 1 + 1 \otimes \delta_{L(i)}$$

by the definition of the comultiplication on $K(\infty)^*$. $\square$

§7-c. **Shapovalov form.** Now focus on a fixed $\lambda \in P_+$. For an $\mathcal{H}_n^\lambda$-module M, we let P_M denote its projective cover in the category $\mathcal{H}_n^\lambda$-mod. Since $\mathcal{H}_n^\lambda$ is a finite dimensional superalgebra, we can identify

$$K(\lambda)^* = \bigoplus_{n \geq 0} K(\mathrm{Proj}\,\mathcal{H}_n^\lambda) \tag{7.14}$$

so that the basis element δ_M corresponds to the isomorphism class $[P_M]$ for each irreducible $\mathcal{H}_n^\lambda$-module M and each $n \geq 0$. Moreover, under this identification, the canonical pairing

$$(.,.) : K(\lambda)^* \times K(\lambda) \to \mathbb{Z} \tag{7.15}$$

satisfies

$$([P_M],[N]) = \begin{cases} \dim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(P_M, N) & \text{if } M \text{ is of type } \mathtt{M}, \\ \frac{1}{2} \dim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(P_M, N) & \text{if } M \text{ is of type } \mathtt{Q}, \end{cases} \tag{7.16}$$

for $\mathcal{H}_n^\lambda$-modules M, N with M irreducible (since the right hand side clearly computes the composition multiplicity $[N : M]$).

There is a homogeneous map

$$\omega : K(\lambda)^* \to K(\lambda) \tag{7.17}$$

induced by the natural maps $K(\mathrm{Proj}\,\mathcal{H}_n^\lambda) \to K(\mathrm{Rep}\,\mathcal{H}_n^\lambda)$ for each n. We warn the reader that we do not yet know that ω is injective.

As explained at the end of §6-c, we can define an action of $e_i^{(r)}$ and $f_i^{(r)}$ on the projective indecomposable modules, hence on $K(\lambda)^*$. We know by Lemma 6.10 that (7.6) holds for the operations on $K(\lambda)^*$ as well as on $K(\lambda)$. Also, the actions of $e_i^{(r)}$ and $f_i^{(r)}$ commute with ω by (6.14).

Lemma 7.6. *The operators e_i, f_i on $K(\lambda)^*$ and $K(\lambda)$ satisfy $(e_i x, y) = (x, f_i y)$, $(f_i x, y) = (x, e_i y)$ for each $x \in K(\lambda)^*$ and $y \in K(\lambda)$.*

Proof. Let M be an irreducible $\mathcal{H}_n^\lambda$-module and N be an irreducible $\mathcal{H}_{n+1}^\lambda$-module. We check that $(f_i[P_M], [N]) = ([P_M], e_i[N])$ in the special case that $i = 0$, M is of type $\mathtt{Q}$ and N is of type $\mathtt{M}$. In this case, Lemmas 6.4, 6.1(i) and 6.3,

$$\begin{aligned}(f_i[P_M], [N]) &= \frac{1}{2}(\mathrm{ind}_i[P_M], [N]) = \frac{1}{2}\dim \mathrm{Hom}_{\mathcal{H}_{n+1}^\lambda}(\mathrm{ind}_i P_M, N)\\ &= \frac{1}{2}\dim \mathrm{Hom}_{\mathcal{H}_n^\lambda}(P_M, \mathrm{res}_i N) = ([P_M], \mathrm{res}_i[N]) = ([P_M], e_i[N]).\end{aligned}$$

All the other situations that need to be considered follow similarly. □

Corollary 7.7. *Suppose*

$$e_i^{(r)}[M] = \sum_{[N]\in B(\lambda)} a_{M,N}[N], \quad f_i^{(r)}[M] = \sum_{[N]\in B(\lambda)} b_{M,N}[N].$$

for $[M] \in B(\lambda)$. Then,

$$e_i^{(r)}[P_N] = \sum_{[M]\in B(\lambda)} b_{M,N}[P_M], \quad f_i^{(r)}[P_N] = \sum_{[M]\in B(\lambda)} a_{M,N}[P_M]$$

for $[N] \in B(\lambda)$.

Using this, the next lemma is an easy consequence of Theorem 6.26(i), see [G_1, Lemma 11.2] for the detailed proof.

Lemma 7.8. *Let M be an irreducible $\mathcal{H}_n^\lambda$-module, set $\varepsilon = \varepsilon_i(M), \varphi = \varphi_i(M)$. Then, for any $m \geq 0$,*

$$e_i^{(m)}[P_M] = \sum_{N \text{ with } \varepsilon_i(N)\geq m} a_N[P_{\tilde{e}_i^m N}]$$

for coefficients $a_N \in \mathbb{Z}_{\geq 0}$. Moreover, in case $m = \varepsilon$,

$$e_i^{(\varepsilon)}[P_M] = \binom{\varepsilon+\varphi}{\varepsilon}[P_{\tilde{e}_i^\varepsilon M}] + \sum_{N \text{ with } \varepsilon_i(N)>\varepsilon} a_N[P_{\tilde{e}_i^\varepsilon N}].$$

We also need:

Theorem 7.9. *Given an irreducible $\mathcal{H}_n^\lambda$-module M, $[P_M] \in K(\mathrm{Proj}\,\mathcal{H}_n^\lambda)$ can be written as an integral linear combination of terms of the form $f_{i_1}^{(r_1)} \dots f_{i_a}^{(r_a)}[\mathbf{1}_\lambda]$.*

Proof. Proceed by induction on n, the conclusion being trivial for $n = 0$. So suppose $n > 0$ and that the result is true for all smaller n. Suppose for a contradiction that we can find an irreducible $\mathcal{H}_n^\lambda$-module M for which the result does not hold. Pick i with $\varepsilon := \varepsilon_i(M) > 0$. Since there are only finitely many irreducible $\mathcal{H}_n^\lambda$-modules, we may assume by the choice of M that the result holds for all irreducible $\mathcal{H}_n^\lambda$-modules L with $\varepsilon_i(L) > \varepsilon$. Write

$$f_i^{(\varepsilon)}[P_{\tilde{e}_i^\varepsilon M}] = \sum_{[L]\in \mathrm{Irr}\,\mathcal{H}_n^\lambda} a_L[P_L]$$

for coefficients $a_L \in \mathbb{Z}$. By Corollary 7.7, $a_L = [e_i^{(\varepsilon)} L : \tilde{e}_i^\varepsilon M]$. In particular, $a_L = 0$ unless $\varepsilon_i(L) \geq \varepsilon$. Moreover, if $a_L \neq 0$ for L with $\varepsilon_i(L) = \varepsilon$, then by Theorem 6.11(i), we have that $e_i^{(\varepsilon)} L \cong \tilde{e}_i^\varepsilon L \cong \tilde{e}_i^\varepsilon M$, whence $L \cong M$ and $a_M = 1$. This shows:

$$[P_M] = f_i^{(\varepsilon)}[P_{\tilde{e}_i^\varepsilon M}] - \sum_{L\ with\ \varepsilon_i(L)>\varepsilon} a_L[P_L] \tag{7.18}$$

for some $a_L \in \mathbb{Z}$. But the inductive hypothesis and choice of M ensure that all terms on the right hand side are integral linear combinations of terms $f_{i_1}^{(r_1)} \dots f_{i_a}^{(r_a)}[\mathbf{1}_{\boldsymbol{\lambda}}]$, hence the same is true for $[P_M]$, a contradiction. □

The next two theorems are fundamental, so we include the proofs even though they are identical to the argument in [G_1, Theorem 11.1].

Theorem 7.10. *The map $\omega : K(\lambda)^* \to K(\lambda)$ from (7.17) is injective.*

Proof. We show by induction on n that the map $\omega : K(\mathrm{Proj}\,\mathcal{H}_n^\lambda) \to K(\mathrm{Rep}\,\mathcal{H}_n^\lambda)$ is injective. This is clear if $n = 0$, so assume $n > 0$ and the result has been proved for all smaller n. Suppose we have a relation

$$\sum_{[M]\in \mathrm{Irr}\,\mathcal{H}_n^\lambda} a_M \omega([P_M]) = 0$$

with not all coefficients a_M being zero. We may choose $i \in I$ and an irreducible module M such that $a_M \neq 0, \varepsilon := \varepsilon_i(M) > 0$ and $a_N = 0$ for all N with $\varepsilon_i(N) > \varepsilon$.

Apply $e_i^{(\varepsilon)}$ to the sum. By Lemma 7.8, we get

$$\sum_{N\ with\ \varepsilon_i(N)=\varepsilon} \binom{\varepsilon + \varphi_i(N)}{\varepsilon} a_N \omega([P_{\tilde{e}_i^\varepsilon N}]) + X = 0$$

where X is a sum of terms of the form $\omega([P_{\tilde{e}_i^\varepsilon L}])$ with $\varepsilon_i(L) > \varepsilon$. Now the inductive hypothesis shows that $X = 0$ and that $a_N = 0$ for each N with $\varepsilon_i(N) = \varepsilon$. In particular, $a_M = 0$, a contradiction. □

In view of Theorem 7.10, we may identify $K(\lambda)^*$ with its image under ω, so $K(\lambda)^* \subseteq K(\lambda)$ are two different lattices in $K(\lambda)_{\mathbb{Q}}$: on tensoring with $\mathbb{Q}$ they become equal. Extending scalars, the pairing (7.15) induces a bilinear form

$$(.,.) : K(\lambda)_{\mathbb{Q}} \times K(\lambda)_{\mathbb{Q}} \to \mathbb{Q} \tag{7.19}$$

with respect to which the operators e_i and f_i are adjoint.

Theorem 7.11. *The form $(.,.) : K(\lambda)_{\mathbb{Q}} \times K(\lambda)_{\mathbb{Q}} \to \mathbb{Q}$ is symmetric and non-degenerate.*

Proof. It is non-degenerate by construction, being induced by the pairing $(.,.)$ from (7.15). So we just need to check that it is symmetric. Proceed by induction on n to show that $(.,.) : K(\mathrm{Rep}\,\mathcal{H}_n^\lambda)_{\mathbb{Q}} \times K(\mathrm{Rep}\,\mathcal{H}_n^\lambda)_{\mathbb{Q}} \to \mathbb{Q}$ is symmetric. In view of Theorem 7.9, any element of $K(\operatorname{Rep}\mathcal{H}_n^\lambda)_\mathbb{Q}$ can be written as $f_i x$ for $x \in K(\operatorname{Rep}\mathcal{H}_{n-1}^\lambda)_\mathbb{Q}$. Then for any other $y \in K(\operatorname{Rep}\mathcal{H}_n^\lambda)_\mathbb{Q}$, we have that

$$(f_i x, y) = (x, e_i y) = (e_i y, x) = (y, f_i x)$$

by the induction hypothesis. □

§7-d. **Chevalley relations.** Continue working with a fixed $\lambda \in P_+$. We turn now to considering the relations satisfied by the operators e_i, f_i on $K(\lambda)$.

Lemma 7.12. *The operators $e_i, f_i : K(\lambda) \to K(\lambda)$ satisfy the Serre relations (4.4).*

Proof. We know the e_i satisfy the Serre relations on all of $K(\infty)$ by Lemma 7.3, so they certainly satisfy the Serre relations on restriction to $K(\lambda)$. Moreover, e_i and f_i are adjoint operators for the bilinear form $(.,.)$ according to Lemma 7.6, and this form is non-degenerate by Theorem 7.11. The lemma follows. □

Now we consider relations between the e_i and f_j. For $i \in I$ and an irreducible $\mathcal{H}_n^\lambda$-module M with central character χ_γ for $\gamma \in \Gamma_n$, define

$$h_i[M] = \langle h_i, \lambda - \gamma\rangle[M]. \tag{7.20}$$

Recall according to Lemma 6.25 that we have equivalently that

$$h_i[M] = (\varphi_i(M) - \varepsilon_i(M))[M]. \tag{7.21}$$

More generally, define

$$\binom{h_i}{r} : K(\lambda) \to K(\lambda), \qquad [M] \mapsto \binom{\varphi_i(M) - \varepsilon_i(M)}{r}[M] \tag{7.22}$$

where $\binom{m}{r}$ denotes $m(m-1)\dots(m-r+1)/(r!)$. Extending linearly, each $\binom{h_i}{r}$ can be viewed as a diagonal linear operator $K(\lambda) \to K(\lambda)$. The definition (7.20) implies immediately that:

Lemma 7.13. *As operators on $K(\lambda)$, $[h_i, e_j] = \langle h_i, \alpha_j\rangle e_j$ and $[h_i, f_j] = -\langle h_i, \alpha_j\rangle f_j$ for all $i, j \in I$.*

For the next lemma, we follow [G_1, Proposition 12.5].

Lemma 7.14. *As operators on $K(\lambda)$, the relation $[e_i, f_j] = \delta_{i,j} h_i$ holds for each $i, j \in I$.*

Proof. Let M be an irreducible $\mathcal{H}_n^\lambda$-module. It follows immediately from Theorems 6.11(i) and 6.26(i) (together with central character considerations in case $i \neq j$) that $[M]$ appears in $e_i f_j[M] - f_j e_i[M]$ with multiplicity $\delta_{i,j}(\varphi_i(M) - \varepsilon_i(M))$. Therefore, it suffices simply to show that $e_i f_j[M] - f_j e_i[M]$ is a multiple of $[M]$. Let us show equivalently that

$$[\mathrm{res}_i \mathrm{ind}_j M - \mathrm{ind}_j \mathrm{res}_i M]$$

is a multiple of $[M]$.

For $m \gg 0$, we have a surjection $\mathrm{ind}_{n,1}^{n+1} M \boxtimes \mathcal{R}_m(j) \twoheadrightarrow \mathrm{ind}_j M$. Apply $\mathrm{pr}^\lambda \circ \mathrm{res}_i$ to get a surjection

$$\mathrm{pr}^\lambda \mathrm{res}_i \mathrm{ind}_{n,1}^{n+1} M \boxtimes \mathcal{R}_m(j) \twoheadrightarrow \mathrm{res}_i \mathrm{ind}_j M. \tag{7.23}$$

By the Mackey Theorem and (4.29), there is an exact sequence

$$0 \to (M \oplus \Pi M)^{\oplus \delta_{i,j}.mc_j} \longrightarrow \mathrm{res}_i \mathrm{ind}_{n,1}^{n+1} M \boxtimes \mathcal{R}_m(j) \longrightarrow \mathrm{ind}_{n-1,1}^n (\mathrm{res}_i M) \boxtimes \mathcal{R}_m(j) \to 0,$$

where c_j is as in Lemma 6.21. For sufficiently large m, $\mathrm{pr}^\lambda \mathrm{ind}^n_{n-1,1}(\mathrm{res}_i M) \boxtimes \mathcal{R}_m(j) = \mathrm{ind}_j \mathrm{res}_i M$. So on applying the right exact functor pr^λ and using the irreducibility of M, this implies that there is an exact sequence

$$0 \longrightarrow M^{\oplus m_1} \oplus \Pi M^{\oplus m_2} \longrightarrow \mathrm{pr}^\lambda \mathrm{res}_i \mathrm{ind}^{n+1}_{n,1} M \boxtimes \mathcal{R}_m(j) \longrightarrow \mathrm{ind}_j \mathrm{res}_i M \longrightarrow 0, \qquad (7.24)$$

for some m_1, m_2. Now let N be any irreducible $\mathcal{H}^\lambda_n$-module with $N \not\cong M$. Combining (7.23) and (7.24) shows that

$$[\mathrm{ind}_j \mathrm{res}_i M - \mathrm{res}_i \mathrm{ind}_j M : N] \geq 0 \qquad (7.25)$$

Now summing over all i, j and using (6.6) gives that $[\mathrm{ind}\,\mathrm{res}\, M - \mathrm{res}\,\mathrm{ind}\, M : N] \geq 0$. But Theorem 3.9 shows that equality holds here, hence it must hold in (7.25) for all $i, j \in I$. This completes the proof. $\square$

To summarize, we have shown in (7.6), Lemmas 7.12, 7.13 and 7.14 that:

Theorem 7.15. *The action of the operators e_i, f_i, h_i on $K(\lambda)$ satisfy the Chevalley relations (4.2), (4.3) and (4.4). Hence, the actions of $e_i^{(r)}, f_i^{(r)}$ and $\binom{h_i}{r}$ for all $i \in I, r \geq 1$ make $K(\lambda)_{\mathbb{Q}}$ into a $U_{\mathbb{Q}}$-module so that $K(\lambda)^*, K(\lambda)$ are $U_{\mathbb{Z}}$-submodules.*

§7-e. **Identification of $K(\infty)^*$, $K(\lambda)^*$ and $K(\lambda)$.** Now we can prove Theorems A and B stated in the introduction. Compare [A$_1$, 4.3, 4.4] and [G$_1$, 14.1, 14.2].

Theorem 7.16. *For any $\lambda \in P_+$,*

(i) *$K(\lambda)_{\mathbb{Q}}$ is precisely the integrable highest weight $U_{\mathbb{Q}}$-module of highest weight λ, with highest weight vector $[\mathbf{1}_\lambda]$;*
(ii) *the bilinear form $(.,.)$ from (7.19) on the highest weight module $K(\lambda)_{\mathbb{Q}}$ coincides with the usual Shapovalov form satisfying $([\mathbf{1}_\lambda], [\mathbf{1}_\lambda]) = 1$;*
(iii) *$K(\lambda)^* \subset K(\lambda)$ are integral forms of $K(\lambda)_{\mathbb{Q}}$ containing $[\mathbf{1}_\lambda]$, with $K(\lambda)^*$ being the minimal lattice $U^-_{\mathbb{Z}}[\mathbf{1}_\lambda]$ and $K(\lambda)$ being its dual under the Shapovalov form.*

Proof. It makes sense to think of $K(\lambda)_{\mathbb{Q}}$ as a $U_{\mathbb{Q}}$-module according to Theorem 7.15. The actions of e_i and f_i are locally nilpotent by Theorems 6.11(i) and 6.26(i). The action of h_i is diagonal by definition. Hence, $K(\lambda)_{\mathbb{Q}}$ is an integrable module. Clearly $[\mathbf{1}_\lambda]$ is a highest weight vector of highest weight λ. Moreover, $K(\lambda)_{\mathbb{Q}} = U^-_{\mathbb{Q}}[\mathbf{1}_\lambda]$ by Theorem 7.9. This completes the proof of (i), and (ii) follows immediately from Lemma 7.6. For (iii), we know already that $K(\lambda)^* \subset K(\lambda)$ are dual lattices of $K(\lambda)_{\mathbb{Q}}$ which are invariant under $U_{\mathbb{Z}}$. Moreover, Theorem 7.9 again shows $K(\lambda)^* = U^-_{\mathbb{Z}}[\mathbf{1}_\lambda]$. $\square$

Theorem 7.17. *The map $\pi : U^+_{\mathbb{Z}} \to K(\infty)^*$ constructed in Lemma 7.5 is an isomorphism.*

Proof. Note by Lemma 7.4 that the action of the $e_i^{(r)} \in U^+_{\mathbb{Z}}$ on $K(\infty)$, hence on each $K(\lambda)$, factors through the map π. So if $x \in \ker \pi$, we have by the previous theorem that x acts as zero on all integrable highest weight modules $K(\lambda), \lambda \in P_+$. Hence $x = 0$ and π is injective. To prove surjectivity, take $x \in K(\infty)$. It suffices to show that $(\pi(u), x) = 0$ for all $u \in U^+_{\mathbb{Z}}$ implies that $x = 0$. Note

$$(\pi(u), x) = (\pi(1)\pi(u), x) = (\pi(1), \pi(u)x) = (\pi(1), ux)$$

where the second equality follows because the right regular action of $K(\infty)^*$ on itself is precisely the dual action to the left action of $K(\infty)^*$ on $K(\infty)$, and the third equality follows from Lemma 7.4. Hence, if $(\pi(u), x) = 0$ for all $u \in U^+_{\mathbb{Z}}$, we have that $(\pi(1), ux) = 0$ for all $u \in U^+_{\mathbb{Z}}$. Now choose $\lambda \in P_+$ sufficiently large so that in fact $x \in K(\lambda) \subset K(\infty)$.

Then, it follows that $([\mathbf{1}_{\boldsymbol{\lambda}}], ux) = 0$ for all $u \in U_{\mathbb{Z}}^+$, where $(.,.)$ now is canonical pairing between $K(\lambda)^*$ and $K(\lambda)$. Hence by Lemma 7.6, $(v[\mathbf{1}_{\boldsymbol{\lambda}}], x) = 0$ for all $v \in U_{\mathbb{Z}}^-$. But then Theorem 7.9 implies that $x = 0$. $\square$

8. Identification of the crystal

§8-a. **Final properties of $B(\infty)$.** Now we follow the ideas of [G₁, §13].

Lemma 8.1. *Let $M \in \mathrm{Rep}_I\,\mathcal{H}_m$ be irreducible.*

(i) *For any $i \in I$, either $\varepsilon_i(\tilde{f}_i^* M) = \varepsilon_i(M)$ or $\varepsilon_i(M) + 1$.*
(ii) *For any $i, j \in I$ with $i \neq j$, $\varepsilon_i(\tilde{f}_j^* M) = \varepsilon_i(M)$.*

Proof. We prove (i), the proof of (ii) being similar. By the Shuffle Lemma, we certainly have that $\varepsilon_i(\tilde{f}_i^* M) \leq \varepsilon_i(M) + 1$. Now let $N = \tilde{f}_i^* M$. Then obviously, $\varepsilon_i(\tilde{e}_i^* N) \leq \varepsilon_i(N)$. Hence, $\varepsilon_i(M) \leq \varepsilon_i(\tilde{f}_i^* M)$. $\square$

Lemma 8.2. *Let $M \in \mathrm{Rep}_I\,\mathcal{H}_m$ be irreducible and $i, j \in I$. If $\varepsilon_i(\tilde{f}_j^* M) = \varepsilon_i(M)$ then, writing $\varepsilon := \varepsilon_i(M)$, we have $\tilde{e}_i^\varepsilon \tilde{f}_j^* M \cong \tilde{f}_j^* \tilde{e}_i^\varepsilon M$.*

Proof. Set $n = m - \varepsilon$. Let $N = \tilde{e}_i^\varepsilon M$, so N is an irreducible $\mathcal{H}_n$-module with $\varepsilon_i(N) = 0$ and $M = \tilde{f}_i^\varepsilon N$. For $0 \leq b \leq \varepsilon$, let $Q_b = \mathrm{res}_i^{\varepsilon - b} \tilde{f}_j^* M$. Theorem 6.11(i) and Lemma 6.3 imply that in the Grothendieck group, Q_b is some number of copies of $\tilde{e}_i^{\varepsilon - b} \tilde{f}_j^* M$ plus terms with strictly smaller ε_i. In particular, $\varepsilon_i(L) \leq b$ for all composition factors L of Q_b, while Q_0 is consists only of copies of $\tilde{e}_i^\varepsilon \tilde{f}_j^* M$.

We will show by decreasing induction on $b = \varepsilon, \varepsilon - 1, \dots, 0$ that there is a non-zero $\mathcal{H}_{n+b+1}$-module homorphism

$$\gamma_b : \mathrm{ind}_{1,n,b}^{n+b+1} L(j) \boxtimes N \boxtimes L(i^b) \to Q_b.$$

In case $b = \varepsilon$, $Q_\varepsilon = \tilde{f}_j^* M$ is a quotient of $\mathrm{ind}_{1,m}^{m+1} L(j) \boxtimes M$ and M is a quotient of $\mathrm{ind}_{n,\varepsilon}^m N \boxtimes L(i^\varepsilon)$, so the induction starts. Now we suppose by induction that we have proved $\gamma_b \neq 0$ exists for $b \geq 1$ and construct γ_{b-1}.

Consider $\mathrm{res}_{n+b,1}^{n+b+1} \mathrm{ind}_{1,n,b}^{n+b+1} L(j) \boxtimes N \boxtimes L(i^b)$. By the Mackey Theorem, this has a filtration $0 \subset F_1 \subset F_2 \subset F_3$ with successive quotients

$$\begin{aligned} F_1 &\simeq \mathrm{ind}_{1,n,b-1,1}^{n+b,1} \mathrm{res}_{1,n,b-1,1}^{1,n,b} L(j) \boxtimes N \boxtimes L(i^b), \\ F_2/F_1 &\simeq \mathrm{ind}_{1,n-1,b,1}^{n+b,1}\, {}^w\mathrm{res}_{1,n-1,1,b}^{1,n,b} L(j) \boxtimes N \boxtimes L(i^b), \\ F_3/F_2 &\simeq \mathrm{ind}_{n,b,1}^{n+b,1} N \boxtimes L(i^b) \boxtimes L(j), \end{aligned}$$

where w is the obvious permutation. As $\gamma_b \neq 0$, Frobenius reciprocity implies that there is a copy of the $\mathcal{H}_{1,n,b}$-module $L(j) \circledast N \circledast L(i^b)$ in the image of γ_b. Now $b > 0$, so the $q(i)$-eigenspace of $X_{n+b+1} + X_{n+b+1}^{-1}$ acting on $L(j) \circledast N \circledast L(i^b)$ is non-trivial. We conclude that the map

$$\tilde{\gamma}_b = \mathrm{res}_i(\gamma_b) : \mathrm{res}_i \mathrm{ind}_{1,n,b}^{n+b+1} L(j) \boxtimes N \boxtimes L(i^b) \to \mathrm{res}_i Q_b = Q_{b-1}$$

is non-zero.

If $i \neq j$, then it follows from the description of F_3/F_2 and F_2/F_1 above that $\mathrm{res}_i(F_3/F_1) = 0$. So in this case, we necessarily have that $\tilde{\gamma}_b(F_1) \neq 0$. Similarly if $i = j$, $\mathrm{res}_i(F_2/F_1) = 0$, so if $\tilde{\gamma}_b(F_1) = 0$ we see that $\tilde{\gamma}_b$ factors to a non-zero homomorphism

$$\mathrm{res}^{n+b,1}_{n+b}\mathrm{ind}^{n+b,1}_{n,b,1} N \boxtimes L(i^b) \boxtimes L(i) \to Q_{b-1}.$$

But this implies that Q_{b-1} has a constituent L with $\varepsilon_i(L) = b$, which we know is not the case. Hence we have that $\tilde{\gamma}_b(F_1) \neq 0$ in the case $i = j$ too.

Hence, the restriction of $\tilde{\gamma}_b$ to F_1 gives us a non-zero homomorphism

$$\mathrm{res}^{n+b,1}_{n+b}\mathrm{ind}^{b+n,1}_{1,n,b-1,1}\mathrm{res}^{1,n,b}_{1,n,b-1,1} L(j) \boxtimes N \boxtimes L(i^b) \to Q_{b-1}.$$

Now finally as all composition factors of $\mathrm{res}^b_{b-1} L(i^b)$ are isomorphic to $L(i^{b-1})$, this implies the existence of a non-zero homomorphism $\gamma_{b-1} : \mathrm{ind}^{b+n}_{1,n,b-1} L(j) \boxtimes N \boxtimes L(i^{b-1}) \to Q_{b-1}$ completing the induction.

Now taking $b = 0$ we have a non-zero map $\gamma_0 : \mathrm{ind}^{n+1}_{1,n} L(j) \boxtimes N \to Q_0$. But the left hand side has irreducible cosocle $\tilde{f}^*_j N = \tilde{f}^*_j \tilde{e}^\varepsilon_i M$ while all composition factors of the right hand side are isomorphic to $\tilde{e}^\varepsilon_i \tilde{f}^*_j M$. This completes the proof. $\square$

Corollary 8.3. *Let $M \in \mathrm{Rep}_I(\mathcal{H}_m)$ be irreducible and $i \in I$. Let $M_1 = \tilde{e}_i^{\varepsilon_i(M)} M$ and $M_2 = (\tilde{e}^*_i)^{\varepsilon^*_i(M)} M$. Then, $\varepsilon^*_i(M) = \varepsilon^*_i(M_1)$ if and only if $\varepsilon_i(M) = \varepsilon_i(M_2)$.*

Proof. Since we can apply the automorphism σ, it suffices to check just one of the implications. So suppose $\varepsilon_i(M) = \varepsilon_i(M_2)$. Clearly, $\varepsilon^*_i(M) \geq \varepsilon^*_i(M_1)$. For the reverse inequality, the preceeding two lemmas show that

$$M_1 \cong \tilde{e}_i^{\varepsilon_i(M)} M \cong \tilde{e}_i^{\varepsilon_i(M)} (\tilde{f}^*_i)^{\varepsilon^*_i(M)} M_2 \cong (\tilde{f}^*_i)^{\varepsilon^*_i(M)} \tilde{e}_i^{\varepsilon_i(M)} M_2.$$

This shows that $\varepsilon^*_i(M_1) \geq \varepsilon^*_i(M)$. $\square$

Lemma 8.4. *Let $M \in \mathrm{Rep}_I \mathcal{H}_m$ be irreducible and $i \in I$ satisfy $\varepsilon_i(\tilde{f}^*_i M) = \varepsilon_i(M) + 1$. Then $\tilde{e}_i \tilde{f}^*_i M = M$.*

Proof. Set $\varepsilon := \varepsilon_i(M)$ and $N = \tilde{e}^\varepsilon_i M$. By the Shuffle Lemma and Theorem 5.12,

$$[\mathrm{ind}^{m+1}_{1,m-\varepsilon,\varepsilon} L(i) \circledast N \circledast L(i^\varepsilon)] = [\mathrm{ind}^{m+1}_{m-\varepsilon,\varepsilon+1} N \circledast L(i^{\varepsilon+1})].$$

Hence by Theorem 6.11(i), $[\mathrm{ind}^{m+1}_{1,m-\varepsilon,\varepsilon} L(i) \circledast N \circledast L(i^\varepsilon)]$ equals $[\tilde{f}^{\varepsilon+1}_i N] = [\tilde{f}_i M]$ plus terms $[L]$ for irreducible L with $\varepsilon_i(L) \leq \varepsilon$. On the other hand, $\mathrm{ind}^{m+1}_{1,m-\varepsilon,\varepsilon} L(i) \circledast N \circledast L(i^\varepsilon)$ surjects onto $\tilde{f}^*_i M$. So the assumption $\varepsilon_i(\tilde{f}^*_i M) = \varepsilon + 1$ implies $\tilde{f}_i M \cong \tilde{f}^*_i M$. $\square$

§8-b. **Crystals.** Let us now recall some definitions from [Ka]. A *crystal* is a set B endowed with maps

$$\varphi_i, \varepsilon_i : B \to \mathbb{Z} \cup \{-\infty\} \quad (i \in I),$$
$$\tilde{e}_i, \tilde{f}_i : B \to B \cup \{0\} \quad (i \in I),$$
$$\mathrm{wt} : B \to P$$

such that

(C1) $\varphi_i(b) = \varepsilon_i(b) + \langle h_i, \mathrm{wt}(b)\rangle$ for any $i \in I$;
(C2) if $b \in B$ satisfies $\tilde{e}_i b \neq 0$, then $\varepsilon_i(\tilde{e}_i b) = \varepsilon_i(b) - 1$, $\varphi_i(\tilde{e}_i b) = \varphi_i(b) + 1$ and $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) + \alpha_i$;
(C3) if $b \in B$ satisfies $\tilde{f}_i b \neq 0$, then $\varepsilon_i(\tilde{f}_i b) = \varepsilon_i(b) + 1$, $\varphi_i(\tilde{f}_i b) = \varphi_i(b) - 1$ and $\mathrm{wt}(\tilde{e}_i b) = \mathrm{wt}(b) - \alpha_i$;
(C4) for $b_1, b_2 \in B$, $b_2 = \tilde{f}_i b_1$ if and only if $b_1 = \tilde{e}_i b_2$;

(C5) if $\varphi_i(b) = -\infty$, then $\tilde{e}_i b = \tilde{f}_i b = 0$.

For example, for each $i \in I$, we have the crystal B_i defined as a set to be $\{b_i(n) \mid n \in \mathbb{Z}\}$ with

$$\varepsilon_j(b_i(n)) = \begin{cases} -n & \text{if } j = i, \\ -\infty & \text{if } j \neq i; \end{cases} \qquad \varphi_j(b_i(n)) = \begin{cases} n & \text{if } j = i, \\ -\infty & \text{if } j \neq i; \end{cases}$$

$$\tilde{e}_j(b_i(n)) = \begin{cases} b_i(n+1) & \text{if } j = i, \\ 0 & \text{if } j \neq i; \end{cases} \qquad \tilde{f}_j(b_i(n)) = \begin{cases} b_i(n-1) & \text{if } j = i, \\ 0 & \text{if } j \neq i; \end{cases}$$

and $\mathrm{wt}(b_i(n)) = n\alpha_i$. We abbreviate $b_i(0)$ by b_i. Also for $\lambda \in P$, we have the crystal T_λ equal as a set to $\{t_\lambda\}$, with $\varepsilon_i(t_\lambda) = \varphi_i(t_\lambda) = -\infty$, $\tilde{e}_i t_\lambda = \tilde{f}_i t_\lambda = 0$ and $\mathrm{wt}(t_\lambda) = \lambda$.

A morphism $\psi : B \to B'$ of crystals is a map $\psi : B \cup \{0\} \to B' \cup \{0\}$ such that

(H1) $\psi(0) = 0$;

(H2) if $\psi(b) \neq 0$ for $b \in B$, then $\mathrm{wt}(\psi(b)) = \mathrm{wt}(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$ and $\varphi_i(\psi(b)) = \psi(b)$;

(H3) for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{e}_i b) \neq 0$, we have that $\psi(\tilde{e}_i b) = \tilde{e}_i \psi(b)$;

(H4) for $b \in B$ such that $\psi(b) \neq 0$ and $\psi(\tilde{f}_i b) \neq 0$, we have that $\psi(\tilde{f}_i b) = \tilde{f}_i \psi(b)$.

A morphism of crystals is called *strict* if ψ commutes with the $\tilde{e}_i$'s and $\tilde{f}_i$'s, and an *embedding* if ψ is injective.

We also need the notion of a tensor product of two crystals B, B'. As a set, $B \otimes B'$ is equal to $\{b \otimes b' \mid b \in B, b' \in B'\}$. This is made into a crystal by

$$\varepsilon_i(b \otimes b') = \max(\varepsilon_i(b), \varepsilon_i(b') - \langle h_i, \mathrm{wt}(b) \rangle), \quad \varphi_i(b \otimes b') = \max(\varphi_i(b) + \langle h_i, \mathrm{wt}(b') \rangle, \varphi_i(b')),$$

$$\tilde{e}_i(b \otimes b') = \begin{cases} \tilde{e}_i b \otimes b' & \text{if } \varphi_i(b) \geq \varepsilon_i(b') \\ b \otimes \tilde{e}_i b' & \text{if } \varphi_i(b) < \varepsilon_i(b') \end{cases}, \quad \tilde{f}_i(b \otimes b') = \begin{cases} \tilde{f}_i b \otimes b' & \text{if } \varphi_i(b) > \varepsilon_i(b') \\ b \otimes \tilde{f}_i b' & \text{if } \varphi_i(b) \leq \varepsilon_i(b') \end{cases},$$

$$\mathrm{wt}(b \otimes b') = \mathrm{wt}(b) + \mathrm{wt}(b').$$

Here, we understand $b \otimes 0 = 0 = 0 \otimes b$.

Having recalled these definitions, we now explain how to make our sets $B(\infty)$ and $B(\lambda)$ from (5.5) and (5.12) into crystals in the above sense. To do this, it just remains to define the weight functions on both $B(\infty)$ and $B(\lambda)$, as well as the function φ_i on $B(\infty)$: set

$$\mathrm{wt}(M) = -\gamma, \tag{8.1}$$

for an irreducible $M \in \mathrm{Rep}_\gamma \mathcal{H}_n$, and

$$\mathrm{wt}^\lambda(N) = \lambda - \gamma, \tag{8.2}$$

for an irreducible $N \in \mathrm{Rep}_\gamma \mathcal{H}_n^\lambda$. Also for an irreducible $[M] \in B(\infty)$, define

$$\varphi_i(M) = \varepsilon_i(M) + \langle h_i, \mathrm{wt}(M) \rangle, \tag{8.3}$$

so that property (C1) is automatic. Thus we have defined $(B(\infty), \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ and $(B(\lambda), \varepsilon_i, \varphi_i, \tilde{e}_i^\lambda, \tilde{f}_i^\lambda, \mathrm{wt}^\lambda)$ purely in terms of the representation theory of the Hecke-Clifford superalgebras.

Lemma 8.5. *$(B(\infty), \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ and each $(B(\lambda), \varepsilon_i, \varphi_i, \tilde{e}_i^\lambda, \tilde{f}_i^\lambda, \mathrm{wt}^\lambda)$ for $\lambda \in P_+$ are crystals in the sense of Kashiwara.*

Proof. Property (C1) is Lemma 6.25 or the definition in the affine case. Property (C4) is Lemma 5.10. The remaining properties are immediate. □

Recall the embedding $\mathrm{infl}^\lambda : B(\lambda) \cup \{0\} \to B(\infty) \cup \{0\}$ from (5.13).

Lemma 8.6. *The map $B(\lambda) \hookrightarrow B(\infty)\otimes T_\lambda$, $[N] \mapsto \mathrm{infl}^\lambda[N]\otimes t_\lambda$ is an embedding of crystals with image*

$$\left\{[M]\otimes t_\lambda \in B(\infty)\otimes T_\lambda \;\middle|\; \varepsilon_i^*(M) \le \langle h_i, \lambda\rangle \text{ for each } i \in I\right\}.$$

Proof. Since $\tilde{e}_i^\lambda$ and $\tilde{f}_i^\lambda$ are restrictions of $\tilde{e}_i, \tilde{f}_i$ from $B(\infty)$ to $B(\lambda)$, respectively, the first statement is immediate. The second is a restatement of Corollary 6.13. □

§8-c. Identification of $B(\infty)$ and $B(\lambda)$.

The first lemma follows directly from Lemmas 8.1 and 8.2, applying the automorphism σ to get (ii).

Lemma 8.7. *Let $M \in \mathrm{Rep}_I\,\mathcal{H}_m$ be irreducible and $i, j \in I$ with $i \ne j$. Set $a = \varepsilon_i^*(M)$.*

(i) $\varepsilon_j(M) = \varepsilon_j((\tilde{e}_i^*)^a M)$.

(ii) *If $\varepsilon_j(M) > 0$, then $\varepsilon_i^*(\tilde{e}_j M) = \varepsilon_i^*(M)$ and $(\tilde{e}_i^*)^a \tilde{e}_j M \cong \tilde{e}_j(\tilde{e}_i^*)^a M$.*

The proof of the next result is taken from [G$_1$, Proposition 10.2].

Lemma 8.8. *Let $M \in \mathrm{Rep}_I\,\mathcal{H}_m$ be irreducible and $i \in I$. Set $a = \varepsilon_i^*(M)$ and $\bar{M} = (\tilde{e}_i^*)^a M$.*

(i) $\varepsilon_i(M) = \max(\varepsilon_i(\bar{M}), a - \langle h_i, \mathrm{wt}(\bar{M})\rangle)$.

(ii) *If $\varepsilon_i(M) > 0$,*

$$\varepsilon_i^*(\tilde{e}_i M) = \begin{cases} a & \text{if } \varepsilon_i(\bar{M}) \ge a - \langle h_i, \mathrm{wt}(\bar{M})\rangle, \\ a-1 & \text{if } \varepsilon_i(\bar{M}) < a - \langle h_i, \mathrm{wt}(\bar{M})\rangle. \end{cases}$$

(iii) *If $\varepsilon_i(M) > 0$,*

$$(\tilde{e}_i^*)^b \tilde{e}_i M \cong \begin{cases} \tilde{e}_i(\bar{M}) & \text{if } \varepsilon_i(\bar{M}) \ge a - \langle h_i, \mathrm{wt}(\bar{M})\rangle, \\ \bar{M} & \text{if } \varepsilon_i(\bar{M}) < a - \langle h_i, \mathrm{wt}(\bar{M})\rangle, \end{cases}$$

where $b = \varepsilon_i^*(\tilde{e}_i M)$.

Proof. Let $\varepsilon = \varepsilon_i(M)$, $n = m - \varepsilon$, and $N = (\tilde{e}_i)^\varepsilon M$.

(i) By twisting with σ, it suffices to prove that (for arbitrary M)

$$\varepsilon_i^*(M) = \max(\varepsilon_i^*(N), \varepsilon - \langle h_i, \mathrm{wt}(N)\rangle). \tag{8.4}$$

Define the weights $\lambda(0), \lambda(1), \dots \in P_+$ by taking the Λ_i-coefficient of $\lambda(r)$ to be $\varepsilon_i^*(N) + r$ and the Λ_j-coefficients of $\lambda(r)$ for $j \ne i$ to be $\gg 0$. Then $\mathcal{I}_{\lambda(r)}N = 0$ for any $r \ge 0$, see Corollory 6.13. Moreover, the same corollary implies that for $k = \varphi_i^{\lambda(r)}(N)$,

$$\varepsilon_i^*(\tilde{f}_i^k N) = \langle h_i, \lambda(r)\rangle \quad\text{and}\quad \varepsilon_i^*(\tilde{f}_i^{k+1}N) = \langle h_i, \lambda(r)\rangle + 1.$$

Now, $\varphi_i^\lambda(N) - \varepsilon_i^\lambda(N) = \langle h_i, \lambda + \mathrm{wt}(N)\rangle$ and $\varepsilon_i(N) = \varepsilon_i^\lambda(N)$. Moreover, by Lemma 8.1(i) (twisted with σ) we have $\varepsilon_i^*(\tilde{f}^k N) \ge \varepsilon_i^*(N)$ for any k. All of these applied consecutively to $\lambda(0), \lambda(1), \dots$ imply:

$$\varepsilon_i^*(\tilde{f}_i^s N) = \begin{cases} \varepsilon_i^*(N) & \text{if } s \le \varepsilon_i^*(N) + \varepsilon_i(N) + \langle h_i, \mathrm{wt}(N)\rangle, \\ s - \varepsilon_i(N) - \langle h_i, \mathrm{wt}(N)\rangle & \text{if } s \ge \varepsilon_i^*(N) + \varepsilon_i(N) + \langle h_i, \mathrm{wt}(N)\rangle \end{cases}$$

for all $s \ge 0$. For $s = \varepsilon$, taking into account $\varepsilon_i(N) = 0$, this gives (8.4).

(ii) Observe by (8.4) that $\varepsilon_i^*(\tilde{e}_i M) = \varepsilon_i^*(M) - 1$ if and only if $\varepsilon > \langle h_i, \mathrm{wt}(N)\rangle + \varepsilon_i^*(N)$, and that otherwise $\varepsilon_i^*(\tilde{e}_i M) = \varepsilon_i^*(M)$. But $\langle h_i, \mathrm{wt}(N)\rangle = \langle h_i, \mathrm{wt}(M)\rangle + 2\varepsilon$ and $\langle h_i, \mathrm{wt}(\bar{M})\rangle = \langle h_i, \mathrm{wt}(M)\rangle + 2a$ so (ii) follows if we show that $\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i^*(N) + \varepsilon < 0$ if and only if $\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i(\bar{M}) + a < 0$. But by (i) and (8.4),

$$\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i^*(N) + \varepsilon = \max(\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i^*(N) + \varepsilon_i(\bar{M}), \varepsilon_i^*(N) - a),$$

$$\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i(\bar{M}) + a = \max(\langle h_i, \mathrm{wt}(M)\rangle + \varepsilon_i(\bar{M}) + \varepsilon_i^*(N), \varepsilon_i(\bar{M}) - \varepsilon).$$

Moreover, obviously $\varepsilon_i^*(N) - a \leq 0$ and $\varepsilon_i(\bar{M}) - \varepsilon \leq 0$, and it remains to observe that $\varepsilon_i^*(N) - a = 0$ if and only if $\varepsilon_i(\bar{M}) - \varepsilon = 0$, thanks to Corollary 8.3.

(iii) This follows from (ii) and Lemmas 8.2 and 8.4 (twisted with σ). □

Now for each $i \in I$, define a map

$$\Psi_i : B(\infty) \to B(\infty) \otimes B_i \tag{8.5}$$

mapping each $[M] \in B(\infty)$ to $[(\tilde{e}_i^*)^a M] \otimes \tilde{f}_i^a b_i$, where $a = \varepsilon_i^*(M)$.

Lemma 8.9. *The following properties hold:*

(i) *for every* $[M] \in B(\infty)$, $\mathrm{wt}(M)$ *is a negative sum of simple roots;*
(ii) $[\mathbf{1}]$ *is the unique element of* $B(\infty)$ *with weight* 0*;*
(iii) $\varepsilon_i(\mathbf{1}) = 0$ *for every* $i \in I$*;*
(iv) $\varepsilon_i(M) \in \mathbb{Z}$ *for every* $[M] \in B(\infty)$ *and every* $i \in I$*;*
(v) *for every* i*, the map* $\Psi_i : B(\infty) \to B(\infty) \otimes B_i$ *defined above is a strict embedding of crystals;*
(vi) $\Psi_i(B(\infty)) \subseteq B(\infty) \times \{\tilde{f}_i^n b_i \mid n \geq 0\}$*;*
(vii) *for any* $[M] \in B(\infty)$ *other than* $[\mathbf{1}]$*, there exists* $i \in I$ *such that* $\Psi_i([M]) = [N] \otimes \tilde{f}_i^n b_i$ *for some* $[N] \in B(\infty)$ *and* $n > 0$*.*

Proof. Properties (i)–(iv) are immediate from our construction of $B(\infty)$. The information required to verify (v) is exactly contained in Lemmas 8.7 and 8.8. Finally, (vi) is immediate from the definition of Ψ_i, and (vii) holds because every such M has $\varepsilon_i^*(M) > 0$ for at least one $i \in I$. □

The properties in Lemma 8.9 exactly characterize the crystal $B(\infty)$ by [KS, Proposition 3.2.3]. Hence, we have proved:

Theorem 8.10. *The crystal* $B(\infty)$ *is isomorphic to Kashiwara's crystal* $B(\infty)$ *associated to the crystal base of* $U_{\mathbb{Q}}^-$*.*

In view of [Ka, Theorem 8.2], we can also identify our maps Ψ_i with those of [Ka]. Taking into account [Ka, Proposition 8.1] we can then identify our functions ε_i^* on $B(\infty)$ with those in [Ka]. It follows from this, Lemma 8.6 and [Ka, Proposition 8.2] that:

Theorem 8.11. *For each* $\lambda \in P_+$*, the crystal* $B(\lambda)$ *is isomorphic to Kashiwara's crystal* $B(\lambda)$ *associated to the integrable highest weight* $U_{\mathbb{Q}}$*-module of highest weight* λ*.*

§8-d. **Blocks.** Let $\lambda \in P_+$. As an application of the theory, we mention here another delightful argument of Grojnowski from [G$_2$], which in our setting classifies the blocks of the cyclotomic Hecke-Clifford superalgebras $\mathcal{H}_n^\lambda$.

Let us recall the definition in the case of a finite dimensional superalgebra A. Let $\sim$ be the equivalence relation on the set of isomorphism classes of irreducible A-modules such that $[L] \sim [M]$ if and only if there exists a chain $L \cong L_0, L_1, \dots, L_m \cong M$ of irreducible A-modules with either $\mathrm{Ext}_A^1(L_i, L_{i+1}) \neq 0$ or $\mathrm{Ext}_A^1(L_{i+1}, L_i) \neq 0$ for each i. Given a $\sim$-equivalence class b, the corresponding *block* $\mathrm{Rep}_b\, A$ is the full subcategory of $\mathrm{Rep}\, A$ consisting of the A-modules all of whose composition factors belong to the equivalence class $\sim$. Thus, there is a decomposition

$$\mathrm{Rep}\, A = \bigoplus_b \mathrm{Rep}_b\, A$$

as b runs over all $\sim$-equivalence classes. As usual, there are various equivalent points of view: for instance, $[L] \sim [M]$ if and only if the even central characters $\chi_L, \chi_M : Z(A)_{\bar{0}} \to F$

arising from the actions on the irreducible modules L, M are equal. Alternatively, the blocks of A can be defined in terms of a decomposition of the identity 1_A into a sum of centrally primitive *even* idempotents.

Theorem 8.12. *Let M, N be irreducible $\mathcal{H}_n^\lambda$-modules with $M \not\cong N$. Let*

$$0 \longrightarrow M \longrightarrow X \longrightarrow N \longrightarrow 0$$

be an exact sequence of $\mathcal{H}_n$-modules. Then, $\mathrm{pr}^\lambda X = X$ and the sequence is also an exact sequence of $\mathcal{H}_n^\lambda$-modules.

Proof. We note that for irreducible modules M, N in $\mathrm{Rep}_I\, \mathcal{H}_n$ with $M \not\cong N$, we have that

$$\mathrm{Hom}_{\mathcal{H}_{n-1}}(\mathrm{res}^n_{n-1} M, \mathrm{res}^n_{n-1} N) = 0.$$

This follows immediately from Corollary 6.12, Lemma 6.3 and (6.6). Using this in place of [G_2, Lemma 2], the proof of the theorem is now completed by exactly the same argument as in [G_2]. □

Note the theorem can be reinterpreted as saying that

$$\mathrm{Ext}^1_{\mathcal{H}_n}(M, N) \simeq \mathrm{Ext}^1_{\mathcal{H}_n^\lambda}(M, N) \tag{8.6}$$

for irreducible $\mathcal{H}_n^\lambda$-modules M, N with $M \not\cong N$. This is certainly not the case if $M \cong N$! Recalling the definitions from §4-f, we immediately deduce the following corollary which determines the blocks of $\mathcal{H}_n^\lambda$:

Corollary 8.13. *The blocks of $\mathcal{H}_n^\lambda$ are precisely the subcategories $\mathrm{Rep}_\gamma\, \mathcal{H}_n^\lambda$ for $\gamma \in \Gamma_n$. Moreover, $\mathrm{Rep}_\gamma\, \mathcal{H}_n^\lambda$ is non-trivial if and only if the $(\lambda - \gamma)$-weight space of the highest weight module $K(\lambda)_\mathbb{Q}$ is non-zero.*

9. Branching rules

In this final section, we deduce some important consequences for modular representations of the double cover $\widehat{S}_n$ of the symmetric group. In particular we obtain the classification of the irreducible modules, describe the blocks of the group algebra $F\widehat{S}_n$ and prove analogues of the modular branching rules of [K_1, K_2, K_3, K_4, B, BK_1]. Actually the results on branching are somewhat weaker here: there is no representation theoretic interpretation at present to the notion of "normal node" introduced below.

§9-a. **Kang's description of the crystal graph.** Now we focus on the fundamental highest weight Λ_0. In this case, Kang [Kg] has given a convenient combinatorial description of the crystal $B(\Lambda_0)$ in terms of Young diagrams, which we now describe.

For any $n \geq 0$, let $\lambda = (\lambda_1, \lambda_2, \dots)$ be a partition of n, i.e. a non-increasing sequence of non-negative integers summing to n. Recall that $\ell \in \mathbb{Z}_{>0} \cup \{\infty\}$ and $h = 2\ell + 1$. We call λ an *h-strict partition* if h divides λ_r whenever $\lambda_r = \lambda_{r+1}$ for $r \geq 1$ (to interpret correctly in case $h = \infty$, we adopt the convention that ∞ divides 0 and nothing else). Let $\mathscr{P}_h(n)$ denote the set of all h-strict partitions of n, and $\mathscr{P}_h := \bigcup_{n \geq 0} \mathscr{P}_h(n)$. We say that $\lambda \in \mathscr{P}_h(n)$ is *restricted* if in addition

$$\begin{cases} \lambda_r - \lambda_{r+1} < h & \text{if } h | \lambda_r, \\ \lambda_r - \lambda_{r+1} \leq h & \text{if } h \nmid \lambda_r \end{cases}$$

for each $r \geq 1$. Let $\mathscr{RP}_h(n)$ denote the set of all restricted h-strict partitions of n, and $\mathscr{RP}_h := \bigcup_{n \geq 0} \mathscr{RP}_h(n)$.

Let $\lambda \in \mathscr{P}_h$ be an h-strict partition. We identify λ with its *Young diagram*

$$\lambda = \{(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0} \mid s \leq \lambda_r\}.$$

Elements $(r, s) \in \mathbb{Z}_{>0} \times \mathbb{Z}_{>0}$ are called *nodes*. We label the nodes of λ with *residues*, which are the elements of the set $I = \{0, 1, \dots, \ell\}$. The labelling depends only on the column and follows the repeating pattern

$$0, 1, \dots, \ell - 1, \ell, \ell - 1, \dots, 1, 0,$$

starting fom the first column and going to the right, see Example 9.1 below. The residue of the node A is denoted res A. Define the *residue content* of λ to be the tuple

$$\operatorname{cont}(\lambda) = (\gamma_i)_{i \in I} \tag{9.1}$$

where for each $i \in I$, γ_i is the number of nodes of residue i contained in the diagram λ.

Let $i \in I$ be some fixed residue. A node $A = (r, s) \in \lambda$ is called *i-removable* (for λ) if one of the following holds:

(R1) res $A = i$ and $\lambda_A := \lambda - \{A\}$ is again an h-strict partition;
(R2) the node $B = (r, s+1)$ immediately to the right of A belongs to λ, res $A =$ res $B = i$, and both $\lambda_B = \lambda - \{B\}$ and $\lambda_{A,B} := \lambda - \{A, B\}$ are h-strict partitions.

Similarly, a node $B = (r, s) \notin \lambda$ is called *i-addable* (for λ) if one of the following holds:

(A1) res $B = i$ and $\lambda^B := \lambda \cup \{B\}$ is again an h-strict partition;
(A2) the node $A = (r, s - 1)$ immediately to the left of B does not belong to λ, res $A =$ res $B = i$, and both $\lambda^A = \lambda \cup \{A\}$ and $\lambda^{A,B} := \lambda \cup \{A, B\}$ are h-strict partitions.

We note that (R2) and (A2) above are only possible in case $i = 0$.

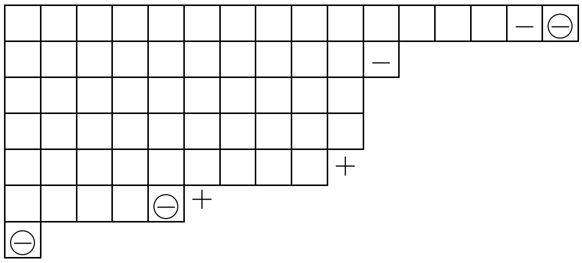
Now label all i-addable nodes of the diagram λ by $+$ and all i-removable nodes by $-$. Then, the *i-signature* of λ is the sequence of pluses and minuses obtained by going along the rim of the Young diagram from bottom left to top right and reading off all the signs. The *reduced i-signature* of λ is obtained from the i-signature by successively erasing all neighbouring pairs of the form $+-$.

Note the reduced i-signature always looks like a sequence of $-$'s followed by $+$'s. Nodes corresponding to a $-$ in the reduced i-signature are called *i-normal*, nodes corresponding to a $+$ are called *i-conormal*. The rightmost i-normal node (corresponding to the rightmost $-$ in the reduced i-signature) is called *i-good*, and the leftmost i-conormal node (corresponding to the leftmost $+$ in the reduced i-signature) is called *i-cogood*.

Example 9.1. Let $h = 5$, so $\ell = 2$. The partition $\lambda = (16, 11, 10, 10, 9, 5, 1)$ belongs to $\mathscr{RP}_h$, and its residues are as follows:

0	1	2	1	0	0	1	2	1	0	0	1	2	1	0	0
0	1	2	1	0	0	1	2	1	0	0					
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1	0						
0	1	2	1	0	0	1	2	1							
0	1	2	1	0											
0															

The 0-addable and 0-removable nodes are as labelled in the diagram:



Hence, the 0-signature of λ is $-,-,+,+,-,-,-$ and the reduced 0-signature is $-,-,-$. Note the nodes corresponding to the $-$'s in the reduced 0-signature have been circled in the above diagram. So, there are three 0-normal nodes, the rightmost of which is 0-good; there are no 0-conormal or 0-cogood nodes.

In general, we define

$$\varepsilon_i(\lambda) = \sharp\{i\text{-normal nodes in } \lambda\} = \sharp\{-\text{'s in the reduced } i\text{-signature of } \lambda\}, \tag{9.2}$$

$$\varphi_i(\lambda) = \sharp\{i\text{-conormal nodes in } \lambda\} = \sharp\{+\text{'s in the reduced } i\text{-signature of } \lambda\}. \tag{9.3}$$

Also set

$$\tilde{e}_i(\lambda) = \begin{cases} \lambda_A & \text{if } \varepsilon_i(\lambda) > 0 \text{ and } A \text{ is the (unique) } i\text{-good node,} \\ 0 & \text{if } \varepsilon_i(\lambda) = 0, \end{cases} \tag{9.4}$$

$$\tilde{f}_i(\lambda) = \begin{cases} \lambda^B & \text{if } \varphi_i(\lambda) > 0 \text{ and } B \text{ is the (unique) } i\text{-cogood node,} \\ 0 & \text{if } \varphi_i(\lambda) = 0. \end{cases} \tag{9.5}$$

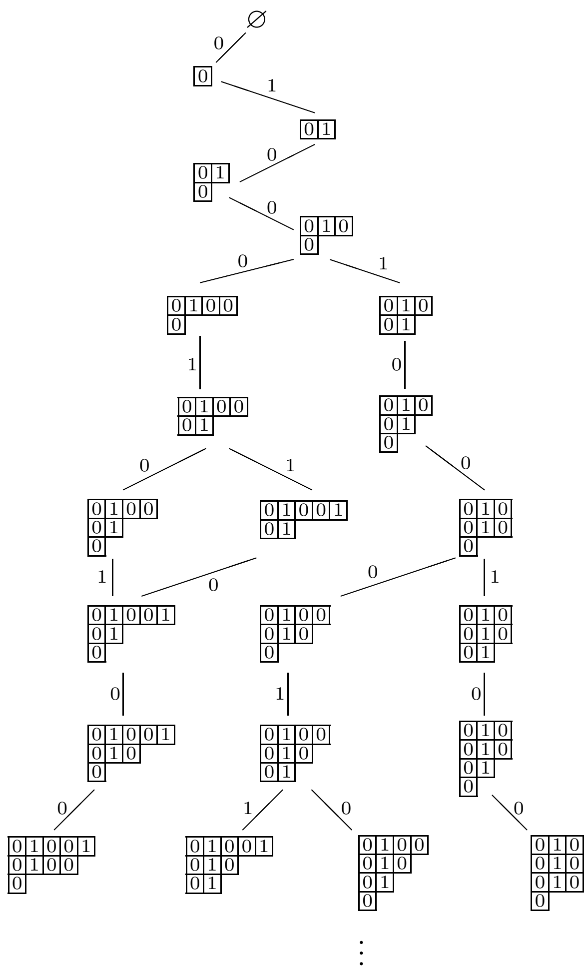
Finally define

$$\mathrm{wt}(\lambda) = \Lambda_0 - \sum_{i \in I} \gamma_i \alpha_i \tag{9.6}$$

where $\mathrm{cont}(\lambda) = (\gamma_i)_{i\in I}$. We have now defined a datum $(\mathscr{P}_h, \varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt})$ which makes the set $\mathscr{P}_h$ of all h-strict partitions into a crystal in the sense of §8-b. The definitions imply that $\tilde{e}_i(\lambda), \tilde{f}_i(\lambda)$ are restricted (or zero) in case λ is itself restricted. Hence, $\mathscr{R}\mathscr{P}_h$ is a sub-crystal of $\mathscr{P}_h$. We can now state the main result of Kang [Kg, 7.1] for type $A^{(2)}_{2\ell}$:

Theorem 9.2. *The set $\mathscr{R}\mathscr{P}_h$ equipped with $\varepsilon_i, \varphi_i, \tilde{e}_i, \tilde{f}_i, \mathrm{wt}$ as above is isomorphic (in the unique way) to the crystal $B(\Lambda_0)$ associated to the integrable highest weight $U_{\mathbb{Q}}$-module of fundamental highest weight Λ_0.*

Example 9.3. The crystal graph of $\mathscr{R}\mathscr{P}_h = B(\Lambda_0)$ in case $h = 3$, up to degree 10, is as follows:



Let us also mention here the extension of Morris' notion of h-bar core [Mo] to an arbitrary h-strict partition $\lambda \in \mathscr{P}_h(n)$. By an *$h$-bar* of λ, we mean one of the following:

(B1) the rightmost h nodes of row i of λ if $\lambda_i \geq h$ and either $h|\lambda_i$ or λ has no row of length $(\lambda_i - h)$;

(B2) the set of nodes in rows i and j of λ if $\lambda_i + \lambda_j = h$.

If λ has no h-bars, it is called an *h-bar core*. In general, the h-bar core $\tilde{\lambda}$ of λ is obtained by successively removing h-bars, reordering the rows each time so that the result still lies in $\mathscr{P}_h$, until it is reduced to a core. The *h-bar weight* of λ, denoted $w(\lambda)$, is then the total number of h-bars that get removed. For a Lie theoretic explanation of these notions, we refer the reader to [Kc, §12.6]. In particular, as observed in [LT$_2$, §4], for $\mu, \lambda \in \mathscr{P}_h(n)$ we have that

$$\mathrm{cont}(\mu) = \mathrm{cont}(\lambda) \text{ if and only if } \tilde{\mu} = \tilde{\lambda}. \tag{9.7}$$

Also, bearing in mind Theorem 9.2, we can state Kac' formula [Kc, (12.13.5)] for the character of the highest weight $U_{\mathbb{Q}}$-module of highest weight Λ_0 as follows: for $\lambda \in \mathscr{RP}_h(n)$,

$$\sharp\{\mu \in \mathscr{RP}_h(n) \mid \mathrm{cont}(\mu) = \mathrm{cont}(\lambda)\} = \mathrm{Par}_\ell(w(\lambda)), \tag{9.8}$$

where $\mathrm{Par}_\ell(N)$ denotes the number of partitions of N as a sum of positive integers of ℓ different colors.

§9-b. **Representations of finite Hecke-Clifford superalgebras.** Now that we have an explicit description of the crystal $B(\Lambda_0)$, we formulate a more combinatorial description of our main results for the representation theory of the finite Hecke-Clifford superalgebras $\mathcal{H}_n^{\mathrm{fin}}$. Recall from Remark 4.2 that this is precisely the cyclotomic Hecke-Clifford superalgebra $\mathcal{H}_n^{\Lambda_0}$. The results of this subsection also hold in the degenerate case, when $\mathcal{H}_n^{\mathrm{fin}}$ is the finite Sergeev superalgebra, see §2-k.

As explained in §5-d, the isomorphism classes of irreducible $\mathcal{H}_n^{\mathrm{fin}}$-modules are parametrized by the nodes of the crystal graph $B(\Lambda_0)$. By Theorems 9.2 and 8.11, we can identify $B(\Lambda_0)$ with $\mathscr{RP}_h$. In other words, we can use the set $\mathscr{RP}_h(n)$ of restricted h-strict partitions of n to parametrize the irreducible $\mathcal{H}_n^{\mathrm{fin}}$-modules for each $n \geq 0$. Let us write $M(\lambda)$ for the irreducible $\mathcal{H}_n^{\mathrm{fin}}$-module corresponding to $\lambda \in \mathscr{RP}_h(n)$. To be precise,

$$M(\lambda) := L(i_1, \dots, i_n)$$

if $\lambda = \tilde{f}_{i_n} \dots \tilde{f}_{i_1} \varnothing$. Here the operator $\tilde{f}_i$ is as defined in (9.5), corresponding under the identification $\mathscr{RP}_h(n) = B(\Lambda_0)$ to the crystal operator denoted $\tilde{f}_i^{\Lambda_0}$ in earlier sections, and $\varnothing$ denotes the empty partition, corresponding to $\mathbf{1}_{\boldsymbol{\lambda}} \in B(\Lambda_0)$.

For $\lambda \in \mathscr{RP}_h(n)$, we also define

$$b(\lambda) := \sharp\{r \geq 1 \mid h \nmid \lambda_r\}, \tag{9.9}$$

the number of parts of λ that are not divisible by h. The definition of residue content immediately gives that

$$b(\lambda) \equiv \gamma_0 \pmod 2, \tag{9.10}$$

where γ_0 denotes the number of 0's in the residue content of λ.

Theorem 9.4. *The modules $\{M(\lambda) \mid \lambda \in \mathscr{RP}_h(n)\}$ form a complete set of pairwise non-isomorphic irreducible $\mathcal{H}_n^{\mathrm{fin}}$-modules. Moreover, for $\lambda, \mu \in \mathscr{RP}_h(n)$,*

(i) *$M(\lambda) \cong M(\lambda)^\tau$;*
(ii) *$M(\lambda)$ is of type* `M` *if $b(\lambda)$ is even, type* `Q` *if $b(\lambda)$ is odd;*
(iii) *$M(\mu)$ and $M(\lambda)$ belong to the same block if and only if $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$;*
(iv) *$M(\lambda)$ is a projective module if and only if λ is an h-bar core.*

Proof. We have already discussed the first statement of the theorem, being a consequence of our main results combined with Kang's Theorem 9.2. For the rest, (i) follows from Corollary 5.13, (ii) is a special case of Lemma 5.14 combined with (9.10), and (iii) is a special case of Corollary 8.13. For (iv), note that if $M(\lambda)$ is projective then it is the only irreducible in its block, hence by (9.8), $\mathrm{Par}_\ell(w(\lambda)) = 1$. So either $w(\lambda) = 0$, or $\ell = 1$ and $w(\lambda) = 1$. Now if $w(\lambda) = 0$ then λ is an h-bar core so the Shapovalov form on the (1-dimensional) $\mathrm{wt}(\lambda)$-weight space of $K(\Lambda_0)_{\mathbb{Z}}$ is 1 (since $\mathrm{wt}(\lambda)$ is conjugate to Λ_0 under the action of the affine Weyl group). Hence, $M(\lambda)$ is projective by Theorem 7.16(ii). To rule out the remaining possibilty $\ell = 1$ and $w(\lambda) = 1$, one checks in that case that the Shapovalov form on the $\mathrm{wt}(\lambda)$-weight space of $K(\Lambda_0)_{\mathbb{Z}}$ is 3. $\square$

The next two theorems summarize earlier results concerning restriction and induction.

Theorem 9.5. *Let $\lambda \in \mathscr{RP}_h(n)$. There exist $\mathcal{H}_{n-1}^{\mathrm{fin}}$-modules $e_i M(\lambda)$ for each $i \in I$, unique up to isomorphism, such that*

(i) $\mathrm{res}^{\mathcal{H}^{\mathrm{fin}}_n}_{\mathcal{H}^{\mathrm{fin}}_{n-1}} M(\lambda) \cong \begin{cases} 2e_0M(\lambda) \oplus 2e_1M(\lambda) \oplus \cdots \oplus 2e_\ell M(\lambda) & \textit{if } b(\lambda) \textit{ is odd,} \\ e_0M(\lambda) \oplus 2e_1M(\lambda) \oplus \cdots \oplus 2e_\ell M(\lambda) & \textit{if } b(\lambda) \textit{ is even;} \end{cases}$

(ii) *for each $i \in I$, $e_iM(\lambda) \neq 0$ if and only if λ has an i-good node A, in which case $e_iM(\lambda)$ is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $M(\lambda_A)$.*

Moreover, if $i \in I$ and λ has an i-good node A, then

(iii) *the multiplicity of $M(\lambda_A)$ in $e_iM(\lambda)$ is $\varepsilon_i(\lambda)$, $\varepsilon_i(\lambda_A) = \varepsilon_i(\lambda)-1$, and $\varepsilon_i(\mu) < \varepsilon_i(\lambda)-1$ for all other composition factors $M(\mu)$ of $e_iM(\lambda)$;*

(iv) $\mathrm{End}_{\mathcal{H}^{\mathrm{fin}}_{n-1}}(e_iM(\lambda)) \simeq \mathrm{End}_{\mathcal{H}^{\mathrm{fin}}_{n-1}}(M(\lambda_A))^{\oplus \varepsilon_i(\lambda)}$ *as a vector superspace;*

(v) $\mathrm{Hom}_{\mathcal{H}^{\mathrm{fin}}_{n-1}}(e_iM(\lambda), e_iM(\mu)) = 0$ *for all $\mu \in \mathscr{RP}_h(n)$ with $\mu \neq \lambda$;*

(vi) *$e_iM(\lambda)$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$.*

Hence, $\mathrm{res}^{\mathcal{H}^{\mathrm{fin}}_n}_{\mathcal{H}^{\mathrm{fin}}_{n-1}} M(\lambda)$ *is completely reducible if and only if $\varepsilon_i(\lambda) \le 1$ for every $i \in I$.*

Proof. The existence of such modules $e_iM(\lambda)$ follows from (6.6), Lemma 6.3 and Theorem 6.6(i), combined as usual with Kang's Theorem 9.2. Uniqueness follows from Krull-Schmidt and the block classification from Theorem 9.4(iii). For the remaining properties, (iii),(iv) and (v) follow from Theorem 6.11 and Corollary 6.12. Finally, (vi) follows from (iii) as $e_iM(\lambda)$ is a module with simple socle and cosocle both isomorphic to $M(\lambda_A)$. □

Theorem 9.6. *Let $\lambda \in \mathscr{RP}_h(n)$. There exist $\mathcal{H}^{\mathrm{fin}}_{n+1}$-modules $f_iM(\lambda)$ for each $i \in I$, unique up to isomorphism, such that*

(i) $\mathrm{ind}^{\mathcal{H}^{\mathrm{fin}}_{n+1}}_{\mathcal{H}^{\mathrm{fin}}_n} M(\lambda) \cong \begin{cases} 2f_0M(\lambda) \oplus 2f_1M(\lambda) \oplus \cdots \oplus 2f_\ell M(\lambda) & \textit{if } b(\lambda) \textit{ is odd,} \\ f_0M(\lambda) \oplus 2f_1M(\lambda) \oplus \cdots \oplus 2f_\ell M(\lambda) & \textit{if } b(\lambda) \textit{ is even;} \end{cases}$

(ii) *for each $i \in I$, $f_iM(\lambda) \neq 0$ if and only if λ has an i-cogood node B, in which case $f_iM(\lambda)$ is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $M(\lambda^B)$.*

Moreover, if $i \in I$ and λ has an i-cogood node B, then

(iii) *the multiplicity of $M(\lambda^B)$ in $f_iM(\lambda)$ is $\varphi_i(\lambda)$, $\varphi_i(\lambda^B) = \varphi_i(\lambda) - 1$, and $\varphi_i(\mu) < \varphi_i(\lambda) - 1$ for all other composition factors $M(\mu)$ of $f_iM(\lambda)$;*

(iv) $\mathrm{End}_{\mathcal{H}^{\mathrm{fin}}_{n+1}}(f_iM(\lambda)) \simeq \mathrm{End}_{\mathcal{H}^{\mathrm{fin}}_{n+1}}(M(\lambda^B))^{\oplus \varphi_i(\lambda)}$ *as a vector superspace;*

(v) $\mathrm{Hom}_{\mathcal{H}^{\mathrm{fin}}_{n+1}}(f_iM(\lambda), f_iM(\mu)) = 0$ *for all $\mu \in \mathscr{RP}_h(n)$ with $\mu \neq \lambda$;*

(vi) *$f_iM(\lambda)$ is irreducible if and only if $\varphi_i(\lambda) = 1$.*

Hence, $\mathrm{ind}^{\mathcal{H}^{\mathrm{fin}}_{n+1}}_{\mathcal{H}^{\mathrm{fin}}_n} M(\lambda)$ *is completely reducible if and only if $\varphi_i(\lambda) \le 1$ for every $i \in I$.*

Proof. The argument is the same as Theorem 9.5, but using (6.6), Lemma 6.4, Theorem 6.6(ii), Corollary 6.19 and Theorem 6.26. □

There is one $\mathcal{H}^{\mathrm{fin}}_n$-module that deserves special mention, the so-called *basic spin module.* Recall from §2-d that the subalgebra $\mathcal{H}^{\mathrm{cl}}_n$ of $\mathcal{H}^{\mathrm{fin}}_n$ generated by $T_1, \dots, T_{n-1}$ is the classical Hecke algebra associated to the symmetric group. It has a one dimensional module denoted $\mathbf{1}$, on which each T_i acts as multiplication by q. For $n \ge 1$, we define

$$I(n) := \mathrm{ind}^{\mathcal{H}^{\mathrm{fin}}_n}_{\mathcal{H}^{\mathrm{cl}}_n} \mathbf{1}, \tag{9.11}$$

giving an $\mathcal{H}^{\mathrm{fin}}_n$-module of dimension 2^n. Also introduce the restricted h-strict partition

$$\omega_n := \begin{cases} (h^a, b) & \text{if } b \neq 0, \\ (h^{a-1}, h-1, 1) & \text{if } b = 0, \end{cases} \tag{9.12}$$

where $n = ah + b$ with $0 \le b < h$.

Lemma 9.7. *If $p \nmid n$ then $I(n) \cong M(\omega_n)$; if $p|n$ then $I(n)$ is an indecomposable module with two composition factors both isomorphic to $M(\omega_n)$. In particular,*

$$\dim M(\omega_n) = \begin{cases} 2^n & \text{if } p \nmid n, \\ 2^{n-1} & \text{if } p \mid n. \end{cases}$$

Proof. This is obvious if $n = 1, 2$ and easy to check directly if $n = 3$. Now for $n > 3$ we proceed by induction using Theorem 9.5 together with the observation that $\mathrm{res}^{\mathcal{H}_n^{\mathrm{fin}}}_{\mathcal{H}_{n-1}^{\mathrm{fin}}} I(n) \simeq I(n-1) \oplus \Pi I(n-1)$. We consider the four cases $n \equiv 0, 1$ or $2 \pmod h$ and $n \not\equiv 0, 1, 2 \pmod h$ separately.

Suppose first that $n \not\equiv 0, 1, 2 \pmod h$. Considering the crystal graph shows that $\tilde{f}_i \omega_{n-1} \neq 0$ only for $i = 0$ and for one other $i \in I$, for which $\tilde{f}_i \omega_{n-1} = \omega_n$. By the induction hypothesis, $\mathrm{res}^{\mathcal{H}_n^{\mathrm{fin}}}_{\mathcal{H}_{n-1}^{\mathrm{fin}}} I(n) \cong 2M(\omega_{n-1})$. Hence by Theorem 9.5, $I(n)$ can only contain $M(\omega_n)$ and $M(\tilde{f}_0 \omega_{n-1})$ as composition factors. But the latter case cannot hold since by Theorem 9.5 again, $\mathrm{res}^{\mathcal{H}_n^{\mathrm{fin}}}_{\mathcal{H}_{n-1}^{\mathrm{fin}}} M(\tilde{f}_0 \omega_{n-1})$ is not isotypic. Hence all composition factors of $I(n)$ are $\cong M(\omega_n)$, and one easily gets that in fact $I(n) \cong M(\omega_n)$ by a dimension argument.

Next suppose that $n \equiv 0 \pmod h$. This time, $\tilde{f}_0 \omega_{n-1} = \omega_n$ and all other $\tilde{f}_i \omega_{n-1}$ are zero. Hence, by the induction hypothesis and the branching rules, $I(n)$ only involves $M(\omega_n)$ as a constituent. But we have that $\mathrm{res}^{\mathcal{H}_n^{\mathrm{fin}}}_{\mathcal{H}_{n-1}^{\mathrm{fin}}} M(\omega_n) = e_0 M(\omega_n) \cong M(\omega_{n-1})$ so in fact that $I(n)$ must have $M(\omega_n)$ as a constituent with multiplicty two. Further consideration of the endomorphism ring of $I(n)$ shows moreover that it is an indecomposable module.

The argument in the remaining two cases $n \equiv 1 \pmod h$ and $n \equiv 2 \pmod h$ is entirely similar. □

§9-c. **Projective representations of S_n.** We specialize for the final applications to the degenerate case. So now F is an algebraically closed field of characteristic $p \neq 2$, $h = p$ (if $p \neq 0$) or $h = \infty$ (if $p = 0$), and $\ell = (h-1)/2$. We are interested in the projective representations of the symmetric group S_n over the field F. Equivalently, see for example [BK$_2$, §3], we consider the representations of the *twisted group algebra* $S(n)$, defined by generators $t_1, \dots, t_{n-1}$ subject to the relations

$$t_i^2 = 1, \qquad t_i t_{i+1} t_i = t_{i+1} t_i t_{i+1}, \qquad t_i t_j = -t_j t_i$$

for all $1 \le i \le n-1$ and all $1 \le j \le n-1$ with $|i-j| > 1$. We view $S(n)$ as a superalgebra, defining the grading by declaring the generators $t_1, \dots, t_{n-1}$ to be of degree $\bar{1}$. All modules in this subsection will be $\mathbb{Z}_2$-graded as usual, see §2-b.

Let $W(n)$ denote the finite Sergeev superalgebra, replacing the notation $\mathcal{H}_n^{\mathrm{fin}}$ used previously. Recall from §2-k that $W(n)$ is a twisted tensor product of the group algebra FS_n of the symmetric group and the Clifford superalgebra $C(n)$. So $W(n)$ has even generators $s_1, \dots, s_{n-1}$ subject to the usual relations of the basic transpositions in S_n, odd generators $c_1, \dots, c_n$ subject to the Clifford relations as in (2.7), (2.8), and the additional relations (2.35). The connection between the superalgebras $S(n)$ and $W(n)$ is explained by the following observation, due originally to Sergeev [S$_2$], see also [BK$_2$, 3.3]:

Lemma 9.8. *There is an isomorphism of superalgebras*

$$\varphi : S(n) \otimes C(n) \xrightarrow{\sim} W(n)$$

such that $\varphi(1 \otimes c_i) = c_i$ *and* $\varphi(t_j \otimes 1) = \frac{1}{\sqrt{-2}} s_j(c_j - c_{j+1})$, $i = 1, \dots, n$, $j = 1, \dots, n-1$.

We will from now on *identify* $W(n)$ with $S(n) \otimes C(n)$ according to the lemma, and view $S(n)$ (resp. $C(n)$) as the subalgebra $S(n) \otimes 1$ (resp. $1 \otimes C(n)$) of $W(n)$. Recall the antiautomorphism τ of $W(n)$ defined as in (2.39) to be the unique antiautomorphism which is the identity on the generators $s_1, \dots, s_{n-1}, c_1, \dots, c_n$. This leaves invariant the subalgebra $S(n)$ of $W(n)$, so induces an antiautomorphism

$$\tau : S(n) \to S(n). \tag{9.13}$$

More explicitly, τ on $S(n)$ is the unique antiautomorphism which maps the generator t_i to $-t_i$ for each $i = 1, \dots, n-1$. Given a finite dimensional $S(n)$- (resp. $W(n)$-) module M we write M^τ for the dual vector superspace viewed as a module by twisting the natural right action into a left action via the antiautomorphism τ, see §2-b.

We also need the *Clifford module* $U(n)$, which is the unique irreducible $C(n)$-module up to isomorphism, see for instance [BK$_2$, 2.10]. We recall that $U(n)$ is of type $\mathtt{M}$ if n is even, type $\mathtt{Q}$ if n is odd, and $\dim U(n) = 2^{\lfloor (n+1)/2 \rfloor}$.

Now consider the exact functors

$$\begin{aligned} \mathcal{F}_n &: \operatorname{Rep} S(n) \to \operatorname{Rep} W(n), & \mathcal{F}_n &:= ? \boxtimes U(n),\\ \mathcal{G}_n &: \operatorname{Rep} W(n) \to \operatorname{Rep} S(n), & \mathcal{G}_n &:= \operatorname{Hom}_{C(n)}(U(n), ?). \end{aligned}$$

Also let

$$\begin{aligned} \operatorname{res}^{W(n)}_{W(n-1)} &: \operatorname{Rep} W(n) \to \operatorname{Rep} W(n-1), & \operatorname{ind}^{W(n)}_{W(n-1)} &: \operatorname{Rep} W(n-1) \to \operatorname{Rep} W(n),\\ \operatorname{res}^{S(n)}_{S(n-1)} &: \operatorname{Rep} S(n) \to \operatorname{Rep} S(n-1), & \operatorname{ind}^{S(n)}_{S(n-1)} &: \operatorname{Rep} S(n-1) \to \operatorname{Rep} S(n) \end{aligned}$$

denote the (exact) induction and restriction functors, where $S(n-1) \subset S(n)$ and $W(n-1) \subset W(n)$ are the natural subalgebras generated by all but the last generators. Recalling that Π denotes the parity change functor (2.1), the following lemma lists some basic properties:

Lemma 9.9. *The functors $\mathcal{F}_n$ and $\mathcal{G}_n$ are left and right adjoint to one another, and both commute with τ-duality. Moreover:*

(i) *Suppose that n is even. Then $\mathcal{F}_n$ and $\mathcal{G}_n$ are inverse equivalences of categories, so induce a type-preserving bijection between the isomorphism classes of irreducible $S(n)$-modules and of irreducible $W(n)$-modules. Also,*

$$\mathcal{F}_{n-1} \circ \operatorname{res}^{S(n)}_{S(n-1)} \simeq \operatorname{res}^{W(n)}_{W(n-1)} \circ \mathcal{F}_n, \tag{9.14}$$

$$\mathcal{G}_{n-1} \circ \operatorname{res}^{W(n)}_{W(n-1)} \simeq \operatorname{res}^{S(n)}_{S(n-1)} \circ \mathcal{G}_n \oplus \Pi \circ \operatorname{res}^{S(n)}_{S(n-1)} \circ \mathcal{G}_n, \tag{9.15}$$

$$\mathcal{F}_{n+1} \circ \operatorname{ind}^{S(n+1)}_{S(n)} \simeq \operatorname{ind}^{W(n+1)}_{W(n)} \circ \mathcal{F}_n, \tag{9.16}$$

$$\mathcal{G}_{n+1} \circ \operatorname{ind}^{W(n+1)}_{W(n)} \simeq \operatorname{ind}^{S(n+1)}_{S(n)} \circ \mathcal{G}_n \oplus \Pi \circ \operatorname{ind}^{S(n+1)}_{S(n)} \circ \mathcal{G}_n. \tag{9.17}$$

(ii) *Suppose that n is odd. Then $\mathcal{F}_n \circ \mathcal{G}_n \simeq \operatorname{Id} \oplus \Pi$ and $\mathcal{G}_n \circ \mathcal{F}_n \simeq \operatorname{Id} \oplus \Pi$. Furthermore, the functor $\mathcal{F}_n$ induces a bijection between isomorphism classes of irreducible $S(n)$-modules of type $\mathtt{M}$ and irreducible $W(n)$-modules of type $\mathtt{Q}$, while the functor $\mathcal{G}_n$ induces a bijection between isomorphism classes of irreducible $W(n)$-modules of type*

$\mathtt{M}$ *and irreducible $S(n)$-modules of type* $\mathtt{Q}$. *Finally,*

$$\mathrm{res}^{W(n)}_{W(n-1)} \circ \mathcal{F}_n \simeq \mathcal{F}_{n-1} \circ \mathrm{res}^{S(n)}_{S(n-1)} \oplus \Pi \circ \mathcal{F}_{n-1} \circ \mathrm{res}^{S(n)}_{S(n-1)}, \tag{9.18}$$

$$\mathrm{res}^{S(n)}_{S(n-1)} \circ \mathcal{G}_n \simeq \mathcal{G}_{n-1} \circ \mathrm{res}^{W(n)}_{W(n-1)}, \tag{9.19}$$

$$\mathrm{ind}^{W(n+1)}_{W(n)} \circ \mathcal{F}_n \simeq \mathcal{F}_{n+1} \circ \mathrm{ind}^{S(n+1)}_{S(n)} \oplus \Pi \circ \mathcal{F}_{n+1} \circ \mathrm{ind}^{S(n+1)}_{S(n)}, \tag{9.20}$$

$$\mathrm{ind}^{S(n+1)}_{S(n)} \circ \mathcal{G}_n \simeq \mathcal{G}_{n+1} \circ \mathrm{ind}^{W(n+1)}_{W(n)}. \tag{9.21}$$

Proof. Most of these facts are proved in [BK$_2$, 3.4,3.5], but we recall some of the details since we need to go slightly further. Let us consider the proof that $\mathcal{F}_n \circ \mathcal{G}_n \simeq \mathrm{Id} \oplus \Pi$ and $\mathcal{G}_n \circ \mathcal{F}_n \simeq \mathrm{Id} \oplus \Pi$, assuming that n is odd. Let I, J be a basis for $\mathrm{End}_{C(n)}(U(n))$ with I being the identity and J being an odd involution. Then, there are natural isomorphisms

$$\eta : \mathcal{F}_n \circ \mathcal{G}_n \xrightarrow{\sim} \mathrm{Id} \oplus \Pi, \qquad \xi : \mathrm{Id} \oplus \Pi \xrightarrow{\sim} \mathcal{G}_n \circ \mathcal{F}_n.$$

The first is defined for each $W(n)$-module M by $\eta_M : \mathrm{Hom}_{C(n)}(U(n), M) \boxtimes U(n) \to M \oplus \Pi M$, $\theta \otimes u \mapsto (\theta(u), (-1)^{\bar\theta}\theta(Ju))$. The second is defined for each $S(n)$-module N by $\xi_N : N \oplus \Pi N \to \mathrm{Hom}_{C(n)}(U(n), N \boxtimes U(n))$, $(n, n') \mapsto \theta_{n,n'}$, where $\theta_{n,n'}(u) = n \otimes u + (-1)^{\bar n'} n' \otimes Ju$ for each $u \in U(n)$. The proof that η and ξ really are isomorphisms is similar to the argument in [BK$_2$, 3.4]. Now consider the composite natural transformations

$$\mathrm{Id} \longrightarrow \mathrm{Id} \oplus \Pi \xrightarrow{\xi} \mathcal{G}_n \circ \mathcal{F}_n, \qquad \mathcal{F}_n \circ \mathcal{G}_n \xrightarrow{\eta} \mathrm{Id} \oplus \Pi \longrightarrow \mathrm{Id}, \tag{9.22}$$

$$\mathrm{Id} \longrightarrow \mathrm{Id} \oplus \Pi \xrightarrow{\eta^{-1}} \mathcal{F}_n \circ \mathcal{G}_n, \qquad \mathcal{G}_n \circ \mathcal{F}_n \xrightarrow{\xi^{-1}} \mathrm{Id} \oplus \Pi \longrightarrow \mathrm{Id}, \tag{9.23}$$

where the unmarked arrows are the obvious even ones. Then, (9.22) (resp. 9.23) gives the unit and counit of the adjunction needed to prove that $\mathcal{F}_n$ is left (resp. right) adoint to $\mathcal{G}_n$. We leave the details to be checked to the reader, see e.g. [ML, IV.1, Theorem 2(v)].

In case n is even, a similar but easier argument shows that $\mathcal{F}_n \circ \mathcal{G}_n \simeq \mathrm{Id}, \mathcal{G}_n \circ \mathcal{F}_n \simeq \mathrm{Id}$ so that $\mathcal{F}_n$ and $\mathcal{G}_n$ are inverse equivalences, hence left and right adjoint to each other. Now the statements in (i) and (ii) about isomorphism classes of irreducible modules follow easily as in [BK$_2$, 3.5].

Let us next prove that $\mathcal{F}_n$ commutes with duality. The antiautomorphism τ of $W(n)$ induces the antiautomorphism τ of the subalgebra $C(n)$ with $\tau(c_i) = c_i$ for each $i = 1, \dots, n$. Let $\varphi : U(n) \to U(n)^\tau, u \mapsto \varphi_u$ be a homogeneous isomorphism; note that it is not always possible to choose φ to be even, since $U(n) \not\simeq U(n)^\tau$ in case $n \equiv 2 \pmod 4$. We get a natural isomorphism

$$\Phi_M : M^\tau \boxtimes U(n) \to (M \boxtimes U(n))^\tau, \quad f \otimes u \mapsto \theta_{f,u}$$

for each finite dimensional $S(n)$-module M, where $\theta_{f,u}(m \otimes v) = (-1)^{\bar m \bar\varphi} f(m) \varphi_u(v)$ for each $m \in M, v \in U(n)$. This shows that $\mathcal{F}_n$ commutes with duality, i.e. $\tau \circ \mathcal{F}_n \circ \tau \cong \mathcal{F}_n$. Hence, using (2.2) and the fact that $\mathcal{G}_n$ is left adjoint to $\mathcal{F}_n$, the composite functor $\tau \circ \mathcal{G}_n \circ \tau$ is right adjoint to $\mathcal{F}_n$. But we already know that $\mathcal{G}_n$ is right adjoint to $\mathcal{F}_n$, so uniqueness of adjoints gives that $\tau \circ \mathcal{G}_n \circ \tau \cong \mathcal{G}_n$. This shows that $\mathcal{G}_n$ commutes with duality too.

It just remains to check the isomorphisms (9.14)–(9.21). Well, (9.14) follows from the definition on noting that $\mathrm{res}^{C(n)}_{C(n-1)} U(n) \simeq U(n-1)$ if n is even. Then (9.15) follows from (9.14) on composing on the left with $\mathcal{G}_{n-1}$ and on the right with $\mathcal{G}_n$. Next, (9.19) follows from the definition and an application of Frobenius reciprocity, using the observation that $U(n) \simeq \mathrm{ind}^{C(n)}_{C(n-1)} U(n-1)$ if n is odd. As before (9.18) then follows, composing with $\mathcal{F}_{n-1}$

and $\mathcal{F}_n$. Finally, (9.16), (9.17), (9.20) and (9.21) follow from (9.19), (9.18), (9.15) and (9.14) respectively by uniqueness of adjoints. □

We are ready to derive the consequences for $S(n)$ of the results of §9-b, or rather, of the analogous results for $W(n)$ in the degenerate case. By Theorem 9.4, we have a parametrization $\{M(\lambda) \mid \lambda \in \mathscr{RP}_p(n)\}$ of the irreducible $W(n)$-modules. Lemma 9.9 shows that the functors $\mathcal{F}_n$ and $\mathcal{G}_n$ set up a natural correspondence between classes of irreducible $S(n)$ and $W(n)$-modules, type-preserving if n is even and type-reversing if n is odd. Hence we have a parametrization $\{D(\lambda) \mid \lambda \in \mathscr{RP}_p(n)\}$ of the irreducible $S(n)$-modules, letting $D(\lambda)$ be an irreducible $S(n)$-module corresponding to $M(\lambda)$ under the correspondence. Also, recalling the definition (9.9), define

$$a(\lambda) := n - b(\lambda) \tag{9.24}$$

for $\lambda \in \mathscr{RP}_p(n)$. We observe by (9.10) that

$$a(\lambda) \equiv \gamma_1 + \cdots + \gamma_\ell \pmod 2, \tag{9.25}$$

where $\gamma_1 + \cdots + \gamma_\ell$ counts the number of nodes in the Young diagram λ of residue *different from* 0. Then:

Theorem 9.10. *The modules $\{D(\lambda) \mid \lambda \in \mathscr{RP}_p(n)\}$ form a complete set of pairwise non-isomorphic irreducible $S(n)$-modules. Moreover, for $\lambda, \mu \in \mathscr{RP}_p(n)$,*

(i) *$D(\lambda) \cong D(\lambda)^\tau$;*
(ii) *$D(\lambda)$ is of type* M *if $a(\lambda)$ is even, type* Q *if $a(\lambda)$ is odd;*
(iii) *$D(\mu)$ and $D(\lambda)$ belong to the same block if and only if* $\mathrm{cont}(\mu) = \mathrm{cont}(\lambda)$*;*
(iv) *$D(\lambda)$ is a projective module if and only if λ is a p-bar core.*

Proof. It just remains to observe that (i)–(iv) follow follow directly from Theorem 9.4(i)–(iv) using Lemma 9.9. □

Remark 9.11. The p-blocks of the ordinary irreducible projective representations of S_n were described by Humphreys [H], in terms of the notion of p-bar core. However, unlike the case of S_n, Humphreys' result does not imply Theorem 9.10(iii) because of the lack of information on decomposition numbers.

Define $\omega_n \in \mathscr{RP}_p(n)$ as in (9.12). Then the irreducible $S(n)$-module $D(\omega_n)$ is the *basic spin module*:

Lemma 9.12. *$D(\omega_n)$ is of dimension $2^{\lfloor n/2 \rfloor}$, unless $p|n$ when its dimension is $2^{\lfloor (n-1)/2 \rfloor}$. Moreover, $D(\omega_n)$ is equal to the reduction modulo p of the basic spin module $D((n))_{\mathbb{C}}$ of $S(n)_{\mathbb{C}}$ over $\mathbb{C}$, except if $p|n$ and n is even when the reduction modulo p of $D((n))_{\mathbb{C}}$ has two composition factors both isomorphic to $D(\omega_n)$.*

Proof. The statement about dimension is immediate from Lemmas 9.7 and 9.9. The final statement is easily proved by working in terms of $W(n)$ and using the explicit construction given in (9.11). □

To motivate the next two theorems, note that the map $[D(\lambda)] \mapsto [M(\lambda)]$ for each $\lambda \in \mathscr{RP}_p(n)$ extends linearly to an isomorphism

$$K(\operatorname{Rep} S(n)) \xrightarrow{\sim} K(\operatorname{Rep} W(n))$$

at the level of Grothendieck groups. Using this identification, we can lift the operators e_i and f_i on $K(\Lambda_0) = \bigoplus_{n \geq 0} K(\operatorname{Rep} W(n))$ defined earlier to define similar operators on $\bigoplus_{n \geq 0} K(\operatorname{Rep} S(n))$. Then all our earlier results about $K(\Lambda_0)$, for instance Theorems 7.15

and 7.16, could be restated purely in terms of the representations of $S(n)$ instead of $W(n)$. In fact, we can do slightly better and define the operators e_i and f_i on irreducible $S(n)$-modules, not just on the Grothendieck group. Theorems 9.13 and 9.14 below should be compared with the parallel results [BK$_1$, Theorems E and E$'$] for the symmetric group.

Theorem 9.13. *Let $\lambda \in \mathscr{RP}_p(n)$. There exist $S(n-1)$-modules $e_iD(\lambda)$ for each $i \in I$, unique up to isomorphism, such that*

(i) $\mathrm{res}^{S(n)}_{S(n-1)} D(\lambda) \cong \begin{cases} e_0D(\lambda) \oplus 2e_1D(\lambda) \oplus \cdots \oplus 2e_\ell D(\lambda) & \text{if } a(\lambda) \text{ is odd,} \\ e_0D(\lambda) \oplus e_1D(\lambda) \oplus \cdots \oplus e_\ell D(\lambda) & \text{if } a(\lambda) \text{ is even;} \end{cases}$

(ii) *for each $i \in I$, $e_iD(\lambda) \neq 0$ if and only if λ has an i-good node A, in which case $e_iD(\lambda)$ is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $D(\lambda_A)$.*

Moreover, if $i \in I$ and λ has an i-good node A, then

(iii) *the multiplicity of $D(\lambda_A)$ in $e_iD(\lambda)$ is $\varepsilon_i(\lambda)$, $\varepsilon_i(\lambda_A) = \varepsilon_i(\lambda)-1$, and $\varepsilon_i(\mu) < \varepsilon_i(\lambda)-1$ for all other composition factors $D(\mu)$ of $e_iD(\lambda)$;*

(iv) $\mathrm{End}_{S(n-1)}(e_iD(\lambda)) \simeq \mathrm{End}_{S(n-1)}(D(\lambda_A))^{\oplus \varepsilon_i(\lambda)}$ *as a vector superspace;*

(v) $\mathrm{Hom}_{S(n-1)}(e_iD(\lambda), e_iD(\mu)) = 0$ *for all $\mu \in \mathscr{RP}_p(n)$ with $\mu \neq \lambda$;*

(vi) *$e_iD(\lambda)$ is irreducible if and only if $\varepsilon_i(\lambda) = 1$.*

Hence, $\mathrm{res}^{S(n)}_{S(n-1)} D(\lambda)$ is completely reducible if and only if $\varepsilon_i(\lambda) \leq 1$ for every $i \in I$.

Proof. If n is odd, we simply define $e_iD(\lambda) := \mathscr{G}_{n-1}(e_iM(\lambda))$ for each $i \in I, \lambda \in \mathscr{RP}_p(n)$. If n is even, take

$$e_iD(\lambda) := \begin{cases} \mathscr{G}_{n-1}(e_iM(\lambda)) & \text{if } a(\lambda) \text{ is even and } i \neq 0, \text{ or } a(\lambda) \text{ is odd and } i = 0, \\ \overline{\mathscr{G}}_{n-1}(e_iM(\lambda)) & \text{if } a(\lambda) \text{ is even and } i = 0, \text{ or } a(\lambda) \text{ is odd and } i \neq 0. \end{cases}$$

We need to explain the notation $\overline{\mathscr{G}}_{n-1}$ used in the last two cases: here, $e_iM(\lambda)$ admits an odd involution by Remark 6.7 and Theorem 9.4(ii), and also $U(n-1)$ has an odd involution since n is even. So in exactly the same way as in the definition of (6.7), we can introduce the space

$$\overline{\mathscr{G}}_{n-1}(e_iM(\lambda)) := \overline{\mathrm{Hom}}_{C(n-1)}(U(n-1), e_iM(\lambda)).$$

It is then the case that $\mathscr{G}_{n-1}(e_iM(\lambda)) \simeq \overline{\mathscr{G}}_{n-1}(e_iM(\lambda)) \oplus \Pi\overline{\mathscr{G}}_{n-1}(e_iM(\lambda))$. Equivalently, by Lemma 9.9(ii) $e_iD(\lambda)$ can be characterized by

$$e_iM(\lambda) \cong \mathscr{F}_{n-1}(e_iD(\lambda))$$

if $a(\lambda)$ is even and $i = 0$, or $a(\lambda)$ is odd and $i \neq 0$.

With these definitions, it is now a straightforward matter to prove (i)–(vi) using Theorem 9.5 and Lemma 9.9. Finally, the uniqueness statement is immediate from Krull-Schmidt and the description of blocks from Theorem 9.10(iii). □

Theorem 9.14. *Let $\lambda \in \mathscr{RP}_p(n)$. There exist $S(n+1)$-modules $f_iD(\lambda)$ for each $i \in I$, unique up to isomorphism, such that*

(i) $\mathrm{ind}^{S(n+1)}_{S(n)} D(\lambda) \cong \begin{cases} f_0D(\lambda) \oplus 2f_1D(\lambda) \oplus \cdots \oplus 2f_\ell D(\lambda) & \text{if } a(\lambda) \text{ is odd,} \\ f_0D(\lambda) \oplus f_1D(\lambda) \oplus \cdots \oplus f_\ell D(\lambda) & \text{if } a(\lambda) \text{ is even;} \end{cases}$

(ii) *for each $i \in I$, $f_iD(\lambda) \neq 0$ if and only if λ has an i-cogood node B, in which case $f_iD(\lambda)$ is a self-dual indecomposable module with irreducible socle and cosocle isomorphic to $D(\lambda^B)$.*

Moreover, if $i \in I$ and λ has an i-cogood node B, then

(iii) *the multiplicity of* $D(\lambda^B)$ *in* $f_i D(\lambda)$ *is* $\varphi_i(\lambda)$, $\varphi_i(\lambda^B) = \varphi_i(\lambda) - 1$, *and* $\varphi_i(\mu) < \varphi_i(\lambda) - 1$ *for all other composition factors* $D(\mu)$ *of* $f_i D(\lambda)$*;*

(iv) $\mathrm{End}_{S(n+1)}(f_i D(\lambda)) \simeq \mathrm{End}_{S(n+1)}(D(\lambda^B))^{\oplus \varphi_i(\lambda)}$ *as a vector superspace;*

(v) $\mathrm{Hom}_{S(n+1)}(f_i D(\lambda), f_i D(\mu)) = 0$ *for all* $\mu \in \mathscr{RP}_p(n)$ *with* $\mu \neq \lambda$*;*

(vi) $f_i D(\lambda)$ *is irreducible if and only if* $\varphi_i(\lambda) = 1$.

Hence, $\mathrm{ind}_{S(n)}^{S(n+1)} M(\lambda)$ *is completely reducible if and only if* $\varphi_i(\lambda) \leq 1$ *for every* $i \in I$.

Proof. This is deduced from Theorem 9.6 by similar argument to the proof of Theorem 9.13. $\square$

Remark 9.15. (i) In [BK$_2$, 10.3], we gave an entirely different construction of the irreducible $S(n)$-modules, which we also denoted by $D(\lambda)$ for $\lambda \in \mathscr{RP}_p(n)$. We warn the reader that we have not yet proved that the modules denoted $D(\lambda)$ here are isomorphic to those in [BK$_2$], though we expect this to be the case.

(ii) Over $\mathbb{C}$, the branching rules in the preceeding two theorems are the same as Morris' branching rules, see [Mo]. In particular using this observation, one easily shows that our labelling of irreducibles over $\mathbb{C}$ agrees with the standard labelling. Hence over $\mathbb{C}$ the labelling here agrees with the labelling in [BK$_2$], compare [S$_1$].

(iii) There is one other case where it is easy to see right away that the labelling here agrees with [BK$_2$]: if λ is a p-bar core then consideration of central characters shows that $D(\lambda)$ is equal to a reduction modulo p of the irreducible representation of $S(n)_{\mathbb{C}}$ over $\mathbb{C}$ with the same label. So it coincides with the module $D(\lambda)$ of [BK$_2$] thanks to [BK$_2$, 10.8].

§9-d. **The Jantzen-Seitz problem.** The results of the previous subsection give a solution to the *Jantzen-Seitz problem* for projective representations of the symmetric and alternating groups. This problem originated in [JS], and is of interest in the study of maximal subgroups of the finite classical groups. To consider the Jantzen-Seitz problem, we first need to switch to studying *ungraded* representations of the twisted group algebra $S(n)$. The goal is to describe all ungraded irreducible $S(n)$-modules which remain irreducible on restriction to the subalgebra $S(n-1)$.

As in [BK$_2$, §10], it is straightforward to obtain a parametrization of the ungraded irreducible $S(n)$-modules from Theorem 9.10: if $a(\lambda)$ is odd then $D(\lambda)$ decomposes as an ungraded module as $D(\lambda) = D(\lambda, +) \oplus D(\lambda, -)$ for two non-isomorphic irreducible $S(n)$-modules $D(\lambda, +)$ and $D(\lambda, -)$. If $a(\lambda)$ is even, then $D(\lambda)$ is irreducible viewed as an ungraded $S(n)$-module, but we denote it instead by $D(\lambda, 0)$ to make it clear that we are no longer considering a $\mathbb{Z}_2$-grading. Then

$$\{D(\lambda, 0) \mid \lambda \in \mathscr{RP}_p(n), a(\lambda) \text{ even}\} \sqcup \{D(\lambda, +), D(\lambda, -) \mid \lambda \in \mathscr{RP}_p(n), a(\lambda) \text{ odd}\}$$

gives a complete set of pairwise non-isomorphic ungraded irreducible $S(n)$-modules.

Remark 9.16. Using Theorem 9.10 and a counting argument involving Humphreys' block classification [H], it is not hard to obtain the following description of the ungraded blocks of the algebra $S(n)$. Let $D(\lambda, \varepsilon)$ and $D(\mu, \delta)$ be ungraded irreducible $S(n)$-modules. Then, with one exception, $D(\lambda, \varepsilon)$ and $D(\mu, \delta)$ lie in the same block if and only if λ and μ have the same p-bar core. The exception is if $\lambda = \mu$ is a p-bar core, $a(\lambda)$ is odd and $\varepsilon = -\delta$, when $D(\lambda, \varepsilon)$ and $D(\mu, \delta)$ are in different blocks.

Now we state the solution to the Jantzen-Seitz problem for projective representations of the symmetric group. The proof is a straightforward consequence of Theorem 9.13.

Theorem 9.17. *Let* $\lambda \in \mathscr{RP}_p(n)$. *Then:*

(i) *If $a(\lambda)$ is even,* $\mathrm{res}^{S(n)}_{S(n-1)} D(\lambda, 0)$ *is irreducible if and only if* $\varepsilon_0(\lambda) = \sum_{i\in I} \varepsilon_i(\lambda) = 1$.

(ii) *If $a(\lambda)$ is odd,* $\mathrm{res}^{S(n)}_{S(n-1)} D(\lambda, \pm)$ *is irreducible if and only if* $\sum_{i\in I} \varepsilon_i(\lambda) = 1$.

Finally let us discuss the analogous problem for the projective representations of the alternating group. As explained in [BK$_2$, §10], it suffices for this to consider the representation theory of the algebra $A(n) := S(n)_{\bar{0}}$, i.e. the twisted group algebra of the alternating group. The irreducible $A(n)$-modules (there being no ambiguity between graded and ungraded modules since $A(n)$ is purely even) are constructed out of those for $S(n)$ as in [BK$_2$, Theorem 10.4]. More precisely, if $\lambda \in \mathscr{RP}_p(n)$ has $a(\lambda)$ even, then $\mathrm{res}^{S(n)}_{A(n)} D(\lambda)$ decomposes as a direct sum $E(\lambda, +) \oplus E(\lambda, -)$ of two non-isomorphic irreducible $A(n)$-modules. If $a(\lambda)$ is odd, then $\mathrm{res}^{S(n)}_{A(n)} D(\lambda)$ decomposes as a direct sum of two copies of a single irreducible $A(n)$-module denoted $E(\lambda, 0)$. Then,

$$\{E(\lambda, 0) \mid \lambda \in \mathscr{RP}_p(n), a(\lambda) \text{ odd}\} \sqcup \{E(\lambda, +), E(\lambda, -) \mid \lambda \in \mathscr{RP}_p(n), a(\lambda) \text{ even}\}$$

gives a complete set of pairwise non-isomorphic irreducible $A(n)$-modules. Now the solution to the Jantzen-Seitz problem in this case, again an easy consequence of Theorem 9.13, is as follows:

Theorem 9.18. *Let $\lambda \in \mathscr{RP}_p(n)$. Then:*

(i) *If $a(\lambda)$ is even,* $\mathrm{res}^{A(n)}_{A(n-1)} E(\lambda, \pm)$ *is irreducible if and only if* $\sum_{i\in I} \varepsilon_i(\lambda) = 1$;

(ii) *If $a(\lambda)$ is odd,* $\mathrm{res}^{A(n)}_{A(n-1)} E(\lambda, 0)$ *is irreducible if and only if* $\varepsilon_0(\lambda) = \sum_{i\in I} \varepsilon_i(\lambda) = 1$.

Appendix A. Results from computer calculations

In this appendix we list the characters of the irreducible integral $\mathcal{H}_n$-modules for $n \leq 4$, or $n \leq 6$ in case $\ell = 1$. This data, which is not needed in proving the main results of the article, was generated in part by lengthy computer calculations checked independently in both the degenerate and quantum cases. The approach is usually along the lines of the calculations in §5-f. We should explain notation in the tables: for $i \in I$, we abbreviate $(i+1)$ by i'. Moreover where different letters are used, it is assumed implicitly that they are not neighbours in the Dynkin diagram. For instance an entry like

$$\mathrm{ch}\, L(iji') = iji' + jii' + ii'j$$

means that $\mathrm{ch}\, L(iji') = [L(i) \circledast L(j) \circledast L(i')] + [L(j) \circledast L(i) \circledast L(i')] + [L(i) \circledast L(i') \circledast L(j)]$ whenever $i, i', j \in I$ satisfy $|i - j| > 1, |i' - j| > 1$. Note iji' is not the only valid label for $L(iji')$: it could also be denoted $L(jii')$ or $L(ii'j)$.

Case $n = 2$: all blocks.

L	$\mathrm{ch}\, L$	Conditions
$L(ii)$	$2.ii$	$i \in I$
$L(ii')$	ii'	$0 \leq i \leq \ell - 1$
$L(i'i)$	$i'i$	”
$L(ij)$	$ij + ji$	$i, j \in I$

Case $n = 3$: all blocks.

L	ch L	Conditions
$L(iii)$	$6.iii$	$i \in I$
$L(iii')$	$2.iii'$	$i = 0$ or $\ell - 1$
$L(ii'i)$	$ii'i$	"
$L(i'ii)$	$2.i'ii$	"
$L(iii')$	$2.iii' + ii'i$	$0 < i < \ell - 1$
$L(i'ii)$	$2.i'ii + ii'i$	"
$L(ii'i')$	$2.ii'i' + i'ii'$	$0 \leq i \leq \ell - 1$
$L(i'ii')$	$2.i'i'i + i'ii'$	"
$L(ii'i'')$	$ii'i''$	$0 \leq i \leq \ell - 2$
$L(ii''i')$	$ii''i' + i''ii'$	"
$L(i'ii'')$	$i'ii'' + i'i''i$	"
$L(i''i'i)$	$i''i'i$	"
$L(iij)$	$2.iij + 2.iji + 2.jii$	$i, j \in I$
$L(iji')$	$iji' + jii' + ii'j$	$0 \leq i \leq \ell - 1, j \in I$
$L(i'ji)$	$i'ji + ji'i + i'ij$	"
$L(ijk)$	$ijk +$ all permutations	$i, j, k \in I$

Case $n = 4$: blocks of rank ≤ 2.

L	ch L	Conditions
$L(iiii)$	$24.iiii$	$i \in I$
$L(iiii')$	$6.iiii'$	$\ell = 1, i = 0$
$L(iii'i)$	$2.iii'i$	"
$L(ii'ii)$	$2.ii'ii$	"
$L(i'iii)$	$6.i'iii$	"
$L(iiii')$	$6.iiii' + 2.iii'i$	$\ell > 1, i = 0$ or $i = \ell - 1$
$L(ii'ii)$	$2.iii'i + 2.ii'ii$	"
$L(i'iii)$	$6.i'iii + 2.ii'ii$	"
$L(iiii')$	$6.iiii' + 4.iii'i + 2.ii'ii$	$\ell > 1, 0 < i < \ell - 1$
$L(i'iii)$	$6.i'iii + 4.ii'ii + 2.iii'i$	"
$L(iii'i')$	$4.iii'i' + 2.ii'ii'$	$i = 0$ or $i = \ell - 1$
$L(ii'i'i)$	$ii'ii' + 2.ii'i'i + i'ii'i$	"
$L(i'iii')$	$2.i'iii'$	"
$L(i'i'ii)$	$4.i'i'ii + 2.i'ii'i$	"
$L(iii'i')$	$4.iii'i' + 2.ii'ii'$	$0 < i < \ell - 1$
$L(ii'i'i)$	$2.i'iii' + 2.ii'i'i + i'ii'i + ii'ii'$	"
$L(i'i'ii)$	$4.i'i'ii + 2.i'ii'i$	"
$L(ii'i'i')$	$6.ii'i'i' + 4.i'ii'i' + 2.i'i'ii'$	$0 \leq i \leq \ell - 1$
$L(i'i'i'i)$	$6.i'i'i'i + 4.i'i'ii' + 2.i'ii'i'$	"
$L(iiij)$	$6(iiij +$ all permutations$)$	$i, j \in I$
$L(iijj)$	$4(iijj +$ all permutations$)$	$i, j \in I$

Case $n = 4$: blocks of rank 3.

L	ch L	Conditions
$L(iii'i'')$	$2.iii'i''$	$i = 0$
$L(ii'ii'')$	$ii'ii'' + ii'i''i$	"
$L(i''i'ii)$	$2.i''i'ii$	"
$L(i'iii'')$	$2.i'iii'' + 2.i'i''ii + 2.i'ii''i$	"
$L(ii''i'i)$	$ii''i'i + i''ii'i$	"
$L(iii''i')$	$2.iii''i' + 2.ii''ii' + 2.i''iii'$	"
$L(iii'i'')$	$2.iii'i'' + ii'ii'' + ii'i''i$	$0 < i \leq \ell - 2$
$L(i''i'ii)$	$2.i''i'ii + i''ii'i + ii''i'i$	"
$L(i'i''ii)$	$2.i'i''ii + 2.i'ii''i + 2.i'iii'' + ii'i''i + ii'ii''$	"
$L(iii''i')$	$2.iii''i' + 2.ii''ii' + 2.i''iii' + ii''i'i + i''ii'i$	"
$L(ii'i'i'')$	$ii'i''i' + 2.ii'i'i'' + i'ii'i''$	$0 \leq i < \ell - 2$
$L(i''i'ii')$	$i''i'ii' + 2.i''i'i'i + i'i''i'i$	"
$L(ii''i'i')$	$2.ii''i'i' + 2.i''ii'i' + ii'i''i' + i''i'ii'$	"
$L(i'ii'i'')$	$2.i'i'i''i + 2.i'i'ii'' + i'i''i'i + i'ii'i''$	"
$L(i'ii''i')$	$i'ii''i' + i'i''ii'$	"
$L(ii'i''i')$	$ii'i''i'$	$i = \ell - 2$
$L(ii''i'i)$	$i'i''i'i$	"
$L(ii'i'i'')$	$2.ii'i'i'' + i'ii'i''$	"
$L(i''i'ii')$	$2.i''i'i'i + i''i'ii'$	"
$L(i'ii''i')$	$i'ii''i' + i'i''ii'$	"
$L(ii''i'i')$	$2.ii''i'i' + 2.i''ii'i' + i''i'ii'$	"
$L(i'i'ii'')$	$2.i'i'ii'' + 2.i'i'i''i + i'ii'i''$	"
$L(ii'i''i'')$	$2.ii'i''i'' + ii''i'i'' + i''ii'i''$	$0 \leq i \leq \ell - 2$
$L(i''i'ii'')$	$2.i''i''i'i + i''i'i''i + i''i'ii''$	"
$L(i'ii''i'')$	$2.i'i''i''i + 2.i'i''ii'' + 2.i'ii''i'' + i''i'i''i + i''i'ii''$	"
$L(ii''i'i'')$	$2.ii''i''i' + 2.i''ii''i' + 2.i''i''ii' + ii''i'i'' + i''ii'i''$	"
$L(ii'jj)$	$2.ii'jj + 2.iji'j + 2.jii'j + 2.ijji' + 2.jiji' + 2.jjii'$	$0 \leq i \leq \ell - 1, j \in I$
$L(i'ijj)$	$2.i'ijj + 2.i'jij + 2.ji'ij + 2.i'jji + 2.ji'ji + 2.jji'i$	"
$L(iii'j)$	$2.iii'j + i.iiji' + 2.ijii' + 2.jiii'$	$i = 0$ or $\ell - 1, j \in I$
$L(ii'ij)$	$ii'ij + ii'ji + iji'i + jii'i$	"
$L(i'iij)$	$2.i'iij + 2.i'iji + 2.i'jii + 2.ji'ii$	"
$L(iii'j)$	$2.iii'j + 2.iiji' + 2.ijii' + 2.jiii' + ii'ij + ii'ji + iji'i + jii'i$	$0 < i < \ell - 1, j \in I$
$L(i'iij)$	$2.i'iij + 2.i'iji + 2.i'jii + 2.ji'ii + ii'ij + ii'ji + iji'i + jii'i$	"
$L(ii'i'j)$	$2.ii'i'j + 2.ii'ji' + 2.iji'i' + 2.jii'i' + i'ii'j + i'iji' + i'jii' + ji'ii'$	$0 \leq i \leq \ell - 1, j \in I$
$L(i'ii'j)$	$2.i'i'ij + 2.i'i'ji + 2.i'ji'i + 2.ji'i'i + i'ii'j + i'iji' + i'jii' + ji'ii'$	"
$L(iijk)$	$2(iijk + \text{all permutations})$	$i, j, k \in I$

Case $n = 4$: blocks of rank 4.

L	$\mathrm{ch}\, L$	Conditions
$L(ii'i''i''')$	$ii'i''i'''$	$0 \le i \le \ell - 3$
$L(i'''i''i'i)$	$i'''i''i'i$	”
$L(ii'i'''i'')$	$ii'i'''i'' + ii'''i'i'' + i'''ii'i''$	”
$L(ii'''i''i')$	$ii'''i''i' + i'''ii''i' + i'''i''ii'$	”
$L(i'i''i'''i)$	$i'i''i'''i + i'i''ii''' + i'ii''i'''$	”
$L(i''i'ii''')$	$i''i'''i'i + i''i'i'''i + i''i'ii'''$	”
$L(ii''i'i''')$	$ii''i'i''' + ii''i'''i' + i''ii'i''' + i''ii'''i' + i''i'''ii'$	”
$L(i'''i'i''i)$	$i'''i'i''i + i'i'''i''i + i'''i'ii'' + i'i'''ii'' + i'ii'''i''$	”
$L(ii'jj')$	$ii'jj' + iji'j' + ijj'i' + jii'j' + jij'i' + jj'ii'$	$0 \le i \le \ell - 1, j \in I$
$L(i'ijj')$	$i'ijj' + i'jij' + i'jj'i + ji'ij' + ji'j'i + jj'i'i$	”
$L(ii'j'j)$	$ii'j'j + ij'i'j + ij'ji' + j'ii'j + j'iji' + j'jii'$	”
$L(i'ij'j)$	$i'ij'j + i'j'ij + i'j'ji + j'i'ij + j'i'ji + j'ji'i$	”
$L(ii'jk)$	$ii'jk + ii'kj + iji'k + iki'j + jii'k + kii'j + ijki' +$ $ikji' + jiki' + kiji' + jkii' + kjii'$	$0 \le i \le \ell - 1, j, k \in I$
$L(i'ijk)$	$i'ijk + i'ikj + i'jik + i'kij + ji'ik + ki'ij + i'jki +$ $i'kji + ji'ki + ki'ji + jki'i + kji'i$	”
$L(ii'i''j)$	$ii'i''j + ii'ji'' + iji'i'' + jii'i''$	$0 \le i \le \ell - 2, j \in I$
$L(ii''i'j)$	$ii''i'j + i''ii'j + ii''ji' + i''iji' + iji''i' + i''jii' +$ $jii''i' + ji''ii'$	”
$L(i'ii''j)$	$ji'ii'' + ji'i''i + ji'ii'' + ji'i''i + ji'ii'' + ji'i''i +$ $ji'ii'' + ji'i''i$	”
$L(i''i'ij)$	$i''i'ij + i''i'ji + i''ji'i + ji''i'i$	”
$L(ijkl)$	$ijkl +$ all permutations	$i, j, k, l \in I$

Case $n = 5, \ell = 1$: all blocks.

L	$\mathrm{ch}\, L$
$L(00000)$	120.00000
$L(00001)$	24.00001
$L(00010)$	6.00010
$L(00100)$	4.00100
$L(01000)$	6.01000
$L(10000)$	24.10000
$L(10001)$	6.10001
$L(01001)$	2.01001
$L(00101)$	$2.00101 + 4.00110 + 2.01010$
$L(00011)$	$12.00011 + 6.00101$
$L(11000)$	$12.11000 + 6.10100$
$L(01010)$	$2.01010 + 4.01100 + 2.10100$
$L(10010)$	2.10010
$L(10011)$	$4.10011 + 2.10101 + 4.11001$
$L(00111)$	$12.00111 + 8.01011 + 4.10011 + 4.01101 + 2.10101$
$L(01110)$	$6.01110 + 4.10110 + 4.01101 + 2.01011 + 2.11010 + 2.10101$
$L(11100)$	$12.11100 + 8.11010 + 4.11001 + 4.10110 + 2.10101$
$L(01111)$	$24.01111 + 18.10111 + 12.11011 + 6.11101$
$L(11110)$	$24.11110 + 18.11101 + 12.11011 + 6.10111$
$L(11111)$	120.11111

Case $n = 6, \ell = 1$*: all blocks.*

L	$\mathrm{ch}\, L$
$L(000000)$	720.000000
$L(000001)$	$24.000010 + 120.000001$
$L(000100)$	$12.000100 + 24.000010$
$L(001000)$	$12.001000 + 12.000100$
$L(010000)$	$12.001000 + 24.010000$
$L(100000)$	$24.010000 + 120.100000$
$L(000011)$	$48.000011 + 24.000101 + 4.001001$
$L(110000)$	$48.110000 + 24.101000 + 4.100100$
$L(000110)$	$12.000110 + 6.000101 + 6.001010$
$L(011000)$	$12.011000 + 6.101000 + 6.010100$
$L(001010)$	$4.001010 + 8.001100 + 4.010100$
$L(001001)$	4.001001
$L(010001)$	6.010001
$L(100001)$	24.100001
$L(100100)$	4.100100
$L(100010)$	6.100010
$L(010010)$	2.010010
$L(100101)$	$4.100110 + 4.110010 + 2.100101 + 2.101010$
$L(010011)$	$4.011001 + 4.010011 + 2.101001 + 2.010101$
$L(100011)$	$6.100101 + 12.100011$
$L(110001)$	$6.101001 + 12.110001$
$L(000111)$	$36.000111 + 24.001011 + 12.010011 + 12.001101 + 6.010101$
$L(011001)$	$12.011100 + 8.101100 + 8.011010 + 2.101001 + 2.010101 + 4.010110 +$ $4.011001 + 4.110100 + 4.101010$
$L(111000)$	$36.111000 + 24.110100 + 12.110010 + 12.101100 + 6.101010$
$L(001011)$	$12.001110 + 8.001101 + 8.010110 + 2.100101 + 2.101010 + 4.011010 +$ $4.100110 + 4.001011 + 4.010101$
$L(001111)$	$48.001111 + 36.010111 + 24.011011 + 12.011101 + 24.100111 + 8.101101 +$ $8.110011 + 16.101011 + 4.110101$
$L(100111)$	$12.111001 + 12.100111 + 16.110011 + 8.101011 + 8.110101 + 4.101101$
$L(011101)$	$24.011110 + 18.101110 + 12.110110 + 6.111010 + 18.011101 + 12.101101 +$ $12.011011 + 6.101011 + 6.110101 + 6.010111$
$L(111001)$	$48.111100 + 36.111010 + 24.110110 + 12.101110 + 24.111001 + 16.110101 +$ $8.101101 + 8.110011 + 4.101011$
$L(011111)$	$120.011111 + 96.101111 + 72.110111 + 48.111011 + 24.111101$
$L(111101)$	$120.111110 + 96.111101 + 72.111011 + 48.110111 + 24.101111$
$L(111111)$	720.111111

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