HIGHEST WEIGHT THEORY FOR FINITE W-ALGEBRAS

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Abstract. We define analogues of Verma modules for finite W-algebras. By the usual ideas of highest weight theory, this is a first step towards the classification of finite dimensional irreducible modules. We also introduce an analogue of the BGG category $\mathcal{O}$. Motivated by known results in type A, we then formulate some precise conjectures in the case of nilpotent orbits of standard Levi type.

1. Introduction

There has been a great deal of recent interest in W-algebras and their representation theory. To each nilpotent element e in the Lie algebra $\mathfrak{g}$ of a complex reductive algebraic group G, one can associate a finite W-algebra $U(\mathfrak{g}, e)$. Up to isomorphism, this algebra depends only on the adjoint orbit $G \cdot e$ of e and can be viewed informally as the "universal enveloping algebra" of the Slodowy slice to this orbit. Finite W-algebras were introduced into the mathematical literature by Premet [P1, §4]; see also [GG]. For nilpotent orbits admitting even good gradings in the sense of [EK], these algebras already appeared in work of Kostant and Lynch [K, Ly] in the context of (generalized) Whittaker modules. At one extreme, $U(\mathfrak{g}, 0) = U(\mathfrak{g})$; at the other extreme, when e is regular, $U(\mathfrak{g}, e)$ is isomorphic to the center $Z(\mathfrak{g})$ of $U(\mathfrak{g})$.

There is much motivation for studying the representation theory of finite W-algebras. For instance, through Skryabin's equivalence [S], there is a relationship between the representation theory of $U(\mathfrak{g}, e)$ and the representation theory of $\mathfrak{g}$. This provides an important connection between the primitive ideals of $U(\mathfrak{g})$ whose associated variety contains $G \cdot e$ and the primitive ideals of $U(\mathfrak{g}, e)$; see [P2, Theorem 3.1] and [L, Theorem 1.2.2]. In another direction, it is shown by Premet [P1, §6] that $U(\mathfrak{g}, e)$ gives rise to a natural non-commutative deformation of the singular variety that arises by intersecting the Slodowy slice to the orbit $G \cdot e$ with the nilpotent cone of $\mathfrak{g}$.

In mathematical physics, finite W-algebras and their affine counterparts have attracted a lot of attention under a slightly different guise; see for example [BT, VD, A, DK]. It has recently been proved in [D^{3}HK] that the definition in the mathematical physics literature via BRST cohomology agrees with Premet's definition; see the discussion in §2.3.

For the case $G = \mathrm{GL}_N(\mathbb{C})$, the first and third authors made a thorough study of the finite dimensional representation theory of $U(\mathfrak{g}, e)$ in [BK2]. In this case $U(\mathfrak{g}, e)$ is isomorphic to a quotient of a shifted Yangian [BK1]. Using

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this connection and the natural triangular decomposition of shifted Yangians, we developed a highest weight theory for $U(\mathfrak{g}, e)$, leading to the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules; see [BK2, §7.2]. On the other hand, using Premet's definition of $U(\mathfrak{g}, e)$ and the so-called Whittaker functor, we obtained character formulae for the finite dimensional irreducibles as a consequence of the Kazhdan–Lusztig conjecture for a certain parabolic category $\mathcal{O}$ attached to $\mathfrak{g}$; see [BK2, §8.5]. This type A theory has already had several other quite striking applications; see [B1, B2, BK3].

In the general case there is little concrete knowledge about representations of $U(\mathfrak{g}, e)$. It has only recently been proved that $U(\mathfrak{g}, e)$ always has "enough" finite dimensional irreducible representations; see [P3, Theorem 1.1] and [L, Theorem 1.2.3]. The purpose of this paper is to set up the framework to study representation theory of $U(\mathfrak{g}, e)$ via highest weight theory. In particular, we define Verma modules for $U(\mathfrak{g}, e)$, which turns out to be surprisingly non-trivial.

Recall in classical Lie theory that Verma modules are "parabolically induced" from irreducible representations of a Cartan subalgebra. The main problem for finite W-algebras is to find a suitable algebra to play the role of Cartan subalgebra. It turns out that this role is played by the "smaller" finite W-algebra $U(\mathfrak{g}_0, e)$ where $\mathfrak{g}_0$ is a minimal Levi subalgebra of $\mathfrak{g}$ containing e, i.e. e is a distinguished nilpotent element of $\mathfrak{g}_0$; see §4.1. Given a parametrization

$$\{V_\Lambda \mid \Lambda \in \mathcal{L}\}$$

of a complete set of pairwise inequivalent finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules, we will construct the Verma modules

$$\{M(\Lambda, e) \mid \Lambda \in \mathcal{L}\}$$

for $U(\mathfrak{g}, e)$ by parabolically inducing the V_Λ's from $U(\mathfrak{g}_0, e)$ to $U(\mathfrak{g}, e)$; see §4.2. We then prove as usual that the Verma module $M(\Lambda, e)$ has a unique irreducible quotient $L(\Lambda, e)$ and that the $L(\Lambda, e)$'s parametrized by the subset

$$\mathcal{L}^+ := \{\Lambda \in \mathcal{L} \mid \dim L(\Lambda, e) < \infty\}$$

give a complete set of pairwise inequivalent finite dimensional irreducible $U(\mathfrak{g}, e)$-modules. Incidentally, all our Verma modules belong to a natural category $\mathcal{O}(e)$ whose objects have composition series with only the $L(\Lambda, e)$'s as composition factors; see §4.4. In the case $e = 0$ this category $\mathcal{O}(e)$ is the usual Bernstein-Gelfand-Gelfand category $\mathcal{O}$ from [BGG].

The general principles just described reduce the problem of classifying the finite dimensional irreducible $U(\mathfrak{g}, e)$-modules to two major problems:

1. Find a natural parametrization of the finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules by some explicit labelling set $\mathcal{L}$.
2. Describe the subset $\mathcal{L}^+$ of $\mathcal{L}$ combinatorially.

We remark that the definition of Verma modules, hence the subset $\mathcal{L}^+$ of $\mathcal{L}$, depends essentially on a choice of positive roots in the restricted root system of $U(\mathfrak{g}, e)$ in the sense of [BG, §2]. Unlike in the classical situation there is often more that one conjugacy classes of such choices. The combinatorial description of the subset $\mathcal{L}^+$ will certainly depend in a significant way on this choice.

In the special case that e is of *standard Levi type*, i.e. e is actually a regular nilpotent element of $\mathfrak{g}_0$, Kostant showed in [K, §2] that $U(\mathfrak{g}_0, e)$ is canonically isomorphic to $Z(\mathfrak{g}_0)$. Hence in this case the solution to problem (1) is very simple: the set $\mathcal{L}$ labelling our Verma modules can be naturally identified with the set $\mathfrak{t}^*/W_0$ of W_0-orbits in $\mathfrak{t}^*$, where W_0 is the Weyl group of $\mathfrak{g}_0$ with respect to a maximal toral subalgebra $\mathfrak{t}$. This resembles a result of Friedlander and Parshall [FP, Corollary 3.5] giving a similar labelling of irreducible representations for reduced enveloping algebras of standard Levi type in characteristic p. In §5.1 we formulate two explicit conjectures concerning the standard Levi type case. The first of these conjectures reduces the solution of problem (2) to the combinatorics of associated varieties of primitive ideals in $\mathfrak{g}$. We prove this conjecture in type A (for a standard choice of positive roots) in §5.2, by translating the results from [BK2] into the general framework. Our second conjecture is quite a bit stronger, and was inspired by Premet's ideas in [P2, §7]. It predicts an explicit link between our category $\mathcal{O}(e)$ and another category $\mathcal{O}(\chi)$ introduced by Miličíc and Soergel [MS]. This conjecture also implies the truth of the Kazhdan–Lusztig conjecture for finite W-algebras of standard Levi type from [VD]. We speculate for e of standard Levi type that every primitive ideal of $U(\mathfrak{g}, e)$ is the annihilator of an irreducible highest weight module in $\mathcal{O}(e)$, though we have no evidence for this beyond Duflo's theorem in the case $e = 0$.

The rest of the article is organized as follows. In Section 2, we explain in detail the relationship between three quite different definitions of finite W-algebra. The key to the new results in this paper actually comes from the third of these definitions, namely, the BRST cohomology definition as formulated in [DK]. We point out especially Theorem 2.8 which makes the link between the second and third definitions quite transparent. In Section 3 we survey various results of Premet describing the associated graded algebra to $U(\mathfrak{g}, e)$ in its two natural filtrations, setting up more essential notation along the way. The main new results of the paper are proved in Section 4, the most important being Theorem 4.3. Finally in Section 5 we discuss standard Levi type and explain how to translate the type A results from [BK2].

Notation. We work throughout over the ground field $\mathbb{C}$. By a *character* of a Lie algebra $\mathfrak{g}$ we mean a Lie algebra homomorphism $\rho : \mathfrak{g} \to \mathbb{C}$. Any such ρ induces a shift automorphism

$$S_\rho : U(\mathfrak{g}) \to U(\mathfrak{g})$$

of the universal enveloping algebra $U(\mathfrak{g})$ with $S_\rho(x) := x + \rho(x)$ for each $x \in \mathfrak{g}$. For the convenience of the reader, we also include here a brief summary of some of the less standard notation to be introduced formally later on:

- (e, h, f) is a fixed $\mathfrak{sl}_2$-triple and $\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$ is a good grading for e;
- $\mathfrak{t}$ is a maximal toral subalgebra of $\mathfrak{g}(0)$ containing h and a maximal toral subalgebra $\mathfrak{t}^e$ of $\mathfrak{g}^e$;
- Φ is the root system of $\mathfrak{g}$ with respect to $\mathfrak{t}$;
- Φ^e is the restricted root system, i.e. the non-zero weights of $\mathfrak{t}^e$ on $\mathfrak{g}^e$;
- $\mathfrak{g}_0$ is the centralizer of $\mathfrak{t}^e$ in $\mathfrak{g}$;

- $\mathfrak{n} = \bigoplus_{j<0} \mathfrak{g}(j)$, $\mathfrak{p} = \bigoplus_{j \geq 0} \mathfrak{g}(j)$;
- $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_+$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$;
- $\Phi = \Phi_- \sqcup \Phi_0 \sqcup \Phi_+$ where Φ_0 and Φ_+ are the roots associated to $\mathfrak{g}_0$ and the nilradical $\mathfrak{g}_+$ of $\mathfrak{q}$, respectively, and $\Phi_- = -\Phi_+$;
- $b_1, \dots, b_r$ is a homogeneous basis for $\mathfrak{n}$ such that b_i is of degree $-d_i$ and weight $\beta_i \in \mathfrak{t}^*$;
- $\beta = \sum_i \beta_i$;
- $\gamma = \sum_{i:\beta_i \in \Phi_-} \beta_i$;
- $\delta = \sum_{i:\beta_i \in \Phi_-, d_i \geq 2} \beta_i + \frac{1}{2} \sum_{i:\beta_i \in \Phi_-, d_i = 1} \beta_i$;
- $\epsilon = \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha + \frac{1}{2} \sum_{i:\beta_i \in \Phi_0} \beta_i$.

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2. Three definitions of finite W-algebras

In this section we give three equivalent definitions of the finite W-algebra $U(\mathfrak{g}, e)$. The first two of these definitions have left- and right-handed versions which are not obviously isomorphic; we establish that they are indeed isomorphic using the third definition. Although not used here, we point out that there is also now a *fourth* important definition of the finite W-algebra, namely, Losev's definition via Fedosov quantization; see [L, §3].

2.1. **Definition via Whittaker models.** Let $\mathfrak{g}$ be the Lie algebra of a connected reductive algebraic group G over $\mathbb{C}$. Let $e \in \mathfrak{g}$ be a nilpotent element. By the Jacobson–Morozov theorem, we can find $h, f \in \mathfrak{g}$ so that (e, h, f) is an $\mathfrak{sl}_2$-triple in $\mathfrak{g}$, i.e. $[h, e] = 2e, [h, f] = -2f$ and $[e, f] = h$. We write $\mathfrak{g}^e$, $\mathfrak{g}^f$ and $\mathfrak{g}^h$ for the centralizers of e, f and h in $\mathfrak{g}$, respectively. Then $\mathfrak{g}^h \cap \mathfrak{g}^e$ is a Levi factor of $\mathfrak{g}^e$. Pick a maximal toral subalgebra $\mathfrak{t}^e$ of this Levi factor, and a maximal toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ containing $\mathfrak{t}^e$ and h. So $\mathfrak{t}^e$ is the centralizer of e in $\mathfrak{t}$. Assume in addition that we are given a *good grading*

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}(j)$$

for e that is compatible with $\mathfrak{t}$, i.e. $e \in \mathfrak{g}(2)$, $\mathfrak{g}^e \subseteq \bigoplus_{j \geq 0} \mathfrak{g}(j)$ and $\mathfrak{t} \subseteq \mathfrak{g}(0)$. Good gradings for e are classified in [EK]; see also [BG]. As $h \in \mathfrak{t}$ we have $h \in \mathfrak{g}(0)$ and, by [BG, Lemma 19], it is automatically the case that $f \in \mathfrak{g}(-2)$. Any element $x \in \mathfrak{g}$ decomposes as $x = \sum_{j \in \mathbb{Z}} x(j)$ with $x(j) \in \mathfrak{g}(j)$; we let $x(<0) := \sum_{j<0} x(j)$ and $x(\geq 0) := \sum_{j \geq 0} x(j)$. From now on, we abbreviate

$$\mathfrak{p} := \bigoplus_{j \geq 0} \mathfrak{g}(j), \quad \mathfrak{m} := \bigoplus_{j \leq -2} \mathfrak{g}(j), \quad \mathfrak{n} := \bigoplus_{j<0} \mathfrak{g}(j), \quad \mathfrak{h} := \mathfrak{g}(0), \quad \mathfrak{k} := \mathfrak{g}(-1).$$

In particular, $\mathfrak{p}$ is a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{h}$ and $\mathfrak{n}$ is the nilradical of the opposite parabolic. If the subspace $\mathfrak{k}$ is non-zero then it is not a subalgebra of $\mathfrak{g}$. If it is zero then the good grading is necessarily even, i.e. $\mathfrak{g}(j) = \{0\}$ for all odd j.

Let $(.|.)$ be a non-degenerate symmetric invariant bilinear form on $\mathfrak{g}$, inducing non-degenerate forms on $\mathfrak{t}$ and $\mathfrak{t}^*$ in the usual way. Define a linear map

$$\chi : \mathfrak{g} \to \mathbb{C}, \qquad x \mapsto (e|x).$$

Also let $\langle .|. \rangle$ be the non-degenerate symplectic form on $\mathfrak{k}$ defined by

$$\langle x|y \rangle := \chi([y, x]).$$

Note that χ restricts to a character of $\mathfrak{m}$. Let I (resp. $\overline{I}$) be the left (resp. right) ideal of $U(\mathfrak{g})$ generated by the elements $\{x - \chi(x) \mid x \in \mathfrak{m}\}$. Set

$$Q := U(\mathfrak{g})/I \qquad (\text{resp. } \overline{Q} := U(\mathfrak{g})/\overline{I}),$$

which is a left (resp. right) $U(\mathfrak{g})$-module by the regular action. The adjoint action of $\mathfrak{n}$ on $U(\mathfrak{g})$ leaves the subspace I (resp. $\overline{I}$) invariant, so induces a well-defined adjoint action of $\mathfrak{n}$ on Q (resp. $\overline{Q}$) such that

$$[x, u + I] := [x, u] + I \qquad (\text{resp. } [x, u + \overline{I}] := [x, u] + \overline{I})$$

for $x \in \mathfrak{n}, u \in U(\mathfrak{g})$. Let $Q^{\mathfrak{n}}$ (resp. $\overline{Q}^{\mathfrak{n}}$) be the corresponding invariant subspace. Then

$$(x - \chi(x))(u + I) = [x, u + I] \qquad (\text{resp. } (u + \overline{I})(x - \chi(x)) = -[x, u + \overline{I}])$$

for all $x \in \mathfrak{m}, u \in U(\mathfrak{g})$. This is all that is needed to check that the multiplication on $U(\mathfrak{g})$ induces a well-defined multiplication on $Q^{\mathfrak{n}}$ (resp. $\overline{Q}^{\mathfrak{n}}$):

$$(u + I)(v + I) := uv + I \qquad (\text{resp. } (u + \overline{I})(v + \overline{I}) := uv + \overline{I})$$

for $u + I, v + I \in Q^{\mathfrak{n}}$ (resp. $u + \overline{I}, v + \overline{I} \in \overline{Q}^{\mathfrak{n}}$). We refer to $Q^{\mathfrak{n}}$ as the *Whittaker model realization* of the finite W-algebra associated to e and the chosen good grading. Up to isomorphism, the algebra $Q^{\mathfrak{n}}$ is known to be independent of the choice of good grading; see [BG, Theorem 1] or [L, Corollary 3.3.3]. Later in the section, we will construct a canonical isomorphism between $Q^{\mathfrak{n}}$ and the right-handed analogue $\overline{Q}^{\mathfrak{n}}$; the existence of such an isomorphism is far from clear at this point.

Remark 2.1. The definition of $Q^{\mathfrak{n}}$ just explained is not quite the same as Premet's definition of the finite W-algebra from [P1]. To explain the connection, we need to fix in addition a Lagrangian subspace $\mathfrak{l}$ of $\mathfrak{k}$ with respect to the form $\langle .|. \rangle$. Note that χ still restricts to a character of $\mathfrak{m} \oplus \mathfrak{l}$ (though it need not restrict to a character of $\mathfrak{n}$). Let $I_{\mathfrak{l}} \supseteq I$ denote the left ideal of $U(\mathfrak{g})$ generated by $\{x - \chi(x) \mid x \in \mathfrak{m} \oplus \mathfrak{l}\}$. Set $Q_{\mathfrak{l}} := U(\mathfrak{g})/I_{\mathfrak{l}}$ and

$$\begin{aligned} Q_{\mathfrak{l}}^{\mathfrak{m}\oplus\mathfrak{l}} : &= \{u + I_{\mathfrak{l}} \in Q_{\mathfrak{l}} \mid [x, u] \in I_{\mathfrak{l}} \text{ for all } x \in \mathfrak{m} \oplus \mathfrak{l}\} \\ &= \{u + I_{\mathfrak{l}} \in Q_{\mathfrak{l}} \mid (x - \chi(x))u \in I_{\mathfrak{l}} \text{ for all } x \in \mathfrak{m} \oplus \mathfrak{l}\}. \end{aligned}$$

Again this inherits a well-defined algebra structure from the multiplication in $U(\mathfrak{g})$; it is even the case that

$$Q_{\mathfrak{l}}^{\mathfrak{m}\oplus\mathfrak{l}} \cong \mathrm{End}_{U(\mathfrak{g})}(Q_{\mathfrak{l}})^{\mathrm{op}}.$$

The algebra $Q_{\mathfrak{l}}^{\mathfrak{m}\oplus\mathfrak{l}}$ is exactly Premet's definition of the finite W-algebra from [P1]. By [GG, Theorem 4.1] the canonical quotient map $Q \twoheadrightarrow Q_{\mathfrak{l}}$ restricts to an algebra isomorphism

$$\nu : Q^{\mathfrak{n}} \xrightarrow{\sim} Q_{\mathfrak{l}}^{\mathfrak{m}\oplus\mathfrak{l}}.$$

Hence our Whittaker model realization is equivalent to Premet's.

2.2. **Definition via non-linear Lie algebras.** The next definition of the finite W-algebra is based on [P2, §2.4], and is the main definition that we will use in the subsequent sections. To formulate it, we will use an easy special case of the notion of a non-linear Lie superalgebra from [DK, Definition 3.1]. For the remainder of this article, a *non-linear Lie superalgebra* means a vector superspace $\mathfrak{a} = \mathfrak{a}_{\bar{0}} \oplus \mathfrak{a}_{\bar{1}}$ equipped with a non-linear Lie bracket $[.,.]$, that is, a parity preserving linear map $\mathfrak{a}\otimes\mathfrak{a} \to T(\mathfrak{a})$ (= the tensor algebra on $\mathfrak{a}$) satisfying the following conditions for all homogeneous $a, b, c \in \mathfrak{a}$:

(1) $[a, b] \in \mathbb{C} \oplus \mathfrak{a}$;
(2) $[a, b] = (-1)^{p(a)p(b)}[b, a]$ (where $p(a) \in \mathbb{Z}_2$ denotes parity);
(3) $[a, [b, c]] = [[a, b], c] + (-1)^{p(a)p(b)}[b, [a, c]]$ (interpreted using the convention that any bracket with a scalar is zero).

This definition agrees with the general notion of non-linear Lie superalgebra from [DK, Definition 3.1] when the grading on $\mathfrak{a}$ in the general setup is concentrated in degree 1.

The *universal enveloping superalgebra* of a non-linear Lie superalgebra $\mathfrak{a}$ is $U(\mathfrak{a}) := T(\mathfrak{a})/M(\mathfrak{a})$ where $M(\mathfrak{a})$ is the two-sided ideal generated by the elements $a\otimes b - (-1)^{p(a)p(b)} b\otimes a - [a, b]$ for all homogeneous $a, b \in \mathfrak{a}$. By a special case of [DK, Theorem 3.3], $U(\mathfrak{a})$ is *PBW generated* by $\mathfrak{a}$ in the sense that if $\{x_i \mid i \in I\}$ is any homogeneous ordered basis of $\mathfrak{a}$ then the ordered monomials

$$\{x_{i_1} \cdots x_{i_s} \mid s \geq 0, i_1 \leq \cdots \leq i_s \text{ and } i_t < i_{t+1} \text{ if } p(x_{i_t}) = \bar{1}\}$$

give a basis for $U(\mathfrak{a})$. By a *subalgebra* of a non-linear Lie superalgebra $\mathfrak{a}$ we mean a graded subspace $\mathfrak{b}$ of $\mathfrak{a}$ such that $[\mathfrak{b}, \mathfrak{b}] \subseteq \mathbb{C}\oplus\mathfrak{b}$. In that case $\mathfrak{b}$ is itself a non-linear Lie superalgebra and $U(\mathfrak{b})$ is identified with the subalgebra of $U(\mathfrak{a})$ generated by $\mathfrak{b}$. We call $\mathfrak{a}$ a *non-linear Lie algebra* if it is purely even.

Now return to the setup of §2.1. Following the language of [DK, §5] and [D^{3}HK], let

$$\mathfrak{k}^{\mathrm{ne}} = \{x^{\mathrm{ne}} \mid x \in \mathfrak{k}\}$$

be a "neutral" copy of $\mathfrak{k}$. We allow ourselves to write x^{ne} for any element $x \in \mathfrak{g}$, meaning $x(-1)^{\mathrm{ne}}$. Make $\mathfrak{k}^{\mathrm{ne}}$ into a non-linear Lie algebra with non-linear Lie bracket defined by

$$[x^{\mathrm{ne}}, y^{\mathrm{ne}}] := \langle x|y\rangle$$

for $x, y \in \mathfrak{k}$, recalling that $\langle x|y\rangle = (e|[y, x])$. Then $U(\mathfrak{k}^{\mathrm{ne}})$ is the Weyl algebra associated to $\mathfrak{k}$ and the symplectic form $\langle.|.\rangle$. Let

$$\widetilde{\mathfrak{g}} := \mathfrak{g} \oplus \mathfrak{k}^{\mathrm{ne}}$$

viewed as a non-linear Lie algebra with bracket obtained by extending the brackets already defined on $\mathfrak{g}$ and $\mathfrak{k}^{\mathrm{ne}}$ to all of $\widetilde{\mathfrak{g}}$ by declaring $[x, y^{\mathrm{ne}}] := 0$ for

$x \in \mathfrak{g}, y \in \mathfrak{k}$. Then $U(\widetilde{\mathfrak{g}}) = U(\mathfrak{g}) \otimes U(\mathfrak{k}^{\mathrm{ne}})$. Also introduce the subalgebra

$$\widetilde{\mathfrak{p}} := \mathfrak{p} \oplus \mathfrak{k}^{\mathrm{ne}}$$

of $\widetilde{\mathfrak{g}}$, whose universal enveloping algebra is identified with $U(\mathfrak{p}) \otimes U(\mathfrak{k}^{\mathrm{ne}})$. For use in §3.2, we record the following crucial lemma which is proved as in [GG, (2.2)].

Lemma 2.2. $\widetilde{\mathfrak{p}} = \mathfrak{g}^e \oplus \bigoplus_{j \geq 2} [f, \mathfrak{g}(j)] \oplus \mathfrak{k}^{\mathrm{ne}}$.

Extend the left (resp. right) regular action of $\mathfrak{g}$ on Q (resp. $\overline{Q}$) to an action of $\widetilde{\mathfrak{g}}$ by setting

$$x^{\mathrm{ne}}(u + I) := ux + I \qquad (\text{resp. } (u + \overline{I})x^{\mathrm{ne}} := xu + \overline{I})$$

for $u \in U(\mathfrak{g})$ and $x \in \mathfrak{k}$. This makes Q into a left $U(\widetilde{\mathfrak{g}})$-module (resp. $\overline{Q}$ into a right $U(\widetilde{\mathfrak{g}})$-module). For $x, y \in \mathfrak{n}$, we have that

$$[x - \chi(x) - x^{\mathrm{ne}}, y - \chi(y) - y^{\mathrm{ne}}] = [x, y] - \chi([x, y]) - [x, y]^{\mathrm{ne}},$$

because $[x, y]^{\mathrm{ne}} = 0$. Hence the map $\mathfrak{n} \to U(\widetilde{\mathfrak{g}})$, $x \mapsto x - \chi(x) - x^{\mathrm{ne}}$ is a Lie algebra homomorphism. So we can make $U(\widetilde{\mathfrak{g}})$ into an $\mathfrak{n}$-module via the *twisted adjoint action* defined by letting $x \in \mathfrak{n}$ act as the derivation $u \mapsto [x - \chi(x) - x^{\mathrm{ne}}, u]$. Since $\chi(x)$ is a scalar, this map can be written more succinctly as $u \mapsto [x - x^{\mathrm{ne}}, u]$.

Lemma 2.3. *The natural multiplication map*

$$U(\widetilde{\mathfrak{g}}) \twoheadrightarrow Q,\ u \mapsto u(1 + I) \qquad (\textit{resp. } U(\widetilde{\mathfrak{g}}) \twoheadrightarrow \overline{Q},\ u \mapsto (1 + \overline{I})u)$$

intertwines the twisted adjoint action of $\mathfrak{n}$ *on* $U(\widetilde{\mathfrak{g}})$ *with the adjoint action of* $\mathfrak{n}$ *on* Q *(resp.* $\overline{Q}$*).*

Proof. We verify this in the left-handed case, the other case being similar. We need to show that $[x - x^{\mathrm{ne}}, u](1 + I) = [x, u(1 + I)]$ for $x \in \mathfrak{n}$ and $u \in U(\widetilde{\mathfrak{g}})$. We may assume that $u = vy_1^{\mathrm{ne}} \cdots y_n^{\mathrm{ne}}$ for $v \in U(\mathfrak{g})$ and $y_1, \ldots, y_n \in \mathfrak{k}$. Then,

$$[x - x^{\mathrm{ne}}, u] = [x, v]y_1^{\mathrm{ne}} \cdots y_n^{\mathrm{ne}} - v[x^{\mathrm{ne}}, y_1^{\mathrm{ne}} \cdots y_n^{\mathrm{ne}}].$$

Acting on $1 + I$, we get that

$$\begin{aligned} [x - x^{\mathrm{ne}}, u](1 + I) &= [x, v]y_n \cdots y_1 + v[x, y_n \cdots y_1] + I \\ &= [x, vy_n \cdots y_1] + I = [x, vy_n \cdots y_1 + I] = [x, u(1 + I)] \end{aligned}$$

as required. □

Let J (resp. $\overline{J}$) be the left (resp. right) ideal of $U(\widetilde{\mathfrak{g}})$ generated by the elements $\{x - \chi(x) - x^{\mathrm{ne}} \mid x \in \mathfrak{n}\}$. By the PBW theorem, we have that

$$U(\widetilde{\mathfrak{g}}) = U(\widetilde{\mathfrak{p}}) \oplus J \qquad (\text{resp. } U(\widetilde{\mathfrak{g}}) = U(\widetilde{\mathfrak{p}}) \oplus \overline{J}).$$

Let $\mathrm{Pr} : U(\widetilde{\mathfrak{g}}) \to U(\widetilde{\mathfrak{p}})$ (resp. $\overline{\mathrm{Pr}} : U(\widetilde{\mathfrak{g}}) \to U(\widetilde{\mathfrak{p}})$) denote the corresponding linear projection. Define

$$\begin{aligned} U(\mathfrak{g}, e) &:= \{u \in U(\widetilde{\mathfrak{p}}) \mid \mathrm{Pr}([x - x^{\mathrm{ne}}, u]) = 0 \text{ for all } x \in \mathfrak{n}\}, \\ \overline{U}(\mathfrak{g}, e) &:= \{u \in U(\widetilde{\mathfrak{p}}) \mid \overline{\mathrm{Pr}}([x - x^{\mathrm{ne}}, u]) = 0 \text{ for all } x \in \mathfrak{n}\}. \end{aligned}$$

The following theorem is very similar to [P2, Proposition 2.2].

Theorem 2.4. *The subspaces $U(\mathfrak{g}, e)$ and $\overline{U}(\mathfrak{g}, e)$ are subalgebras of $U(\widetilde{\mathfrak{p}})$, and the maps*

$$U(\mathfrak{g}, e) \to Q^{\mathfrak{n}},\ u \mapsto u(1+I), \qquad \overline{U}(\mathfrak{g}, e) \to \overline{Q}^{\mathfrak{n}},\ u \mapsto (1+\overline{I})u$$

are well-defined algebra isomorphisms.

Proof. We deal with the left-handed case, the right-handed analogue being similar. Note for $x \in \mathfrak{n}$ that $(x - \chi(x) - x^{\mathrm{ne}})(1+I) \subseteq I$, so $J(1+I) \subseteq I$. Hence $\mathrm{Pr}(u)(1+I) = u(1+I)$ for every $u \in U(\widetilde{\mathfrak{g}})$. Given this, Lemma 2.3 implies that

$$\mathrm{Pr}([x - x^{\mathrm{ne}}, u])(1+I) = [x - x^{\mathrm{ne}}, u](1+I) = [x, u(1+I)]$$

for $x \in \mathfrak{n}$ and $u \in U(\widetilde{\mathfrak{p}})$. By the PBW theorem, the map $U(\widetilde{\mathfrak{p}}) \to Q$, $u \mapsto u(1+I)$ is a vector space isomorphism. Putting our observations together, $u \in U(\widetilde{\mathfrak{p}})$ belongs to $U(\mathfrak{g}, e)$ if and only if $u(1+I)$ belongs to $Q^{\mathfrak{n}}$. Thus the map $U(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} Q$ restricts to a vector space isomorphism $U(\mathfrak{g}, e) \xrightarrow{\sim} Q^{\mathfrak{n}}$.

Now let $z_1, \dots, z_{2s}$ be a basis for $\mathfrak{k}$. For $0 \leq k \leq 2s$, let Ω_k denote the set of all tuples $\boldsymbol{i} = (i_1, \dots, i_k)$ with $1 \leq i_1 < \cdots < i_k \leq 2s$. Take general elements

$$\sum_{\substack{0 \leq k \leq 2s \\ \boldsymbol{i} \in \Omega_k}} u_{\boldsymbol{i}} z_{i_1}^{\mathrm{ne}} \cdots z_{i_k}^{\mathrm{ne}}, \quad \sum_{\substack{0 \leq l \leq 2s \\ \boldsymbol{j} \in \Omega_l}} v_{\boldsymbol{j}} z_{j_1}^{\mathrm{ne}} \cdots z_{j_l}^{\mathrm{ne}} \in U(\mathfrak{g}, e),$$

where $u_{\boldsymbol{i}}, v_{\boldsymbol{j}} \in U(\mathfrak{p})$. The image in Q of their product in $U(\widetilde{\mathfrak{p}})$ is equal to

$$\sum_{\substack{0 \leq k \leq 2s \\ \boldsymbol{i} \in \Omega_k}} \sum_{\substack{0 \leq l \leq 2s \\ \boldsymbol{j} \in \Omega_l}} u_{\boldsymbol{i}} v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1} z_{i_k} \cdots z_{i_1} + I.$$

We claim that this is equal to the product of their images in $Q^{\mathfrak{n}}$, namely,

$$\sum_{\substack{0 \leq k \leq 2s \\ \boldsymbol{i} \in \Omega_k}} \sum_{\substack{0 \leq l \leq 2s \\ \boldsymbol{j} \in \Omega_l}} u_{\boldsymbol{i}} z_{i_k} \cdots z_{i_1} v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1} + I.$$

To see this, note that $\sum v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1} + I$ belongs to $Q^{\mathfrak{n}}$. Hence each commutator $[z_{i_r}, v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1}]$ belongs to I. Also observe that $I z_{i_{r-1}} \cdots z_{i_1} \subseteq I$. Using these two facts applied successively to $r = 1, \dots, k$, we get that

$$z_{i_k} \cdots z_{i_1} \sum_{\substack{0 \leq l \leq 2s \\ \boldsymbol{j} \in \Omega_l}} v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1} + I = \sum_{\substack{0 \leq l \leq 2s \\ \boldsymbol{j} \in \Omega_l}} v_{\boldsymbol{j}} z_{j_l} \cdots z_{j_1} z_{i_k} \cdots z_{i_1} + I$$

and the claim follows.

The claim shows that the product of two elements in $U(\mathfrak{g}, e)$ is again an element of $U(\mathfrak{g}, e)$, because the image of the product lies in $Q^{\mathfrak{n}}$. Hence $U(\mathfrak{g}, e)$ is indeed a subalgebra of $U(\widetilde{\mathfrak{p}})$. At the same time, the claim establishes that our vector space isomorphism is actually an algebra isomorphism. □

We refer to $U(\mathfrak{g}, e)$ simply as the *finite W-algebra* associated to e.

2.3. **Definition via BRST cohomology.** We now turn to the third definition of the W-algebra. This has been proved to be equivalent to the Whittaker model definition above in [D^{3}HK]. Let

$$\mathfrak{n}^{\mathrm{ch}} = \{x^{\mathrm{ch}} \mid x \in \mathfrak{n}\}$$

be a "charged" copy of $\mathfrak{n}$. As before, we allow ourselves to write x^{ch} for any $x \in \mathfrak{g}$, meaning $x(<0)^{\mathrm{ch}}$. Recalling that $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathfrak{k}^{\mathrm{ne}}$, let

$$\widehat{\mathfrak{g}} := \widetilde{\mathfrak{g}} \oplus \mathfrak{n}^* \oplus \mathfrak{n}^{\mathrm{ch}}$$

viewed as a non-linear Lie superalgebra with even part equal to $\widetilde{\mathfrak{g}}$, odd part equal to $\mathfrak{n}^* \oplus \mathfrak{n}^{\mathrm{ch}}$, and non-linear Lie bracket $[.,.]$ defined as follows. It is equal to the non-linear Lie bracket defined above on $\widetilde{\mathfrak{g}}$. It is identically zero on $\mathfrak{n}^*$, $\mathfrak{n}^{\mathrm{ch}}$ or between elements of $\widetilde{\mathfrak{g}}$ and $\mathfrak{n}^* \oplus \mathfrak{n}^{\mathrm{ch}}$. Finally $[f, x^{\mathrm{ch}}] := \langle f, x\rangle$ for $f \in \mathfrak{n}^*, x \in \mathfrak{n}$, where $\langle f, x\rangle$ denotes the natural pairing of $f \in \mathfrak{n}^*$ with $x \in \mathfrak{n}$. We also have the subalgebra

$$\widehat{\mathfrak{p}} := \widetilde{\mathfrak{p}} \oplus \mathfrak{n}^* \oplus \mathfrak{n}^{\mathrm{ch}}$$

of $\widehat{\mathfrak{g}}$. We put the *cohomological grading* on $\widehat{\mathfrak{g}}$, hence also on $\widehat{\mathfrak{p}}$, consistent with the $\mathbb{Z}_2$-grading, by declaring that elements of $\widetilde{\mathfrak{g}}$ are in degree 0, elements of $\mathfrak{n}^*$ are in degree 1, and elements of $\mathfrak{n}^{\mathrm{ch}}$ are in degree -1. It induces gradings

$$U(\widehat{\mathfrak{g}}) = \bigoplus_{i\in\mathbb{Z}} U(\widehat{\mathfrak{g}})^i, \qquad U(\widehat{\mathfrak{p}}) = \bigoplus_{i\in\mathbb{Z}} U(\widehat{\mathfrak{p}})^i.$$

Fix a basis $b_1, \dots, b_r$ for $\mathfrak{n}$ such that b_i lies in the β_i-weight space of $\mathfrak{g}(-d_i)$ for some $\beta_i \in \mathfrak{t}^*$ and $d_i > 0$. Define the structure constants $\gamma_{i,j,k} \in \mathbb{C}$ from

$$[b_i, b_j] = \sum_{k=1}^{r} \gamma_{i,j,k} b_k.$$

Let $f_1, \dots, f_r$ be the dual basis for $\mathfrak{n}^*$. The coadjoint action of $\mathfrak{n}$ on $\mathfrak{n}^*$ defined by $\langle(\mathrm{ad}^* b)(f), b'\rangle = -\langle f, (\mathrm{ad}\, b)(b')\rangle$ for $b, b' \in \mathfrak{n}$ and $f \in \mathfrak{n}^*$ satisfies

$$(\mathrm{ad}^* b_i)(f_j) = \sum_{k=1}^{r} \gamma_{k,i,j} f_k.$$

Let $d : U(\widehat{\mathfrak{g}}) \to U(\widehat{\mathfrak{g}})$ be the superderivation of cohomological degree 1 defined by taking the supercommutator with the odd element

$$\sum_{i=1}^{r} f_i(b_i - \chi(b_i) - b_i^{\mathrm{ne}}) - \tfrac{1}{2} \sum_{i,j=1}^{r} f_i f_j [b_i, b_j]^{\mathrm{ch}}.$$

As in [D^{3}HK], one checks:

$$d(x) = \sum_{i=1}^{r} f_i [b_i, x] \qquad\qquad (x \in \mathfrak{g}),$$

$$d(f) = \tfrac{1}{2} \sum_{i=1}^{r} f_i\, (\mathrm{ad}^* b_i)(f) \qquad\qquad (f \in \mathfrak{n}^*),$$

$$d(x^{\mathrm{ch}}) = x - \chi(x) - x^{\mathrm{ne}} + \sum_{i=1}^{r} f_i [b_i, x]^{\mathrm{ch}} \qquad (x \in \mathfrak{n}),$$
$$d(x^{\mathrm{ne}}) = \sum_{i=1}^{r} f_i \chi([b_i, x]) \qquad (x \in \mathfrak{k}).$$

Using these formulae it is easy to check that $d^2 = 0$, i.e. $(U(\widehat{\mathfrak{g}}), d)$ is a differential graded superalgebra. Let $H^\bullet(U(\widehat{\mathfrak{g}}), d)$ be its cohomology. It is known from [D^{3}HK] that this is concentrated in cohomological degree 0. So

$$H^\bullet(U(\widehat{\mathfrak{g}}), d) = \ker(d : U(\widehat{\mathfrak{g}})^0 \to U(\widehat{\mathfrak{g}})^1) \,\big/\, \mathrm{im}(d : U(\widehat{\mathfrak{g}})^{-1} \to U(\widehat{\mathfrak{g}})^0).$$

Note by the PBW theorem that

$$U(\widehat{\mathfrak{g}})^0 = U(\widetilde{\mathfrak{g}}) \oplus \mathfrak{n}^* U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^{\mathrm{ch}} \qquad (\text{resp. } U(\widehat{\mathfrak{g}})^0 = U(\widetilde{\mathfrak{g}}) \oplus \mathfrak{n}^{\mathrm{ch}} U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^*),$$

with $\mathfrak{n}^* U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^{\mathrm{ch}}$ (resp. $\mathfrak{n}^{\mathrm{ch}} U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^*$) being a two-sided ideal. So we can define a linear map

$$q : U(\widehat{\mathfrak{g}})^0 \twoheadrightarrow Q \qquad (\text{resp. } \overline{q} : U(\widehat{\mathfrak{g}})^0 \twoheadrightarrow \overline{Q})$$

such that $q(u) = u(1 + I)$ (resp. $\overline{q}(u) = (1 + \overline{I})u$) for $u \in U(\widetilde{\mathfrak{g}})$ and

$$\ker q = J \oplus \mathfrak{n}^* U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^{\mathrm{ch}} \qquad (\text{resp. } \ker \overline{q} = \overline{J} \oplus \mathfrak{n}^{\mathrm{ch}} U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^*).$$

By the above explicit formulae for the differential d, it follows that d maps $U(\widehat{\mathfrak{g}})^{-1}$ into $J \oplus \mathfrak{n}^* U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^{\mathrm{ch}}$ (resp. $\overline{J} \oplus \mathfrak{n}^{\mathrm{ch}} U(\widehat{\mathfrak{g}})^0 \mathfrak{n}^*$). So the restriction of q (resp. $\overline{q}$) to $\ker(d : U(\widehat{\mathfrak{g}}^0) \to U(\widehat{\mathfrak{g}})^1)$ induces a well-defined linear map

$$q : H^\bullet(U(\widehat{\mathfrak{g}}), d) \to Q \qquad (\text{resp. } \overline{q} : H^\bullet(U(\widehat{\mathfrak{g}}), d) \to \overline{Q}).$$

In [D^{3}HK], it is proved that q is an *algebra isomorphism* between $H^\bullet(U(\widehat{\mathfrak{g}}), d)$ and $Q^{\mathfrak{n}}$. A similar argument shows that $\overline{q}$ is an algebra isomorphism between $H^\bullet(U(\widehat{\mathfrak{g}}), d)$ and $\overline{Q}^{\mathfrak{n}}$. This already shows that $Q^{\mathfrak{n}} \cong \overline{Q}^{\mathfrak{n}}$ as algebras, though it does not give the isomorphism as explicitly as we would like.

To remedy this, note that the projection along the decomposition

$$U(\widehat{\mathfrak{p}})^0 = U(\widetilde{\mathfrak{p}}) \oplus \mathfrak{n}^* U(\widehat{\mathfrak{p}})^0 \mathfrak{n}^{\mathrm{ch}} \qquad (\text{resp. } U(\widehat{\mathfrak{p}})^0 = U(\widetilde{\mathfrak{p}}) \oplus \mathfrak{n}^{\mathrm{ch}} U(\widehat{\mathfrak{p}})^0 \mathfrak{n}^*)$$

defines a surjective algebra homomorphism

$$p : U(\widehat{\mathfrak{p}})^0 \twoheadrightarrow U(\widetilde{\mathfrak{p}}) \qquad (\text{resp. } \overline{p} : U(\widehat{\mathfrak{p}})^0 \twoheadrightarrow U(\widetilde{\mathfrak{p}})).$$

Moreover the following diagrams commute:

$$\begin{tikzcd} U(\widehat{\mathfrak{p}})^0 \arrow[r] \arrow[d, "p"'] & U(\widehat{\mathfrak{g}})^0 \arrow[d, "q"] \\ U(\widetilde{\mathfrak{p}}) \arrow[r, "\sim"] & Q \end{tikzcd} \qquad \begin{tikzcd} U(\widehat{\mathfrak{p}})^0 \arrow[r] \arrow[d, "\overline{p}"'] & U(\widehat{\mathfrak{g}})^0 \arrow[d, "\overline{q}"] \\ U(\widetilde{\mathfrak{p}}) \arrow[r, "\sim"] & \overline{Q} \end{tikzcd}$$

where the top maps are the inclusions and the bottom maps are the multiplication maps defined like in Lemma 2.3. For the next lemma, recall that the basis element $b_i \in \mathfrak{n}$ is of $\mathfrak{t}$-weight β_i.

Lemma 2.5. $\beta := \sum_{i=1}^r \beta_i \in \mathfrak{t}^*$ *extends uniquely to a character of* $\mathfrak{p}$.

Proof. For $x \in \mathfrak{h}$, the linear map $\operatorname{ad} x : \mathfrak{g} \to \mathfrak{g}$ leaves $\mathfrak{n}$ invariant, so the map

$$\beta : \mathfrak{h} \to \mathbb{C}, \qquad x \mapsto \operatorname{tr}(\operatorname{ad} x|_{\mathfrak{n}})$$

is a well-defined Lie algebra homomorphism. Extending by zero on the nilradical of $\mathfrak{p}$ we get the desired Lie algebra homomorphism $\beta : \mathfrak{p} \to \mathbb{C}$. The uniqueness is clear. $\square$

In view of the lemma, we can define shift automorphisms

$$S_{\pm\beta} : U(\widetilde{\mathfrak{p}}) \to U(\widetilde{\mathfrak{p}}), \quad x \mapsto x \pm \beta(x), \; y^{\mathrm{ne}} \mapsto y^{\mathrm{ne}} \qquad (x \in \mathfrak{p}, y \in \mathfrak{k}).$$

Of course, we have that $(S_{\pm\beta})^{-1} = S_{\mp\beta}$. The definition of the function ϕ in the next lemma is based on a construction of Arakawa [A, 2.5] for regular nilpotents.

Lemma 2.6. *There are well-defined algebra homomorphisms*

$$\phi : U(\widetilde{\mathfrak{p}}) \hookrightarrow U(\widehat{\mathfrak{p}})^0, \qquad \overline{\phi} : U(\widetilde{\mathfrak{p}}) \hookrightarrow U(\widehat{\mathfrak{p}})^0$$

such that

$$\phi(x) = x + \textstyle\sum_i f_i[b_i, x]^{\mathrm{ch}}, \qquad \overline{\phi}(x) = x - \textstyle\sum_i [b_i, x]^{\mathrm{ch}} f_i,$$
$$\phi(y^{\mathrm{ne}}) = y^{\mathrm{ne}}, \qquad \overline{\phi}(y^{\mathrm{ne}}) = y^{\mathrm{ne}},$$

for $x \in \mathfrak{p}$ and $y \in \mathfrak{k}$. Moreover,

(1) $p \circ \phi = \overline{p} \circ \overline{\phi} = \mathrm{id}_{U(\widetilde{\mathfrak{p}})}$*;*
(2) $\overline{\phi} = \phi \circ S_\beta$*.*

Hence ϕ and $\overline{\phi}$ are injective, $p \circ \overline{\phi} = S_\beta$ and $\overline{p} \circ \phi = S_{-\beta}$.

Proof. To see that there is an algebra homomorphism $\phi : U(\widetilde{\mathfrak{p}}) \to U(\widehat{\mathfrak{p}})^0$ defined on generators as in the statement of the lemma, we need to show that $\phi(x)\phi(y) - \phi(y)\phi(x) = \phi([x, y])$ for all $x, y \in \widetilde{\mathfrak{p}}$. If $x, y \in \mathfrak{k}^{\mathrm{ne}}$, the result is clear. If $x \in \mathfrak{p}$ and $y \in \mathfrak{k}^{\mathrm{ne}}$ or vice versa, both sides are obviously zero. It remains to consider the case that $x, y \in \mathfrak{p}$. Then

$$\phi(x)\phi(y) = xy + \textstyle\sum_i x f_i [b_i, y]^{\mathrm{ch}} + \sum_i f_i [b_i, x]^{\mathrm{ch}} y + \sum_{i,j} f_i [b_i, x]^{\mathrm{ch}} f_j [b_j, y]^{\mathrm{ch}},$$
$$\phi(y)\phi(x) = yx + \textstyle\sum_i y f_i [b_i, x]^{\mathrm{ch}} + \sum_i f_i [b_i, y]^{\mathrm{ch}} x + \sum_{i,j} f_j [b_j, y]^{\mathrm{ch}} f_i [b_i, x]^{\mathrm{ch}}.$$

Now note that

$$\begin{aligned} &\textstyle\sum_{i,j} f_i [b_i, x]^{\mathrm{ch}} f_j [b_j, y]^{\mathrm{ch}} \\ &\qquad = -\textstyle\sum_{i,j} f_i f_j [b_i, x]^{\mathrm{ch}} [b_j, y]^{\mathrm{ch}} + \sum_{i,j} f_i \langle f_j, [b_i, x](<0) \rangle [b_j, y]^{\mathrm{ch}} \\ &\qquad = -\textstyle\sum_{i,j} f_i f_j [b_i, x]^{\mathrm{ch}} [b_j, y]^{\mathrm{ch}} + \sum_i f_i [[b_i, x](<0), y]^{\mathrm{ch}} \\ &\qquad = -\textstyle\sum_{i,j} f_i f_j [b_i, x]^{\mathrm{ch}} [b_j, y]^{\mathrm{ch}} + \sum_i f_i [[b_i, x], y]^{\mathrm{ch}}. \end{aligned}$$

Similarly,

$$\begin{aligned} \textstyle\sum_{i,j} f_j [b_j, y]^{\mathrm{ch}} f_i [b_i, x]^{\mathrm{ch}} &= -\textstyle\sum_{i,j} f_j f_i [b_j, y]^{\mathrm{ch}} [b_i, x]^{\mathrm{ch}} + \sum_j f_j [[b_j, y], x]^{\mathrm{ch}} \\ &= -\textstyle\sum_{i,j} f_i f_j [b_i, x]^{\mathrm{ch}} [b_j, y]^{\mathrm{ch}} - \sum_i f_i [x, [b_i, y]]^{\mathrm{ch}}. \end{aligned}$$

Thus, we finally get the required equality:

$$\begin{aligned}\phi(x)\phi(y)-\phi(y)\phi(x) &= [x,y]+\textstyle\sum_i f_i\left([[b_i,x],y]^{\mathrm{ch}}+[x,[b_i,y]]^{\mathrm{ch}}\right)\\ &= [x,y]+\textstyle\sum_i f_i[b_i,[x,y]]^{\mathrm{ch}}=\phi([x,y]).\end{aligned}$$

Now *define* an algebra homomorphism $\overline{\phi}:=\phi\circ S_\beta$. For $y\in\mathfrak{k}$ and $x\in\mathfrak{p}$ we have that

$$\begin{aligned}\overline{\phi}(y^{\mathrm{ne}}) &= \phi(S_\beta(y^{\mathrm{ne}}))=y^{\mathrm{ne}},\\ \overline{\phi}(x) &= \phi(S_\beta(x))=\phi(x+\beta(x))=x+\beta(x)+\textstyle\sum_i f_i[b_i,x]^{\mathrm{ch}}\\ &= x+\beta(x)+\textstyle\sum_i\langle f_i,[b_i,x](<0)\rangle-\sum_i[b_i,x]^{\mathrm{ch}}f_i.\end{aligned}$$

If x belongs to the nilradical of $\mathfrak{p}$, then $\langle f_i,[b_i,x](<0)\rangle=0=-\beta(x)$ by degree considerations. Instead if $x\in\mathfrak{h}$ then

$$\textstyle\sum_i\langle f_i,[b_i,x](<0)\rangle=-\sum_i\langle f_i,(\operatorname{ad}x)(b_i)\rangle=-\operatorname{tr}(\operatorname{ad}x|_{\mathfrak{n}})=-\beta(x),$$

recalling the proof of Lemma 2.5. Hence $\overline{\phi}(x)=x-\sum_i[b_i,x]^{\mathrm{ch}}f_i$ as in the statement of the lemma. Finally the property (1) is obvious. □

Lemma 2.7. *For $u\in U(\widetilde{\mathfrak{p}})$, we have that*

$$d(\phi(u))=\textstyle\sum_i f_i\phi(\operatorname{Pr}([b_i-b_i^{\mathrm{ne}},u])),\quad d(\overline{\phi}(u))=\sum_i\overline{\phi}(\overline{\operatorname{Pr}}([b_i-b_i^{\mathrm{ne}},u]))f_i.$$

Proof. We deal with the left-handed version, the argument for the right-handed analogue being similar. We first check the result for any $u\in U(\mathfrak{k}^{\mathrm{ne}})$ by induction on the natural filtration. The base case is when u is a scalar, which is trivial as both sides are zero. For the induction step, take $u=y^{\mathrm{ne}}v$ for $y\in\mathfrak{k}$ and $v\in U(\mathfrak{k}^{\mathrm{ne}})$. Since $[b_i,v]=0$, the induction hypothesis gives

$$d(\phi(v))=\textstyle\sum_i f_i\phi(\operatorname{Pr}([b_i-b_i^{\mathrm{ne}},v]))=\sum_i f_i\phi(\operatorname{Pr}([-b_i^{\mathrm{ne}},v])).$$

Hence,

$$\begin{aligned}d(\phi(y^{\mathrm{ne}}v)) &= d(\phi(y^{\mathrm{ne}})\phi(v))=d(\phi(y^{\mathrm{ne}}))\phi(v)+\phi(y^{\mathrm{ne}})d(\phi(v))\\ &= \textstyle\sum_i f_i\chi([b_i,y])\phi(v)+\phi(y^{\mathrm{ne}})\sum_i f_i\phi(\operatorname{Pr}([-b_i^{\mathrm{ne}},v]))\\ &= \textstyle\sum_i f_i\phi\big(\chi([b_i,y])v+y^{\mathrm{ne}}\operatorname{Pr}([-b_i^{\mathrm{ne}},v])\big)\\ &= \textstyle\sum_i f_i\phi\big([-b_i^{\mathrm{ne}},y^{\mathrm{ne}}]v+\operatorname{Pr}(y^{\mathrm{ne}}[-b_i^{\mathrm{ne}},v])\big)\\ &= \textstyle\sum_i f_i\phi\big(\operatorname{Pr}([-b_i^{\mathrm{ne}},y^{\mathrm{ne}}]v+y^{\mathrm{ne}}[-b_i^{\mathrm{ne}},v])\big)\\ &= \textstyle\sum_i f_i\phi\big(\operatorname{Pr}([-b_i^{\mathrm{ne}},y^{\mathrm{ne}}v])\big)=\sum_i f_i\phi\big(\operatorname{Pr}([b_i-b_i^{\mathrm{ne}},y^{\mathrm{ne}}v])\big)\end{aligned}$$

as we wanted.

Next we prove the result for $x\in\mathfrak{p}$, when

$$\begin{aligned}d(\phi(x)) &= d(x)+\textstyle\sum_i d(f_i[b_i,x]^{\mathrm{ch}})\\ &= \textstyle\sum_i f_i[b_i,x]+\frac{1}{2}\sum_{i,j}f_j(\operatorname{ad}^*b_j)(f_i)[b_i,x]^{\mathrm{ch}}-\sum_i f_i[b_i,x](<0)\\ &\quad+\textstyle\sum_i f_i\chi([b_i,x])+\sum_i f_i[b_i,x]^{\mathrm{ne}}-\sum_{i,j}f_if_j[b_j,[b_i,x](<0)]^{\mathrm{ch}}\\ &= \textstyle\sum_i f_i[b_i,x](\geq 0)+\sum_i f_i\chi([b_i,x])+\sum_i f_i[b_i,x]^{\mathrm{ne}}\\ &\quad+\textstyle\frac{1}{2}\sum_{i,j}f_j(\operatorname{ad}^*b_j)(f_i)[b_i,x]^{\mathrm{ch}}-\sum_{i,j}f_if_j[b_j,[b_i,x](<0)]^{\mathrm{ch}}\end{aligned}$$

$$
\begin{aligned}
&= \textstyle\sum_i f_i \operatorname{Pr}([b_i,x]) + \frac{1}{2}\sum_{i,j,k} \gamma_{k,j,i} f_j f_k [b_i,x]^{\mathrm{ch}} \\
&\qquad - \textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x](<0)]^{\mathrm{ch}} \\
&= \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i,x])) + \frac{1}{2}\sum_{i,j,k} \gamma_{k,j,i} f_j f_k [b_i,x]^{\mathrm{ch}} \\
&\qquad - \textstyle\sum_{i,j} f_i f_j [b_j,\operatorname{Pr}([b_i,x])]^{\mathrm{ch}} - \sum_{i,j} f_i f_j [b_j,[b_i,x](<0)]^{\mathrm{ch}} \\
&= \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i,x])) + \frac{1}{2}\sum_{i,j,k} \gamma_{k,j,i} f_j f_k [b_i,x]^{\mathrm{ch}} \\
&\qquad - \textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x](\geq 0)]^{\mathrm{ch}} - \sum_{i,j} f_i f_j [b_j,[b_i,x](<0)]^{\mathrm{ch}} \\
&= \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i,x])) + \frac{1}{2}\sum_{i,j,k} \gamma_{j,i,k} f_i f_j [b_k,x]^{\mathrm{ch}} \\
&\qquad - \textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x]]^{\mathrm{ch}}.
\end{aligned}
$$

Finally we observe that

$$
\begin{aligned}
\textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x]]^{\mathrm{ch}} &= \textstyle\sum_{i,j} f_i f_j [b_i,[b_j,x]]^{\mathrm{ch}} + \sum_{i,j} f_i f_j [[b_j,b_i],x]^{\mathrm{ch}} \\
&= -\textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x]]^{\mathrm{ch}} + \sum_{i,j,k} \gamma_{j,i,k} f_i f_j [b_k,x]^{\mathrm{ch}}.
\end{aligned}
$$

Hence

$$\textstyle\sum_{i,j} f_i f_j [b_j,[b_i,x]]^{\mathrm{ch}} = \frac{1}{2}\sum_{i,j,k} \gamma_{j,i,k} f_i f_j [b_k,x]^{\mathrm{ch}}$$

and we have proved that

$$d(\phi(x)) = \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i,x])) = \sum_i f_i \phi(\operatorname{Pr}([b_i - b_i^{\mathrm{ne}},x]))$$

as required.

To finish the proof, we use induction on the standard filtration on $U(\widetilde{\mathfrak{g}})$. Take any $x \in \mathfrak{p}$ and $u \in U(\widetilde{\mathfrak{p}})$. Note that $[\phi(x), f_i] = \sum_j f_j \langle f_i, [b_j,x](<0)\rangle$. Using this and the induction hypothesis, we get that

$$
\begin{aligned}
d(\phi(xu)) &= d(\phi(x)\phi(u)) = d(\phi(x))\phi(u) + \phi(x)d(\phi(u)) \\
&= \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i - b_i^{\mathrm{ne}},x]))\phi(u) + \sum_i \phi(x) f_i \phi(\operatorname{Pr}([b_i - b_i^{\mathrm{ne}},u])) \\
&= \textstyle\sum_i f_i \phi(\operatorname{Pr}([b_i,x])u) + \sum_i f_i \phi(x \operatorname{Pr}([b_i - b_i^{\mathrm{ne}},u])) \\
&\qquad + \textstyle\sum_{i,j} f_i \phi\big(\langle f_j,[b_i,x](<0)\rangle \operatorname{Pr}([b_j - b_j^{\mathrm{ne}},u])\big) \\
&= \textstyle\sum_i f_i \phi\big(\operatorname{Pr}([b_i,x])u + x\operatorname{Pr}([b_i - b_i^{\mathrm{ne}},u]) \\
&\qquad + \operatorname{Pr}([[b_i,x](<0),u]) - \operatorname{Pr}([[b_i,x]^{\mathrm{ne}},u])\big).
\end{aligned}
$$

We are trying to show that this equals

$$\textstyle\sum_i f_i \phi\big(\operatorname{Pr}([b_i - b_i^{\mathrm{ne}},xu])\big) = \sum_i f_i \phi\big(\operatorname{Pr}([b_i,x]u) + \operatorname{Pr}(x[b_i - b_i^{\mathrm{ne}},u])\big).$$

So it remains to show that

$$\operatorname{Pr}([b_i,x]u) - \operatorname{Pr}([b_i,x])u - \operatorname{Pr}([[b_i,x](<0),u]) + \operatorname{Pr}([[b_i,x]^{\mathrm{ne}},u]) = 0.$$

To see this, we expand each term separately:

$$
\begin{aligned}
\operatorname{Pr}([b_i,x]u) &= [b_i,x](\geq 0)u + \operatorname{Pr}([b_i,x](<0)u), \\
-\operatorname{Pr}([b_i,x])u &= -[b_i,x](\geq 0)u - [b_i,x]^{\mathrm{ne}}u - \chi([b_i,x])u, \\
-\operatorname{Pr}([[b_i,x](<0),u]) &= -\operatorname{Pr}([b_i,x](<0)u) + u[b_i,x]^{\mathrm{ne}} + u\chi([b_i,x]), \\
\operatorname{Pr}([[b_i,x]^{\mathrm{ne}},u]) &= [b_i,x]^{\mathrm{ne}}u - u[b_i,x]^{\mathrm{ne}}.
\end{aligned}
$$

Adding these together gives zero, completing the induction step. $\square$

Theorem 2.8. *We have that*

$$U(\mathfrak{g},e) = \{u \in U(\widetilde{\mathfrak{p}}) \mid d(\phi(u)) = 0\}, \quad \overline{U}(\mathfrak{g},e) = \{u \in U(\widetilde{\mathfrak{p}}) \mid d(\overline{\phi}(u)) = 0\}.$$

Moreover, we have that $\ker d = \phi(U(\mathfrak{g},e)) \oplus \operatorname{im} d = \overline{\phi}(\overline{U}(\mathfrak{g},e)) \oplus \operatorname{im} d$.

Proof. As usual, we just prove the left-handed version. By definition,

$$U(\mathfrak{g},e) = \{u \in U(\widetilde{\mathfrak{p}}) \mid \Pr([b_i - b_i^{\mathrm{ne}}, u]) = 0 \text{ for each } i = 1,\dots,r\}.$$

The injectivity of ϕ, implies that $\Pr([b_i - b_i^{\mathrm{ne}}, u]) = 0$ for each $i = 1,\dots,r$ if and only if $\sum_{i=1}^r f_i \phi(\Pr([b_i - b_i^{\mathrm{ne}}, u])) = 0$. By Lemma 2.7, this is precisely the statement that $d(\phi(u)) = 0$. In particular, this shows that $\phi(U(\mathfrak{g},e)) \subseteq \ker d$. Finally, recalling the commutative diagram immediately before Lemma 2.5, we consider the induced commutative diagram

$$\begin{tikzcd} \phi(U(\mathfrak{g},e)) \arrow[r] \arrow[d, "p"'] & H^\bullet(U(\widehat{\mathfrak{g}}), d) \arrow[d, "q"] \\ U(\mathfrak{g},e) \arrow[r, "\sim"] & Q^{\mathfrak{n}} \end{tikzcd}$$

where the top map is the map $u \mapsto u + \ker d$. The left hand map is an isomorphism by Lemma 2.6. We have already observed, the right hand map is an isomorphism by [D^{3}HK]. Hence the top map is an isomorphism too, showing that $\ker d = \phi(U(\mathfrak{g},e)) \oplus \operatorname{im} d$. □

We are now in a position to give the promised explicit isomorphism between $U(\mathfrak{g},e)$ and $\overline{U}(\mathfrak{g},e)$.

Corollary 2.9. *The restrictions of the automorphisms $S_{\pm\beta}$ of $U(\widetilde{\mathfrak{p}})$ define mutually inverse algebra isomorphisms*

$$S_\beta : \overline{U}(\mathfrak{g},e) \xrightarrow{\sim} U(\mathfrak{g},e), \qquad S_{-\beta} : U(\mathfrak{g},e) \xrightarrow{\sim} \overline{U}(\mathfrak{g},e).$$

Proof. By Theorem 2.8, $d(\phi(U(\mathfrak{g},e))) = \{0\}$ and $d(\overline{\phi}(\overline{U}(\mathfrak{g},e))) = \{0\}$. Thus by Lemma 2.6, $d(\phi(S_\beta(\overline{U}(\mathfrak{g},e)))) = \{0\}$. So by Theorem 2.8 again, we have $S_\beta(\overline{U}(\mathfrak{g},e)) \subseteq U(\mathfrak{g},e)$. Similarly, we have $S_{-\beta}(U(\mathfrak{g},e)) \subseteq \overline{U}(\mathfrak{g},e)$. □

Using Corollary 2.9 it is an easy matter to translate statements about $U(\mathfrak{g},e)$ into analogous statements about $\overline{U}(\mathfrak{g},e)$. So for the remainder of the article we will just formulate things in the left-handed case.

3. Associated graded algebras

Finite W-algebras possess two important filtrations. In this section we review the fundamental theorems describing the corresponding associated graded algebras; almost all of these results are due to Premet [P1, P2].

3.1. **Restricted roots.** For $\alpha \in (\mathfrak{t}^e)^*$, let $\mathfrak{g}_\alpha = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_\alpha(j)$ denote the α-weight space of $\mathfrak{g}$ with respect to $\mathfrak{t}^e$. So

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi^e} \mathfrak{g}_\alpha$$

where $\Phi^e \subset (\mathfrak{t}^e)^*$ denotes the set of non-zero weights of $\mathfrak{t}^e$ on $\mathfrak{g}$. Similarly each of the spaces $\mathfrak{p}, \mathfrak{m}, \mathfrak{n}, \mathfrak{h}$ and $\mathfrak{k}$ decomposes into $\mathfrak{t}^e$-weight spaces. In the language of [BG, §2], Φ^e is a *restricted root system.* It is not a root system in the usual sense; for example, for $\alpha \in \Phi^e$ there may be multiples of α other than $\pm\alpha$ that belong to Φ^e. There is an induced restricted root decomposition

$$\mathfrak{g}^e = \mathfrak{g}^e_0 \oplus \bigoplus_{\alpha \in \Phi^e} \mathfrak{g}^e_\alpha$$

of the centralizer $\mathfrak{g}^e$. By [BG, Lemma 13], Φ^e is also the set of non-zero weights of $\mathfrak{t}^e$ on $\mathfrak{g}^e$, so all the subspaces $\mathfrak{g}^e_\alpha = \bigoplus_{j \geq 0} \mathfrak{g}^e_\alpha(j)$ in this decomposition are non-zero. Moreover $\mathfrak{g}^e_\alpha(j)$ is of the same dimension as $\mathfrak{g}^e_{-\alpha}(j)$; if $j = 0$ this dimension is either 0 or 1, but for $j > 0$ the space $\mathfrak{g}^e_\alpha(j)$ can be bigger.

Recall that $\mathfrak{t}^e$ is contained in $\mathfrak{h}^e = \mathfrak{g}^e(0)$. We define a *dot action* of $\mathfrak{h}^e$, hence also of $\mathfrak{t}^e$, on the vector space $\widehat{\mathfrak{g}}$ by setting

$$t \cdot x := [t, x], \quad t \cdot y^{\mathrm{ne}} := [t, y]^{\mathrm{ne}}, \quad t \cdot z^{\mathrm{ch}} := [t, z]^{\mathrm{ch}}, \quad \langle t \cdot f, z \rangle := -\langle f, [t, z] \rangle$$

for $t \in \mathfrak{h}^e, x \in \mathfrak{g}, y \in \mathfrak{k}, z \in \mathfrak{n}$ and $f \in \mathfrak{n}^*$. Using the following lemma, it is routine to check that this extends uniquely to an action of $\mathfrak{t}^e$ on $U(\widehat{\mathfrak{g}})$ by derivations. Moreover the dot action leaves all the subspaces $\widehat{\mathfrak{p}}, \widetilde{\mathfrak{p}}$ and $\mathfrak{g}^e$ invariant, so there are induced dot actions on $U(\widehat{\mathfrak{p}})$, $U(\widetilde{\mathfrak{p}})$ and $U(\mathfrak{g}^e)$ too.

Lemma 3.1. *The adjoint action of $\mathfrak{h}^e$ on $\mathfrak{k}$ preserves the form $\langle .|. \rangle$.*

Proof. We calculate

$$\begin{aligned} \langle [t, x] | y \rangle + \langle x | [t, y] \rangle &= \chi([y, [t, x]] + [[t, y], x]) = \chi([t, [y, x]]) \\ &= (e | [t, [y, x]]) = ([e, t] | [y, x]) = 0, \end{aligned}$$

for $t \in \mathfrak{h}^e$ and $x, y \in \mathfrak{k}$. □

Let $\mathbb{Z}\Phi^e$ denote the $\mathbb{Z}$-submodule of $(\mathfrak{t}^e)^*$ generated by Φ^e. Using the dot action of $\mathfrak{t}^e$ we can also decompose all of the universal enveloping (super)algebras $U(\widehat{\mathfrak{g}})$, $U(\widehat{\mathfrak{p}})$, $U(\widetilde{\mathfrak{p}})$ and $U(\mathfrak{g}^e)$ into weight spaces. For example:

$$U(\widetilde{\mathfrak{p}}) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} U(\widetilde{\mathfrak{p}})_\alpha$$

where $U(\widetilde{\mathfrak{p}})_\alpha = \{u \in U(\widetilde{\mathfrak{p}}) \mid t \cdot u = \alpha(t) u \text{ for all } t \in \mathfrak{t}^e\}$. Recall the W-algebra $U(\mathfrak{g}, e)$ is a subalgebra of $U(\widetilde{\mathfrak{p}})$.

Lemma 3.2. *The dot action of $\mathfrak{h}^e$ on $U(\widetilde{\mathfrak{p}})$ leaves $U(\mathfrak{g}, e)$ invariant.*

Proof. We first check that $t \cdot J \subseteq J$ for $t \in \mathfrak{h}^e$. For this we need to show for any $x \in \mathfrak{n}$ that $t \cdot (x - \chi(x) - x^{\mathrm{ne}})$ belongs to J. We have that $\chi([t, x]) = (e | [t, x]) = ([e, t] | x) = 0$ as t centralizes e. Hence,

$$t \cdot (x - \chi(x) - x^{\mathrm{ne}}) = [t, x] - [t, x]^{\mathrm{ne}} = [t, x] - \chi([t, x]) - [t, x]^{\mathrm{ne}} \in J.$$

Now to prove the lemma, we take $u \in U(\mathfrak{g}, e)$, so $[x - x^{\mathrm{ne}}, u] \in J$ for all $x \in \mathfrak{n}$. We need to show for $t \in \mathfrak{h}^e$ that $[x - x^{\mathrm{ne}}, t \cdot u] \in J$ for all $x \in \mathfrak{n}$ too. Using the Leibniz rule, we have that

$$t \cdot [x - x^{\mathrm{ne}}, u] = [t \cdot (x - x^{\mathrm{ne}}), u] + [x - x^{\mathrm{ne}}, t \cdot u] = [[t, x] - [t, x]^{\mathrm{ne}}, u] + [x - x^{\mathrm{ne}}, t \cdot u].$$

As $[t,x] \in \mathfrak{n}$, we know already that $[[t,x]-[t,x]^{\mathrm{ne}},u] \in J$. As $[x-x^{\mathrm{ne}},u] \in J$, we deduce that $t\cdot[x-x^{\mathrm{ne}},u] \in J$ too. Hence $[x-x^{\mathrm{ne}}, t\cdot u] \in J$ as required. □

Hence the dot action of $\mathfrak{t}^e$ induces a *restricted root space decomposition* of $U(\mathfrak{g},e)$:

$$U(\mathfrak{g},e) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} U(\mathfrak{g},e)_\alpha.$$

Actually, we can do rather better: the following theorem due to Premet constructs a canonical embedding of $\mathfrak{t}^e$ into $U(\mathfrak{g},e)$ such that the dot action of $\mathfrak{t}^e$ on $U(\mathfrak{g},e)$ coincides with its adjoint action via this embedding. This means that $\mathfrak{t}^e$ also acts on any $U(\mathfrak{g},e)$-module; we will use this later on to define weight space decompositions of $U(\mathfrak{g},e)$-modules compatible with the above restricted root space decomposition. To formulate the theorem following [P2, §2.5], let $z_1,\dots,z_{2s}$ be a symplectic basis for $\mathfrak{k}$, so that $\langle z_i|z_j^*\rangle = \delta_{i,j}$ for all $1 \le i,j \le 2s$ where

$$z_j^* := \begin{cases} z_{j+s} & \text{for } j=1,\dots,s,\\ -z_{j-s} & \text{for } j=s+1,\dots,2s.\end{cases}$$

Theorem 3.3. *There is a $\mathfrak{t}^e$-equivariant linear map $\theta : \mathfrak{g}^e \hookrightarrow U(\widetilde{\mathfrak{p}})$ such that*

$$\theta(x) = \begin{cases} x + \frac{1}{2}\sum_{i=1}^{2s}[x,z_i^*]^{\mathrm{ne}} z_i^{\mathrm{ne}} & \textit{if } x \in \mathfrak{h}^e,\\ x & \textit{otherwise,}\end{cases}$$

for each homogeneous element $x \in \mathfrak{g}^e$. The map θ does not depend on the choice of basis $z_1,\dots,z_{2s}$. Moreover:

1. *$[\theta(x),u] = x\cdot u$ for any $x \in \mathfrak{h}^e$ and $u \in U(\widetilde{\mathfrak{p}})$;*
2. *$[\theta(x),\theta(y)] = \theta([x,y])$ for each $x,y \in \mathfrak{g}^e$;*
3. *$\theta(x)$ belongs to $U(\mathfrak{g},e)$ for every $x \in \mathfrak{h}^e$.*

Proof. This is proved in [P2, §2.5] but we repeat the details since our setup is slightly different. It is routine that θ is independent of the choice of the symplectic basis. Moreover, as the symplectic form on $\mathfrak{k}$ is $\mathfrak{t}^e$-invariant, we may choose the symplectic basis to be a $\mathfrak{t}^e$-weight basis with the weight of z_i being the negative of the weight of z_i^* for each i. It is then clear that θ is $\mathfrak{t}^e$-equivariant.

We next check (1) for a fixed $x \in \mathfrak{h}^e$. Both the maps $u \mapsto [\theta(x),u]$ and $u \mapsto x\cdot u$ are derivations, so it suffices to check that $[\theta(x),y] = x\cdot y$ for each $y \in \mathfrak{p}$ and that $[\theta(x),(z_j^*)^{\mathrm{ne}}] = x\cdot(z_j^*)^{\mathrm{ne}}$ for each $j=1,\dots,2s$. The first of these statements is obvious, since both $[\theta(x),y]$ and $x\cdot y$ equal $[x,y]$. For the second, we calculate using Lemma 3.1:

$$\begin{aligned}[\theta(x),(z_j^*)^{\mathrm{ne}}] &= \tfrac{1}{2}\textstyle\sum_i \langle [x,z_i^*]|z_j^*\rangle z_i^{\mathrm{ne}} + \tfrac{1}{2}[x,z_j^*]^{\mathrm{ne}}\\ &= \tfrac{1}{2}\textstyle\sum_i \langle [x,z_j^*]|z_i^*\rangle z_i^{\mathrm{ne}} + \tfrac{1}{2}[x,z_j^*]^{\mathrm{ne}} = [x,z_j^*]^{\mathrm{ne}}.\end{aligned}$$

For (2), take homogeneous $x,y \in \mathfrak{g}^e$. If x or y belongs to $\mathfrak{g}^e(j)$ for some $j>0$ then (2) is obvious so assume $x,y \in \mathfrak{h}^e$. Then using (1) we get

$$[\theta(x),\theta(y)] = x\cdot\theta(y) = [x,y] + \tfrac{1}{2}\textstyle\sum_i [x,[y,z_i^*]]^{\mathrm{ne}} z_i^{\mathrm{ne}} + \tfrac{1}{2}\sum_i [y,z_i^*]^{\mathrm{ne}}[x,z_i]^{\mathrm{ne}}.$$

Now we simplify the last term:

$$\begin{aligned}\textstyle\sum_i [y,z_i^*]^{\mathrm{ne}}[x,z_i]^{\mathrm{ne}} &= -\textstyle\sum_i [y,z_i]^{\mathrm{ne}}[x,z_i^*]^{\mathrm{ne}}\\ &= -\textstyle\sum_{i,j}\langle [x,z_i^*]|z_j^*\rangle [y,z_i]^{\mathrm{ne}} z_j^{\mathrm{ne}}\\ &= -\textstyle\sum_{i,j}\langle [x,z_j^*]|z_i^*\rangle [y,z_i]^{\mathrm{ne}} z_j^{\mathrm{ne}}\\ &= -\textstyle\sum_i [y,[x,z_i^*]]^{\mathrm{ne}} z_i^{\mathrm{ne}}\\ &= \textstyle\sum_i [[x,y],z_i^*]^{\mathrm{ne}} z_i^{\mathrm{ne}} - \sum_i [x,[y,z_i^*]]^{\mathrm{ne}} z_i^{\mathrm{ne}}.\end{aligned}$$

Thus

$$[\theta(x),\theta(y)] = [x,y] + \tfrac{1}{2}\textstyle\sum_i [[x,y],z_i^*]^{\mathrm{ne}} z_i^{\mathrm{ne}} = \theta([x,y]).$$

To check (3), take $x \in \mathfrak{h}^e$. We need to show that $[y - y^{\mathrm{ne}}, \theta(x)] \in J$ for all $y \in \mathfrak{n}$. If $y \in \mathfrak{m}$, then $[y - y^{\mathrm{ne}}, \theta(x)] = [y,x] \in J$, as $\chi([y,x]) = 0$ using the fact that $x \in \mathfrak{g}^e$. If instead $y \in \mathfrak{k}$, then using (1) we have that

$$\begin{aligned}[y - y^{\mathrm{ne}}, \theta(x)] &= [y,x] + [\theta(x), y^{\mathrm{ne}}] = [y,x] + x\cdot y^{\mathrm{ne}}\\ &= [y,x] + [x,y]^{\mathrm{ne}} = [y,x] - \chi([y,x]) - [y,x]^{\mathrm{ne}}\end{aligned}$$

which does belong to J. $\square$

Remark 3.4. One can modify the map θ in Theorem 3.3 by composing with an automorphism of $U(\widetilde{\mathfrak{p}})$ of the form $x \mapsto x+\eta(x), y^{\mathrm{ne}} \mapsto y^{\mathrm{ne}}$ for some character η of $\mathfrak{p}$. Providing the character δ in §4.1 below is replaced by $\delta - \eta$, all subsequent results remain true exactly as formulated for θ modified in this way. This extra degree of freedom will be useful when we discuss type A in §5.2 below.

3.2. **The Kazhdan filtration.** Now we introduce the first important filtration, working to start with in terms of the Whittaker model realization $Q^{\mathfrak{n}}$ of the W-algebra following [GG]. The *Kazhdan filtration*

$$\cdots \subseteq \mathrm{F}_i\, U(\mathfrak{g}) \subseteq \mathrm{F}_{i+1}\, U(\mathfrak{g}) \subseteq \cdots$$

on $U(\mathfrak{g})$ is defined by declaring that $x \in \mathfrak{g}(j)$ is of *Kazhdan degree* $(j+2)$. So $\mathrm{F}_i\, U(\mathfrak{g})$ is the span of the monomials $x_1 \cdots x_n$ for $n \geq 0$ and $x_1 \in \mathfrak{g}(j_1)$, $\dots$, $x_n \in \mathfrak{g}(j_n)$ such that $(j_1+2)+\cdots+(j_n+2) \leq i$. The associated graded algebra $\mathrm{gr}\, U(\mathfrak{g})$ is the symmetric algebra $S(\mathfrak{g})$ viewed as a graded algebra via the *Kazhdan grading* on $\mathfrak{g}$ in which $x \in \mathfrak{g}(j)$ is of degree $(j+2)$.

The Kazhdan filtration on $U(\mathfrak{g})$ induces a filtration on the left ideal I and on the quotient $Q = U(\mathfrak{g})/I$ such that $\mathrm{gr}\, Q = S(\mathfrak{g})/\mathrm{gr}\, I$. We use the bilinear form $(.|.)$ to identify $S(\mathfrak{g})$ with the algebra $\mathbb{C}[\mathfrak{g}]$ of regular functions on the affine variety $\mathfrak{g}$. Then $\mathrm{gr}\, I$ is the ideal generated by the functions $\{x - \chi(x) \mid x \in \mathfrak{m}\}$, i.e. the ideal of all functions in $\mathbb{C}[\mathfrak{g}]$ vanishing on the closed subvariety $e + \mathfrak{m}^{\perp}$ of $\mathfrak{g}$. In this way, we have identified

$$\mathrm{gr}\, Q = \mathbb{C}[e + \mathfrak{m}^{\perp}].$$

Let N be the closed subgroup of G corresponding to the subalgebra $\mathfrak{n}$. The adjoint action of N on $\mathfrak{g}$ leaves $e + \mathfrak{m}^{\perp}$ invariant, so we get induced an action of N on $\mathbb{C}[e + \mathfrak{m}^{\perp}]$ by automorphisms; the resulting action of $\mathfrak{n}$ on $\mathbb{C}[e + \mathfrak{m}^{\perp}]$ by derivations coincides under the above identification with the action of $\mathfrak{n}$ on

$\operatorname{gr} Q$ induced by the adjoint action of $\mathfrak{n}$ on Q itself. By [GG, Lemma 2.1], the action of N on $e + \mathfrak{m}^\perp$ is regular and the *Slodowy slice*

$$e + \mathfrak{g}^f \subseteq e + \mathfrak{m}^\perp$$

gives a set of representatives for the orbits of N on $e + \mathfrak{m}^\perp$. So restriction of functions defines an isomorphism from the invariant subalgebra $\mathbb{C}[e+\mathfrak{m}^\perp]^N$ onto $\mathbb{C}[e + \mathfrak{g}^f]$. Finally, the Kazhdan filtration on Q induces an algebra filtration on $Q^{\mathfrak{n}}$ so that $\operatorname{gr}(Q^{\mathfrak{n}})$ is identified with a graded subalgebra of $\operatorname{gr} Q = \mathbb{C}[e + \mathfrak{m}^\perp]$. Now the *PBW theorem* for $Q^{\mathfrak{n}}$ [GG, Theorem 4.1] gives:

$$\operatorname{gr}(Q^{\mathfrak{n}}) = \mathbb{C}[e + \mathfrak{m}^\perp]^N \cong \mathbb{C}[e + \mathfrak{g}^f].$$

The next goal is to reformulate this PBW theorem for $U(\mathfrak{g}, e)$ directly. To do this, extend the Kazhdan filtration on $U(\mathfrak{g})$ to $U(\widetilde{\mathfrak{g}})$ by declaring that elements of $\mathfrak{k}^{\mathrm{ne}}$ are of degree 1. Setting $\mathrm{F}_i\, U(\widetilde{\mathfrak{p}}) := U(\widetilde{\mathfrak{p}}) \cap \mathrm{F}_i\, U(\widetilde{\mathfrak{g}})$, we get an induced Kazhdan filtration on the subalgebra $U(\widetilde{\mathfrak{p}})$. We can obviously identify $\operatorname{gr} U(\widetilde{\mathfrak{g}})$ and $\operatorname{gr} U(\widetilde{\mathfrak{p}})$ with $S(\widetilde{\mathfrak{g}})$ and $S(\widetilde{\mathfrak{p}})$, both graded via the analogous *Kazhdan grading* in which $x \in \mathfrak{g}(j)$ is of Kazhdan degree $j + 2$ and $y^{\mathrm{ne}} \in \mathfrak{k}^{\mathrm{ne}}$ is of Kazhdan degree 1. The Kazhdan grading on $\widetilde{\mathfrak{p}}$ only involves positive degrees, so the Kazhdan filtration on $U(\widetilde{\mathfrak{p}})$ is strictly positive in the sense that $\mathrm{F}_0\, U(\widetilde{\mathfrak{p}}) = \mathbb{C}$ and $\mathrm{F}_i\, U(\widetilde{\mathfrak{p}}) = 0$ for $i < 0$. We get an induced strictly positive filtration

$$\mathrm{F}_0\, U(\mathfrak{g}, e) \subseteq \mathrm{F}_1\, U(\mathfrak{g}, e) \subseteq \cdots$$

on $U(\mathfrak{g}, e)$ such that $\operatorname{gr} U(\mathfrak{g}, e)$ is canonically identified with a graded subalgebra of $\operatorname{gr} U(\widetilde{\mathfrak{p}}) = S(\widetilde{\mathfrak{p}})$. The point now is that there is a commutative diagram:

$$\begin{tikzcd}
& \operatorname{gr} U(\widetilde{\mathfrak{g}}) = S(\widetilde{\mathfrak{g}}) \arrow[d, two heads, "\mathrm{pr}"'] \arrow[dr, two heads] & & \\
\operatorname{gr} U(\mathfrak{g}, e) = S(\widetilde{\mathfrak{p}})^{\mathfrak{n}} \arrow[r, hook] \arrow[dr, "\sim"'] & \operatorname{gr} U(\widetilde{\mathfrak{p}}) = S(\widetilde{\mathfrak{p}}) \arrow[r, "\sim"] \arrow[d, "\zeta"'] & \mathbb{C}[e + \mathfrak{m}^\perp] \arrow[d, "\mathrm{res}"] & \mathbb{C}[e + \mathfrak{m}^\perp]^N \arrow[l, hook'] \arrow[dl, "\sim"] \\
& S(\mathfrak{g}^e) \arrow[r, "\sim"] & \mathbb{C}[e + \mathfrak{g}^f] &
\end{tikzcd}$$

There are several things still to be explained in this diagram. The algebra homomorphisms $S(\widetilde{\mathfrak{g}}) \twoheadrightarrow \mathbb{C}[e + \mathfrak{m}^\perp]$ and $S(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^\perp]$ are induced by the multiplication maps $U(\widetilde{\mathfrak{g}}) \twoheadrightarrow Q, u \mapsto u(1 + I)$ and $U(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} Q, u \mapsto u(1 + I)$, on passing to the associated graded objects. Explicitly, they send $x \in \mathfrak{g}$ or $\mathfrak{p}$ to the function $z \mapsto (x|z)$ and $y^{\mathrm{ne}} \in \mathfrak{k}^{\mathrm{ne}}$ to the function $z \mapsto (y|z)$.

The homomorphism $\mathrm{pr} : S(\widetilde{\mathfrak{g}}) \twoheadrightarrow S(\widetilde{\mathfrak{p}})$ is induced by $\mathrm{Pr} : U(\widetilde{\mathfrak{g}}) \twoheadrightarrow U(\widetilde{\mathfrak{p}})$ on passing to the associated graded algebras. Explicitly, pr is the identity on elements of $\widetilde{\mathfrak{p}}$ and maps $x \in \mathfrak{n}$ to $\chi(x) + x^{\mathrm{ne}}$. The top triangle in the diagram commutes even before passing to the associated graded objects, as we observed that $\mathrm{Pr}(u)(1 + I) = u(1 + I)$ for $u \in U(\widetilde{\mathfrak{g}})$ in the proof of Theorem 2.4.

The twisted adjoint action of $\mathfrak{n}$ on $U(\widetilde{\mathfrak{g}})$ induces a graded action of $\mathfrak{n}$ on $S(\widetilde{\mathfrak{g}})$ by derivations, such that $x \in \mathfrak{n}$ maps $y \in \widetilde{\mathfrak{g}}$ to $[x - x^{\mathrm{ne}}, y]$. This action factors through the map pr to induce an action of $\mathfrak{n}$ on $S(\widetilde{\mathfrak{p}})$ by derivations, such that $x \in \mathfrak{n}$ maps $y \in \widetilde{\mathfrak{p}}$ to $\mathrm{pr}([x - x^{\mathrm{ne}}, y])$. We let $S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$ denote the invariant subalgebra for this action.

By Lemma 2.3, the map $S(\widetilde{\mathfrak{g}}) \twoheadrightarrow \mathbb{C}[e + \mathfrak{m}^{\perp}]$ intertwines the twisted adjoint action of $\mathfrak{n}$ on $S(\widetilde{\mathfrak{g}})$ with the action of $\mathfrak{n}$ on $\mathbb{C}[e + \mathfrak{m}^{\perp}]$ derived from the action of the group N. Hence the map $S(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^{\perp}]$ restricts to an isomorphism $S(\widetilde{\mathfrak{p}})^{\mathfrak{n}} \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^{\perp}]^N$. By Theorem 2.4 and the PBW theorem for $Q^{\mathfrak{n}}$, the isomorphism $S(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^{\perp}]$ maps $\mathrm{gr}\, U(\mathfrak{g}, e)$ isomorphically onto $\mathrm{gr}(Q^{\mathfrak{n}}) = \mathbb{C}[e + \mathfrak{m}^{\perp}]^N$. This shows that $\mathrm{gr}\, U(\mathfrak{g}, e) = S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$.

Let $\zeta : S(\widetilde{\mathfrak{p}}) \twoheadrightarrow S(\mathfrak{g}^e)$ be the homomorphism induced by the projection $\widetilde{\mathfrak{p}} \twoheadrightarrow \mathfrak{g}^e$ along the decomposition from Lemma 2.2. We have that

$$([f, x]|z) = (x|[z, f]) = (x|[e, f]) = (x|h) = 0$$

for all $x \in \bigoplus_{j \geq 1} \mathfrak{g}(j)$ and $z \in e + \mathfrak{g}^f$. So the image of any element of $\ker \zeta$ under the isomorphism $S(\widetilde{\mathfrak{p}}) \xrightarrow{\sim} \mathbb{C}[e + \mathfrak{m}^{\perp}]$ annihilates $e + \mathfrak{g}^f$. Hence there is an induced homomorphism $S(\mathfrak{g}^e) \to \mathbb{C}[e + \mathfrak{g}^f]$ making the bottom square in the diagram commute. This homomorphism maps $x \in \mathfrak{g}^e$ to the function $z \mapsto (x|z)$, from which it is easy to check that it is actually an isomorphism.

Finally, we have already noted that the map res sends $\mathbb{C}[e + \mathfrak{m}^{\perp}]^N$ isomorphically onto the coordinate algebra $\mathbb{C}[e + \mathfrak{g}^f]$ of the Slodowy slice. So the restriction of ζ gives an isomorphism $S(\widetilde{\mathfrak{p}})^{\mathfrak{n}} \xrightarrow{\sim} S(\mathfrak{g}^e)$. This completes the justification of the above diagram, and we have now derived the following convenient reformulation of [GG, Theorem 4.1]:

Lemma 3.5. *We have that* $\mathrm{gr}\, U(\mathfrak{g}, e) = S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$. *Moreover, the restriction of* ζ *is an isomorphism of graded algebras*

$$\zeta : S(\widetilde{\mathfrak{p}})^{\mathfrak{n}} \xrightarrow{\sim} S(\mathfrak{g}^e),$$

where the grading on $S(\mathfrak{g}^e)$ *is induced by the Kazhdan grading on* $\mathfrak{g}^e$.

The dot action of $\mathfrak{t}^e$ on $\widetilde{\mathfrak{p}}$ extends to an action on $S(\widetilde{\mathfrak{p}})$ by derivations, which coincides with the action induced by the dot action of $\mathfrak{t}^e$ on $U(\widetilde{\mathfrak{p}})$ on passing to the associated graded algebra. Hence the $\mathfrak{t}^e$-weight space decomposition

$$S(\widetilde{\mathfrak{p}}) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} S(\widetilde{\mathfrak{p}})_\alpha$$

satisfies $\mathrm{gr}(U(\widetilde{\mathfrak{p}})_\alpha) = S(\widetilde{\mathfrak{p}})_\alpha$. In view of Lemma 3.2, $U(\mathfrak{g}, e)$ is a $\mathfrak{t}^e$-submodule of $U(\widetilde{\mathfrak{p}})$, hence $\mathrm{gr}\, U(\mathfrak{g}, e) = S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$ is a $\mathfrak{t}^e$-submodule of $S(\widetilde{\mathfrak{p}})$ and the induced decomposition

$$\mathrm{gr}\, U(\mathfrak{g}, e) = \bigoplus_{\alpha \in \mathbb{Z}\Phi^e} S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}_\alpha$$

satisfies $\mathrm{gr}(U(\mathfrak{g}, e)_\alpha) = S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}_\alpha$. Now we can deduce the following *PBW theorem* for $U(\mathfrak{g}, e)$, which is essentially [P1, Theorem 4.6]. We remark that the original proof in [P1] involved lifting from characteristic p, whereas we are deducing it ultimately from [GG, Theorem 4.1].

Theorem 3.6. *There exists a (non-unique)* $\mathfrak{t}^e$*-equivariant linear map*

$$\Theta : \mathfrak{g}^e \hookrightarrow U(\mathfrak{g}, e)$$

such that $\Theta(x) \in \mathrm{F}_{j+2}\, U(\mathfrak{g}, e)$ and $\zeta(\mathrm{gr}_{j+2}\,\Theta(x)) = x$ for each $x \in \mathfrak{g}^e(j)$, and $\Theta(t) = \theta(t)$ for each $t \in \mathfrak{h}^e$. Moreover, if $x_1, \dots, x_t$ is a homogeneous basis of $\mathfrak{g}^e$ with $x_i \in \mathfrak{g}^e(n_i)$ then the monomials

$$\{\Theta(x_{i_1}) \dots \Theta(x_{i_k}) \mid k \geq 0, 1 \leq i_1 \leq \dots \leq i_k \leq t, n_{i_1} + \dots + n_{i_k} + 2k \leq j\}$$

form a basis for $\mathrm{F}_j\, U(\mathfrak{g}, e)$ $(j \geq 0)$.

Proof. Let $x_1, \dots, x_t$ be a basis for $\mathfrak{g}^e$ with $x_i \in \mathfrak{g}(n_i)$ of $\mathfrak{t}^e$-weight γ_i. As the decomposition from Lemma 2.2 is a direct sum of $\mathfrak{t}^e$-modules, the isomorphism $\zeta : S(\widetilde{\mathfrak{p}})^{\mathfrak{n}} \xrightarrow{\sim} S(\mathfrak{g}^e)$ from Lemma 3.5 is a $\mathfrak{t}^e$-equivariant isomorphism of Kazhdan graded vector spaces. For each i, let $\hat{x}_i \in S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}_{\gamma_i}$ be the unique element with $\zeta(\hat{x}_i) = x_i$. So $\hat{x}_i$ belongs to $\mathrm{gr}_{n_i+2}(U(\mathfrak{g}, e)_{\gamma_i})$. Hence there is a lift $\Theta(x_i) \in \mathrm{F}_{n_i+2}\, U(\mathfrak{g}, e)_{\gamma_i}$ with $\mathrm{gr}_{n_i+2}\,\Theta(x_i) = \hat{x}_i$, i.e. $\zeta(\mathrm{gr}_{n_i+2}\,\Theta(x_i)) = x_i$. Moreover if $n_i = 0$ then by Lemma 3.3(3) we know that $\theta(x_i) \in \mathrm{F}_2\, U(\mathfrak{g}, e)_{\gamma_i}$ and $\zeta(\mathrm{gr}_2\,\theta(x_i)) = x_i$, so we can choose $\Theta(x_i)$ to be $\theta(x_i)$. Now extend by linearity to obtain the desired map Θ. The final statement then follows immediately from Lemma 3.5. $\square$

3.3. **The good filtration.** Now we turn our attention to the second filtration on $U(\mathfrak{g}, e)$, which we refer to as the *good filtration*. To define it, the good grading on $\mathfrak{p}$ induces a grading on $U(\mathfrak{p})$. We extend this to the *good grading*

$$U(\widetilde{\mathfrak{p}}) = \bigoplus_{j \geq 0} U(\widetilde{\mathfrak{p}})(j)$$

on $U(\widetilde{\mathfrak{p}})$ by declaring that elements of $\mathfrak{k}^{\mathrm{ne}}$ are of degree 0. The subalgebra $U(\mathfrak{g}, e)$ is *not* a graded subalgebra in general, but the good grading at least induces the *good filtration*

$$\mathrm{F}'_0\, U(\mathfrak{g}, e) \subseteq \mathrm{F}'_1\, U(\mathfrak{g}, e) \subseteq \cdots$$

on $U(\mathfrak{g}, e)$, where $\mathrm{F}'_j\, U(\mathfrak{g}, e) = U(\mathfrak{g}, e) \cap \bigoplus_{i \leq j} U(\widetilde{\mathfrak{p}})(i)$. The associated graded algebra $\mathrm{gr}'\, U(\mathfrak{g}, e)$ is then canonically identified with a graded subalgebra of $U(\widetilde{\mathfrak{p}})$. Our goal is to identify this associated graded algebra with $U(\mathfrak{g}^e)$, a result which is a slight variation on [P2, Proposition 2.1].

Lemma 3.7. *Let θ and Θ be as in Theorems 3.3 and 3.6. For $x \in \mathfrak{g}^e(j)$ we have that $\Theta(x) \in \mathrm{F}'_j\, U(\mathfrak{g}, e)$ and $\mathrm{gr}'_j\,\Theta(x) = \theta(x)$.*

Proof. Take $0 \neq x \in \mathfrak{g}^e(j)$ for $j \geq 0$. Let $\hat{x} \in S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$ be the unique element such that $\zeta(\hat{x}) = x$. We claim to start with that

$$\hat{x} \equiv x \pmod{\textstyle\bigoplus_{n \geq 2} S^n(\widetilde{\mathfrak{p}})}.$$

To see this, note ζ is a graded map with respect to the Kazhdan grading and x is in Kazhdan degree $j+2$, so we certainly have that $\hat{x} \equiv y \pmod{\bigoplus_{n \geq 2} S^n(\widetilde{\mathfrak{p}})}$ for some unique $y \in \mathfrak{g}(j)$ with $\zeta(y) = x$. Now we just need to show that y centralizes e, i.e. $\zeta(y) = y$. If not then $[e, y] \neq 0$, so we can find some $z \in \mathfrak{g}(-j-2)$ such that $([e, y]|z) \neq 0$. Then

$$\mathrm{pr}([y, z]) = \chi([y, z]) = (e|[y, z]) = ([e, y]|z) \neq 0.$$

On the other hand if $x_1 \cdots x_n$ is any monomial in $S^n(\widetilde{\mathfrak{p}})$ of Kazhdan degree $j+2$ for some $n \geq 2$, we have that

$$\mathrm{pr}([x_1 \cdots x_n, z]) = \sum_{i=1}^{n} x_1 \cdots x_{i-1} \,\mathrm{pr}([x_i, z]) x_{i+1} \cdots x_n.$$

As $n \geq 2$ we know that x_i is of Kazhdan degree strictly smaller than $j+2$, so $[x_i, z]$ lies in $\mathfrak{g}(k)$ for some $k < -2$. Hence $\mathrm{pr}([x_i, z]) = 0$ so $\mathrm{pr}([x_1 \cdots x_n, z]) = 0$ too. So $\mathrm{pr}([\hat{x}, z]) = \mathrm{pr}([y, z]) \neq 0$. But $\hat{x}$ lies in $S(\widetilde{\mathfrak{p}})^{\mathfrak{n}}$, so by the definition of the $\mathfrak{n}$-action we have that $\mathrm{pr}([z, \hat{x}]) = 0$, giving the desired contradiction.

Now to prove the lemma, take $x \in \mathfrak{g}^e(j)$. If $j = 0$ then $\Theta(x) = \theta(x)$ by Theorem 3.6, and the conclusion is clear. So assume that $j > 0$. We need to show that $\Theta(x) = \theta(x) + (\dagger)$ where $(\dagger)$ belongs to $\bigoplus_{i<j} U(\widetilde{\mathfrak{p}})(i)$. By the claim, the lift $\hat{x}$ is equal to x plus a linear combination of monomials of the form $y_1 \cdots y_r z_1^{\mathrm{ne}} \cdots z_s^{\mathrm{ne}}$ for $r + s \geq 2$ and elements $y_i \in \mathfrak{p}(k_i)$ and $z_i \in \mathfrak{k}$ with $(k_1 + 2) + \cdots + (k_r + 2) + s = j + 2$. Hence the lift $\Theta(x)$ of $\hat{x}$ to $U(\mathfrak{g}, e)$ is equal to x plus a linear combination of such monomials $y_1 \cdots y_r z_1^{\mathrm{ne}} \cdots z_s^{\mathrm{ne}}$ plus some element $u \in \mathrm{F}_{j+1} U(\widetilde{\mathfrak{p}})$. Each $y_1 \cdots y_r z_1^{\mathrm{ne}} \cdots z_s^{\mathrm{ne}}$ is of degree $k_1 + \cdots + k_r = j - 2r - s + 2$. As $r + s \geq 2$, we deduce that $k_1 + \cdots + k_r$ is either zero or it is strictly less than j, so all the $y_1 \cdots y_r z_1^{\mathrm{ne}} \cdots z_s^{\mathrm{ne}}$ terms belong to $\bigoplus_{i<j} U(\widetilde{\mathfrak{p}})(i)$. Finally the element u is itself a linear combination of monomials $y_1 \cdots y_r z_1^{\mathrm{ne}} \cdots z_s^{\mathrm{ne}}$ with $y_i \in \mathfrak{p}(k_i)$ and $z_i \in \mathfrak{k}$ such that $(k_1 + 2) + \cdots + (k_r + 2) + s \leq j + 1$. Hence $k_1 + \cdots + k_r \leq j - 2r - s + 1$, so again we have that $k_1 + \cdots + k_r$ is either zero or strictly less than j. This shows that u belongs to $\bigoplus_{i<j} U(\widetilde{\mathfrak{p}})(i)$ too. $\square$

Theorem 3.8. *The homomorphism $U(\mathfrak{g}^e) \to U(\widetilde{\mathfrak{p}})$ induced by the Lie algebra homomorphism θ from Theorem 3.3 defines a $\mathfrak{t}^e$-equivariant graded algebra isomorphism*

$$\theta : U(\mathfrak{g}^e) \xrightarrow{\sim} \mathrm{gr}' U(\mathfrak{g}, e),$$

viewing $U(\mathfrak{g}^e)$ as a graded algebra via the good grading.

Proof. Pick Θ and a homogeneous basis $x_1, \dots, x_t$ for $\mathfrak{g}^e$ as in Theorem 3.6, so that the monomials $\Theta(x_{i_1}) \cdots \Theta(x_{i_k})$ for all $1 \leq i_1 \leq \cdots \leq i_k \leq t$ form a basis for $U(\mathfrak{g}, e)$. Lemma 3.7 implies that

$$\Theta(x_{i_1}) \cdots \Theta(x_{i_k}) = \theta(x_{i_1} \cdots x_{i_k}) + (\dagger)$$

where the first term on the right hand side lies in $U(\widetilde{\mathfrak{p}})(n_{i_1} + \cdots + n_{i_k})$ and the term $(\dagger)$ lies in the sum of all strictly lower graded components. Hence the monomials $\{\theta(x_{i_1} \cdots x_{i_k}) \mid 1 \leq i_1 \leq \cdots \leq i_k \leq t\}$ give a homogeneous basis for $\mathrm{gr}' U(\mathfrak{g}, e)$. By the PBW theorem for $U(\mathfrak{g}^e)$, the same monomials give a homogeneous basis for $\theta(U(\mathfrak{g}^e))$. $\square$

4. Highest weight theory

This section contains the main new results of the paper. We are going to define Verma modules and explain their relevance to the problem of classifying finite dimensional irreducible $U(\mathfrak{g}, e)$-modules.

4.1. **"Cartan subalgebra".** Recall from §3.1 that the restricted root system Φ^e is the set of non-zero weights of $\mathfrak{t}^e$ on $\mathfrak{g} = \mathfrak{g}_0 \oplus \bigoplus_{\alpha\in\Phi^e} \mathfrak{g}_\alpha$. The zero weight space $\mathfrak{g}_0$ is the centralizer of the toral subalgebra $\mathfrak{t}^e$ in $\mathfrak{g}$, so it is a Levi factor of a parabolic subalgebra of $\mathfrak{g}$. According to Bala–Carter theory [C, 5.9.3, 5.9.4], e is a distinguished nilpotent element of $\mathfrak{g}_0$, i.e. the only semisimple elements of $\mathfrak{g}_0$ that centralize e belong to the center of $\mathfrak{g}_0$. It is also clear that h and f lie in $\mathfrak{g}_0$.

By [BG, Lemma 19], our fixed good grading $\mathfrak{g} = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}(j)$ coincides with the eigenspace decomposition of $\mathrm{ad}\,(h+p)$ for some element $p \in \mathfrak{t}^e$. As p centralizes $\mathfrak{g}_0$ it follows that the induced grading $\mathfrak{g}_0 = \bigoplus_{j\in\mathbb{Z}} \mathfrak{g}_0(j)$ coincides simply with the $\mathrm{ad}\,h$-eigenspace decomposition of $\mathfrak{g}_0$. Of course this is another good grading for e, viewed now as an element of the smaller reductive Lie algebra $\mathfrak{g}_0$. Moreover by [C, 5.7.6] it is an even grading, which means that $\widetilde{\mathfrak{g}}_0 = \mathfrak{g}_0$, $\widetilde{\mathfrak{p}}_0 = \mathfrak{p}_0$, $\mathfrak{m}_0 = \mathfrak{n}_0$ and $\mathfrak{k}_0 = \{0\}$. In particular, the finite W-algebra $U(\mathfrak{g}_0, e)$ associated to $e \in \mathfrak{g}_0$ is defined simply by

$$U(\mathfrak{g}_0, e) := \{u \in U(\mathfrak{p}_0) \mid \mathrm{Pr}_0([x, u]) = 0 \text{ for all } x \in \mathfrak{m}_0\},$$

where I_0 is the left ideal of $U(\mathfrak{g}_0)$ generated by the elements $\{x - \chi(x) \,|\, x \in \mathfrak{m}_0\}$, and Pr_0 is the projection along the decomposition $U(\mathfrak{g}_0) = U(\mathfrak{p}_0) \oplus I_0$. This finite W-algebra is going to play the role of Cartan subalgebra in our highest weight theory. However, unlike in the case $e = 0$ it does not embed obviously as a subalgebra of $U(\mathfrak{g}, e)$; instead we will realize it as a certain section.

Before we can do this, we need to fix one more critically important choice: let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}$ with Levi factor $\mathfrak{g}_0$. We stress that there is often more than one conjugacy class of choices for $\mathfrak{q}$, unlike in the case $e = 0$ when there is just one conjugacy class of Borel subalgebras containing $\mathfrak{t}$. In the language of [BG, §2], the choice of $\mathfrak{q}$ determines a system Φ^e_+ of positive roots in the restricted root system Φ^e, namely, $\Phi^e_+ := \{\alpha \in \Phi^e \mid \mathfrak{g}_\alpha \subseteq \mathfrak{q}\}$. Setting $\Phi^e_- := -\Phi^e_+$, we define $\mathfrak{g}_\pm := \bigoplus_{\alpha\in\Phi^e_\pm} \mathfrak{g}_\alpha$, so that

$$\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+, \qquad \mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{g}_+.$$

The choice Φ^e_+ of positive roots induces a *dominance ordering* $\leq$ on $(\mathfrak{t}^e)^*$: $\mu \leq \lambda$ if $\lambda - \mu \in \mathbb{Z}_{\geq 0}\Phi^e_+$.

In this paragraph, we let $\mathfrak{a}$ denote one of $\widehat{\mathfrak{g}}, \widehat{\mathfrak{p}}, \widetilde{\mathfrak{p}}$ or $\mathfrak{g}^e$. Recall from §3.1 that the dot actions of $\mathfrak{t}^e$ on $\mathfrak{a}$ and its universal enveloping (super)algebra $U(\mathfrak{a})$ induce decompositions $\mathfrak{a} = \mathfrak{a}_0 \oplus \bigoplus_{\alpha\in\Phi^e} \mathfrak{a}_\alpha$ and $U(\mathfrak{a}) = \bigoplus_{\alpha\in\mathbb{Z}\Phi^e} U(\mathfrak{a})_\alpha$. In particular, $U(\mathfrak{a})_0$, the zero weight space of $U(\mathfrak{a})$ with respect to the dot action, is a subalgebra of $U(\mathfrak{a})$. Also let $U(\mathfrak{a})_\sharp$ (resp. $U(\mathfrak{a})_\flat$) denote the left (resp. right) ideal of $U(\mathfrak{a})$ generated by the root spaces $\mathfrak{a}_\alpha$ for $\alpha \in \Phi^e_+$ (resp. $\alpha \in \Phi^e_-$). Let

$$U(\mathfrak{a})_{0,\sharp} := U(\mathfrak{a})_0 \cap U(\mathfrak{a})_\sharp, \qquad U(\mathfrak{a})_{\flat,0} := U(\mathfrak{a})_\flat \cap U(\mathfrak{a})_0,$$

which are obviously left and right ideals of $U(\mathfrak{a})_0$, respectively. By the PBW theorem for non-linear Lie algebras, we actually have that

$$U(\mathfrak{a})_{0,\sharp} = U(\mathfrak{a})_{\flat,0},$$

hence $U(\mathfrak{a})_{0,\sharp}$ is a two-sided ideal of $U(\mathfrak{a})_0$. Moreover $\mathfrak{a}_0$ is a subalgebra of $\mathfrak{a}$, and by the PBW theorem again we have that $U(\mathfrak{a})_0 = U(\mathfrak{a}_0) \oplus U(\mathfrak{a})_{0,\sharp}$. The projection along this decomposition defines a surjective algebra homomorphism

$$\pi : U(\mathfrak{a})_0 \twoheadrightarrow U(\mathfrak{a}_0)$$

with $\ker \pi = U(\mathfrak{a})_{0,\sharp}$. Hence $U(\mathfrak{a})_0/U(\mathfrak{a})_{0,\sharp} \cong U(\mathfrak{a}_0)$.

We can repeat some but not all of the preceding discussion for the W-algebra $U(\mathfrak{g}, e)$ itself. To make things as explicit as possible, let us choose a homogeneous $\mathfrak{t}^e$-weight basis $f_1, \dots, f_m, h_1, \dots, h_l, e_1, \dots, e_m$ of $\mathfrak{g}^e$ so that the weight of f_i is $-\gamma_i \in \Phi^e_-$, the weight of e_i is $\gamma_i \in \Phi^e_+$, and $h_1, \dots, h_l \in \mathfrak{g}^e_0$; the weights γ_i here are not necessarily distinct. Choosing an embedding Θ as in Theorem 3.6, we get the corresponding elements $F_i := \Theta(f_i) \in U(\mathfrak{g}, e)_{-\gamma_i}$, $E_i := \Theta(e_i) \in U(\mathfrak{g}, e)_{\gamma_i}$, and $H_j := \Theta(h_j) \in U(\mathfrak{g}, e)_0$. For $\mathbf{a} \in \mathbb{Z}^m_{\geq 0}$, we write $F^{\mathbf{a}}$ for $F_1^{a_1} \dots F_m^{a_m}$, and define $H^{\mathbf{b}}$ and $E^{\mathbf{c}}$ for $\mathbf{b} \in \mathbb{Z}^l_{\geq 0}$ and $\mathbf{c} \in \mathbb{Z}^m_{\geq 0}$ similarly. We get the following PBW basis for $U(\mathfrak{g}, e)$:

$$\{F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}} \mid \mathbf{a}, \mathbf{c} \in \mathbb{Z}^m_{\geq 0}, \mathbf{b} \in \mathbb{Z}^l_{\geq 0}\}.$$

The subspace $U(\mathfrak{g}, e)_\alpha$ in the restricted root space decomposition has basis given by all the PBW monomials $F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}}$ such that $\sum_i (c_i - a_i)\gamma_i = \alpha$.

Now we define $U(\mathfrak{g}, e)_\sharp$ (resp. $U(\mathfrak{g}, e)_\flat$) to be the left (resp. right) ideal of $U(\mathfrak{g}, e)$ generated by $E_1, \dots, E_m$ (resp. $F_1, \dots, F_m$). Note that $U(\mathfrak{g}, e)_\sharp$ (resp. $U(\mathfrak{g}, e)_\flat$) is equivalently the left (resp. right) ideal of $U(\mathfrak{g}, e)$ generated by all $U(\mathfrak{g}, e)_\alpha$ for $\alpha \in \Phi^e_+$ (resp. $\alpha \in \Phi^e_-$), so it does not depend on the explicit choice of the basis. Set

$$U(\mathfrak{g}, e)_{0,\sharp} := U(\mathfrak{g}, e)_0 \cap U(\mathfrak{g}, e)_\sharp, \qquad U(\mathfrak{g}, e)_{\flat,0} := U(\mathfrak{g}, e)_\flat \cap U(\mathfrak{g}, e)_0,$$

which are obviously left and right ideals of the zero weight space $U(\mathfrak{g}, e)_0$, respectively. The PBW monomials $F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}}$ with $\mathbf{c} \neq \mathbf{0}$ (resp. $\mathbf{a} \neq \mathbf{0}$) form a basis of $U(\mathfrak{g}, e)_\sharp$ (resp. $U(\mathfrak{g}, e)_\flat$), and the PBW monomials $F^{\mathbf{a}} H^{\mathbf{b}} E^{\mathbf{c}}$ with $\sum_i (c_i - a_i)\gamma_i = 0$ form a basis of $U(\mathfrak{g}, e)_0$. It follows that

$$U(\mathfrak{g}, e)_{0,\sharp} = U(\mathfrak{g}, e)_{\flat,0},$$

hence it is a two-sided ideal of $U(\mathfrak{g}, e)_0$. Moreover the cosets of the PBW monomials of the form $H^{\mathbf{b}}$ form a basis of the quotient algebra $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\sharp}$. However the PBW monomials $H^{\mathbf{b}}$ need not span a *subalgebra* of $U(\mathfrak{g}, e)$, unlike the situation for the algebras $U(\mathfrak{a})$ discussed earlier.

The goal now is to prove using the BRST cohomology definition of the W-algebra that $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\sharp}$ is canonically isomorphic to $U(\mathfrak{g}_0, e)$. The isomorphism involves a shift in the spirit of Corollary 2.9: let

$$\gamma := \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi^e_-}} \beta_i, \qquad \delta := \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi^e_- \\ d_i \geq 2}} \beta_i + \tfrac{1}{2} \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} \in \Phi^e_- \\ d_i = 1}} \beta_i,$$

recalling from §2 that $b_1, \dots, b_r$ is a homogeneous basis for $\mathfrak{n}$ with $b_i \in \mathfrak{g}(-d_i)$ of weight $\beta_i \in \mathfrak{t}^*$.

Lemma 4.1. *γ and δ extend uniquely to characters of $\mathfrak{p}_0$.*

Proof. For $x \in \mathfrak{h}_0$, $\operatorname{ad} x$ leaves the subspace $\mathfrak{n}_- = \bigoplus_{\alpha \in \Phi^e_-} \mathfrak{n}_\alpha$ invariant, so the map

$$\gamma : \mathfrak{h}_0 \to \mathbb{C}, \qquad x \mapsto \operatorname{tr}(\operatorname{ad} x|_{\mathfrak{n}_-})$$

is a well-defined Lie algebra homomorphism with $\gamma(x) := \operatorname{tr}(\operatorname{ad} x|_{\mathfrak{n}_-})$. Extending by zero on the nilradical of $\mathfrak{p}_0$, this defines the required homomorphism $\gamma : \mathfrak{p}_0 \to \mathbb{C}$. The construction of δ is similar. □

Lemma 4.2. *The following diagram commutes:*

$$\begin{array}{ccc} U(\mathfrak{g}_0^e) & \xrightarrow{\theta} & U(\widetilde{\mathfrak{p}})_0 \\ {\scriptstyle S_{-\delta}}\downarrow & & \downarrow{\scriptstyle \pi} \\ U(\mathfrak{p}_0) & \xleftarrow[S_{-\gamma}]{} & U(\mathfrak{p}_0). \end{array}$$

Proof. Take $x \in \mathfrak{g}_0^e(j)$. If $j > 0$ then $S_{-\gamma}(\pi(\theta(x))) = x = S_{-\delta}(x)$. Now assume that $j = 0$, i.e. $x \in \mathfrak{h}_0^e$. As $\mathfrak{k}_0 = \{0\}$, we can choose the elements $z_1, \dots, z_{2s}$ from Theorem 3.3 so that $z_1, \dots, z_s$ (resp. $z_{s+1}, \dots, z_{2s}$) belong to negative (resp. positive) $\mathfrak{t}^e$-root spaces. Then

$$\begin{aligned} \pi(\theta(x)) &= \pi(x + \tfrac{1}{2}\textstyle\sum_{i=1}^{2s}[x, z_i^*]^{\mathrm{ne}} z_i^{\mathrm{ne}}) = \pi(x + \tfrac{1}{2}\sum_{i=1}^{s}[x, z_i^*]^{\mathrm{ne}} z_i^{\mathrm{ne}}) \\ &= \pi(x + \tfrac{1}{2}\textstyle\sum_{i=1}^{s} z_i^{\mathrm{ne}}[x, z_i^*]^{\mathrm{ne}} + \tfrac{1}{2}\sum_{i=1}^{s}\langle [x, z_i^*] | z_i\rangle) \\ &= x + \tfrac{1}{2}\textstyle\sum_{i=1}^{s}\langle [x, z_i] | z_i^*\rangle = x + \gamma(x) - \delta(x), \end{aligned}$$

noting that $\gamma(x) - \delta(x)$ is the trace of $\frac{1}{2}\operatorname{ad} x$ on $\mathfrak{k}_- = \bigoplus_{\alpha \in \Phi^e_-} \mathfrak{k}_\alpha$. Hence $S_{-\gamma}(\pi(\theta(x))) = S_{-\delta}(x)$ as required. □

Theorem 4.3. *The restriction of $S_{-\gamma} \circ \pi : U(\widetilde{\mathfrak{p}})_0 \twoheadrightarrow U(\mathfrak{p}_0)$ defines a surjective algebra homomorphism*

$$\pi_{-\gamma} : U(\mathfrak{g}, e)_0 \twoheadrightarrow U(\mathfrak{g}_0, e)$$

with $\ker \pi_{-\gamma} = U(\mathfrak{g}, e)_{0,\sharp}$. *Hence* $U(\mathfrak{g}, e)_0 / U(\mathfrak{g}, e)_{0,\sharp} \cong U(\mathfrak{g}_0, e)$.

Proof. Consider the following diagram:

$$\begin{array}{ccccccc} U(\mathfrak{g}, e)_0 & \hookrightarrow & U(\widetilde{\mathfrak{p}})_0 & \xrightarrow{S_{-\gamma}\circ\pi} & U(\mathfrak{p}_0) & \hookleftarrow & U(\mathfrak{g}_0, e) \\ & & {\scriptstyle \phi}\downarrow & & \downarrow{\scriptstyle \phi_0} & & \\ & & U(\hat{\mathfrak{p}})_0 & \xrightarrow{\pi} & U(\hat{\mathfrak{p}}_0) & & \\ & & {\scriptstyle d}\downarrow & & \downarrow{\scriptstyle d_0} & & \\ & & U(\hat{\mathfrak{g}})_0 & \xrightarrow{\pi} & U(\hat{\mathfrak{g}}_0) & & \end{array}$$

We have already constructed the horizontal maps. From top to bottom, they have kernels $U(\widetilde{\mathfrak{p}})_{0,\sharp}, U(\widehat{\mathfrak{p}})_{0,\sharp}$ and $U(\widehat{\mathfrak{g}})_{0,\sharp}$. For the vertical maps, recall the derivation $d : U(\widehat{\mathfrak{g}}) \to U(\widehat{\mathfrak{g}})$ and the homomorphism $\phi : U(\widetilde{\mathfrak{p}}) \to U(\widehat{\mathfrak{p}})$ from §2. One checks that both of these are $\mathfrak{t}^e$-equivariant, hence they restrict to give the maps d and ϕ in the diagram. The maps d_0 and ϕ_0 come from the analogous maps for the reductive subalgebra $\mathfrak{g}_0$.

We first verify that the top square commutes. As ϕ is $\mathfrak{t}^e$-invariant, it maps $U(\widetilde{\mathfrak{p}})_{0,\sharp}$ into $U(\widehat{\mathfrak{p}})_{0,\sharp}$. Hence the top square commutes on restriction to $U(\widetilde{\mathfrak{p}})_{0,\sharp}$, as we get zero both ways round. Since $U(\widetilde{\mathfrak{p}})_0 = U(\mathfrak{p})_0 \oplus U(\widetilde{\mathfrak{p}})_{0,\sharp}$ it remains to check the square commutes on restriction to $U(\mathfrak{p}_0)$. So take $x \in \mathfrak{p}_0$. Introduce the shorthands $\sum_i^{\pm}$ and $\sum_i^0$ for the sums over all $1 \leq i \leq r$ such that $\beta_i|_{\mathfrak{t}^e} \in \Phi^e_{\pm}$ and $\beta_i|_{\mathfrak{t}^e} = 0$, respectively. Both $\sum_i^+ f_i[b_i,x]^{\mathrm{ch}}$ and $\sum_i^- [b_i,x]^{\mathrm{ch}} f_i$ belong to $U(\widehat{\mathfrak{p}})_{0,\sharp}$, hence map to zero under π. So we get

$$\begin{aligned}\pi(\phi(x)) &= \pi(x + \textstyle\sum_i f_i[b_i,x]^{\mathrm{ch}}) = \pi(x + \sum_i^0 f_i[b_i,x]^{\mathrm{ch}} + \sum_i^- f_i[b_i,x]^{\mathrm{ch}})\\ &= \pi(\phi_0(x) + \textstyle\sum_i^- \langle f_i,[b_i,x]\rangle + \sum_i^- [b_i,x]^{\mathrm{ch}} f_i)\\ &= \phi_0(x) - \textstyle\sum_i^- \langle f_i,[x,b_i]\rangle = \phi_0(x) - \gamma(x) = \phi_0(S_{-\gamma}(\pi(x))).\end{aligned}$$

Next we check that the bottom square commutes. Again d is $\mathfrak{t}^e$-equivariant so maps $U(\widehat{\mathfrak{p}})_{0,\sharp}$ into $U(\widehat{\mathfrak{g}})_{0,\sharp}$, hence the bottom square commutes on restriction to $U(\widehat{\mathfrak{p}})_{0,\sharp}$. It remains to check it commutes on restriction to $U(\widehat{\mathfrak{p}}_0)$. Recalling $\widehat{\mathfrak{p}}_0 = \mathfrak{p}_0 \oplus \mathfrak{m}_0^* \oplus \mathfrak{m}_0^{\mathrm{ch}}$, it suffices to consider elements $x \in \mathfrak{p}_0$, $f \in \mathfrak{m}_0^*$ and $y^{\mathrm{ch}} \in \mathfrak{m}_0^{\mathrm{ch}}$. In the first case we calculate:

$$\begin{aligned}\pi(d(x)) &= \pi(\textstyle\sum_i^0 f_i[b_i,x] + \sum_i^+ f_i[b_i,x] + \sum_i^- f_i[b_i,x])\\ &= \pi(d_0(x) + \textstyle\sum_i^- [b_i,x] f_i) = d_0(\pi(x)).\end{aligned}$$

The second case is very similar. The calculation in the third case is as follows:

$$\begin{aligned}\pi(d(y^{\mathrm{ch}})) &= \pi(y - \chi(y) - y^{\mathrm{ne}} + \textstyle\sum_i f_i[b_i,y]^{\mathrm{ch}}) = \pi(d_0(y^{\mathrm{ch}}) + \sum_i^- f_i[b_i,y]^{\mathrm{ch}})\\ &= d_0(y^{\mathrm{ch}}) - \pi(\textstyle\sum_i^- [b_i,y]^{\mathrm{ch}} f_i + \sum_i^- \langle f_i,[y,b_i]\rangle) = d_0(\pi(y^{\mathrm{ch}})),\end{aligned}$$

noting each $\langle f_i,[y,b_i]\rangle = 0$ by degree considerations.

Now Theorem 2.8 gives that $d(\phi(u)) = 0$ for all $u \in U(\mathfrak{g},e)_0$. By the commutativity of the diagram we deduce that $d_0(\phi_0(S_{-\gamma}(\pi(u)))) = 0$ for all such u. By Theorem 2.8 again, we have that $U(\mathfrak{g}_0,e) = \{u \in U(\mathfrak{p}_0) \,|\, d_0(\phi_0(u)) = 0\}$. Hence $S_{-\gamma}(\pi(U(\mathfrak{g},e)_0)) \subseteq U(\mathfrak{g}_0,e)$, showing that the restriction of $S_{-\gamma}\circ\pi$ defines an algebra homomorphism $\pi_{-\gamma} : U(\mathfrak{g},e)_0 \to U(\mathfrak{g}_0,e)$. Moreover $U(\mathfrak{g},e)_{0,\sharp} \subseteq U(\widetilde{\mathfrak{p}})_{0,\sharp}$ so $U(\mathfrak{g},e)_{0,\sharp} \subseteq \ker \pi_{-\gamma}$. Recall that the quotient $U(\mathfrak{g},e)_0/U(\mathfrak{g},e)_{0,\sharp}$ has basis given by the cosets of the PBW monomials $H^{\mathbf{b}}$ with $\mathbf{b} \in \mathbb{Z}^l_{\geq 0}$. Therefore to complete the proof it suffices to show that the monomials $\pi_{-\gamma}(H^{\mathbf{b}}) \in U(\mathfrak{g}_0,e)$ actually form a basis for for $U(\mathfrak{g}_0,e)$, since that shows simultaneously that $\pi_{-\gamma}$ is surjective and that its kernel is no bigger than $U(\mathfrak{g},e)_{0,\sharp}$.

By Lemma 3.7, $H^{\mathbf{b}} = \theta(h^{\mathbf{b}}) + (\dagger)$ where $h^{\mathbf{b}} := h_1^{b_1}\cdots h_l^{b_l}$ and $(\dagger)$ denotes a linear combination of terms of strictly smaller degree in the good grading. Applying $\pi_{-\gamma}$ using Lemma 4.2, we deduce that $\pi_{-\gamma}(H^{\mathbf{b}}) = S_{-\delta}(h^{\mathbf{b}}) + (\ddagger)$ where $(\ddagger)$ consists of lower degree terms. Recalling that $\mathrm{gr}'\, U(\mathfrak{g}_0,e) = U(\mathfrak{g}_0^e)$ by Theorem 3.8, we see by the PBW theorem for $U(\mathfrak{g}_0^e)$ that the monomials $\pi_{-\gamma}(H^{\mathbf{b}})$ for all $\mathbf{b} \in \mathbb{Z}^l_{\geq 0}$ do indeed form a basis for $U(\mathfrak{g}_0,e)$. □

4.2. **Verma modules.** Recall the embedding $\theta : \mathfrak{t}^e \hookrightarrow U(\mathfrak{g},e)$ from Theorem 3.3 and the weight δ from §4.1. For a $U(\mathfrak{g},e)$-module V and $\lambda \in (\mathfrak{t}^e)^*$, we define the λ-*weight space*

$$V_\lambda := \{v \in V \mid (\theta + \delta)(t)v = \lambda(t)v \text{ for all } t \in \mathfrak{t}^e\}.$$

By Theorem 3.3(1) we have that $U(\mathfrak{g}, e)_\alpha V_\lambda \subseteq V_{\lambda+\alpha}$. In particular each V_λ is invariant under the action of the subalgebra $U(\mathfrak{g}, e)_0$. We say that V_λ is a *maximal weight space* of V if $U(\mathfrak{g}, e)_\sharp V_\lambda = \{0\}$. For example, if λ is any *maximal weight* of V in the dominance ordering, i.e. $V_\lambda \neq \{0\}$ and $V_\mu = \{0\}$ for all $\mu > \lambda$, then V_λ is a maximal weight space of V.

Let V_λ be a maximal weight space in a $U(\mathfrak{g}, e)$-module V. Then the action of $U(\mathfrak{g}, e)_0$ on V_λ factors through the map $\pi_{-\gamma}$ from Theorem 4.3 to make V_λ into a $U(\mathfrak{g}_0, e)$-module such that $um = \pi_{-\gamma}(u)m$ for $u \in U(\mathfrak{g}, e)_0$ and $m \in V_\lambda$. Note also that $\mathfrak{t}^e$ is a Lie subalgebra of $U(\mathfrak{g}_0, e)$ (since $\mathfrak{t}^e$ even lies in the center of $U(\mathfrak{p}_0)$), so we get another action of $\mathfrak{t}^e$ on V_λ by restricting the $U(\mathfrak{g}_0, e)$-action. By Lemma 4.2 this new action satisfies

$$tv = \lambda(t)v$$

for all $t \in \mathfrak{t}^e$. This explains why we included the additional shift by δ in the definition of the λ-weight space of a $U(\mathfrak{g}, e)$-module.

We say that a $U(\mathfrak{g}, e)$-module V is a *highest weight module* if it is generated by a maximal weight space V_λ such that V_λ is finite dimensional and irreducible as a $U(\mathfrak{g}_0, e)$-module. In that case, as we will see shortly, λ is the unique maximal weight of V in the dominance ordering. Let

$$\{V_\Lambda \mid \Lambda \in \mathcal{L}\}$$

be a complete set of pairwise inequivalent finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules for some set $\mathcal{L}$. If V is a highest weight module generated by a maximal weight space V_λ and $V_\lambda \cong V_\Lambda$ for $\Lambda \in \mathcal{L}$, we say that V is of *type* Λ.

Since $U(\mathfrak{g}, e)_\sharp$ is invariant under left multiplication by $U(\mathfrak{g}, e)$ and right multiplication by $U(\mathfrak{g}, e)_0$, the quotient $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp$ is a $(U(\mathfrak{g}, e), U(\mathfrak{g}, e)_0)$-bimodule. Moreover the right action of $U(\mathfrak{g}, e)_0$ factors through the map $\pi_{-\gamma}$ from Theorem 4.3 to make $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp$ into a $(U(\mathfrak{g}, e), U(\mathfrak{g}_0, e))$-bimodule. For $\Lambda \in \mathcal{L}$ we define the *Verma module of type* Λ by setting

$$M(\Lambda, e) := (U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp) \otimes_{U(\mathfrak{g}_0, e)} V_\Lambda.$$

We are going show that this is a universal highest weight module of type Λ, meaning that $M(\Lambda, e)$ is a highest weight module of type Λ (in particular it is non-zero) and moreover if V is another highest weight module generated by a maximal weight space V_λ and $f : V_\Lambda \xrightarrow{\sim} V_\lambda$ is a $U(\mathfrak{g}_0, e)$-module isomorphism then there is a unique $U(\mathfrak{g}, e)$-module homomorphism $\tilde{f} : M(\Lambda, e) \twoheadrightarrow V$ extending f. Recall the PBW basis for $U(\mathfrak{g}, e)$ fixed in §4.1.

Lemma 4.4. *As a right $U(\mathfrak{g}_0, e)$-module, $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp$ is free with basis $\{F^{\mathbf{a}} \mid \mathbf{a} \in \mathbb{Z}^m_{\geq 0}\}$.*

Proof. This follows because the cosets of the PBW monomials of the form $F^{\mathbf{a}} H^{\mathbf{b}}$ give a basis for the quotient $U(\mathfrak{g}, e)/U(\mathfrak{g}, e)_\sharp$ and the cosets of the monomials of the form $H^{\mathbf{b}}$ give a basis for $U(\mathfrak{g}, e)_0/U(\mathfrak{g}, e)_{0,\sharp} \cong U(\mathfrak{g}_0, e)$. □

Theorem 4.5. *Given $\Lambda \in \mathcal{L}$, let $v_1, \dots, v_k$ be a basis for V_Λ and λ be its $\mathfrak{t}^e$-weight.*

1. *The vectors $\{F^{\mathbf{a}} \otimes v_i \mid \mathbf{a} \in \mathbb{Z}^m_{\geq 0}, 1 \leq i \leq k\}$ give a basis of $M(\Lambda, e)$.*
2. *The weight λ is the unique maximal weight of $M(\Lambda, e)$ in the dominance ordering, $M(\Lambda, e)$ is generated by the maximal weight space $M(\Lambda, e)_\lambda$, and $M(\Lambda, e)_\lambda \cong V_\Lambda$ as $U(\mathfrak{g}_0, e)$-modules.*
3. *The module $M(\Lambda, e)$ is a universal highest weight module of type Λ.*
4. *There is a unique maximal proper submodule $R(\Lambda, e)$ in $M(\Lambda, e)$,*
$$L(\Lambda, e) := M(\Lambda, e)/R(\Lambda, e)$$
is irreducible, and $\{L(\Lambda, e) \mid \Lambda \in \mathcal{L}\}$ is a complete set of pairwise inequivalent irreducible highest weight modules over $U(\mathfrak{g}, e)$.

Proof. (1) This is clear from Lemma 4.4.

(2) The basis vector $F^{\mathbf{a}} \otimes v_i$ is of weight $\lambda - \sum_i a_i \gamma_i$. Hence the λ-weight space of $M(\Lambda, e)$ is $1 \otimes V_\Lambda$ and all other weights of $M(\Lambda, e)$ are strictly smaller in the dominance ordering.

(3) By (1)–(2) $M(\Lambda, e)$ is a highest weight module of type Λ. Let V be another highest weight module generated by a maximal weight space V_μ and $f : V_\Lambda \to V_\mu$ be a $U(\mathfrak{g}_0, e)$-module isomorphism. By comparing $\mathfrak{t}^e$-actions we get that $\mu = \lambda$. By adjointness of tensor and hom f extends uniquely to a $U(\mathfrak{g}, e)$-module homomorphism $\tilde{f} : M(\Lambda, e) \to V$ such that $\tilde{f}(1 \otimes v_i) = f(v_i)$ for each i. As $\tilde{f}(1 \otimes V_\Lambda) = f(V_\Lambda)$ generates V, we get that $\tilde{f}$ is surjective.

(4) Let N be a submodule of $M(\Lambda, e)$. Then N is the direct sum of its $\mathfrak{t}^e$-weight spaces. If $N_\lambda \neq 0$ then N_λ generates all of $1 \otimes V_\Lambda$ as a $U(\mathfrak{g}_0, e)$-module, hence it generates all of $M(\Lambda, e)$ as a $U(\mathfrak{g}, e)$-module. This shows that if N is a proper submodule then it is contained in $\bigoplus_{\mu < \lambda} M(\Lambda, e)_\mu$. Hence the sum of all proper submodules of $M(\Lambda, e)$ is still a proper submodule, so $M(\Lambda, e)$ has a unique maximal submodule $R(\Lambda, e)$ as claimed. By (3) any irreducible highest weight module V of type Λ is a quotient of $M(\Lambda, e)$, hence $V \cong L(\Lambda, e)$. Moreover λ is the unique maximal weight of $L(\Lambda, e)$ by (2) and $L(\Lambda, e)_\lambda \cong V_\Lambda$ as $U(\mathfrak{g}_0, e)$-modules. Hence Λ is uniquely determined by V. □

Corollary 4.6. *Let $\mathcal{L}^+ := \{\Lambda \in \mathcal{L} \mid \dim L(\Lambda, e) < \infty\}$. Then the modules $\{L(\Lambda, e) | \Lambda \in \mathcal{L}^+\}$ give a complete set of pairwise inequivalent finite dimensional irreducible $U(\mathfrak{g}, e)$-modules.*

Proof. Any finite dimensional irreducible $U(\mathfrak{g}, e)$-module L has a maximal weight λ. Moreover by irreducibility L is generated by any irreducible $U(\mathfrak{g}_0, e)$-submodule of L_λ. Hence L is an irreducible highest weight module. Now apply Theorem 4.5(4). □

Unfortunately we have absolutely no idea how to give an explicit combinatorial parametrization $\{V_\Lambda \mid \Lambda \in \mathcal{L}\}$ of a complete set of pairwise inequivalent finite dimensional irreducible $U(\mathfrak{g}_0, e)$-modules unless the distinguished nilpotent element e of $\mathfrak{g}_0$ is actually regular. We will discuss the regular case in more detail in §5.1.

4.3. **Central characters.** Let $Z(\mathfrak{g})$ denote the center of $U(\mathfrak{g})$ and $Z(\mathfrak{g}, e)$ denote the center of $U(\mathfrak{g}, e)$. It is easy to see that the restriction of the linear map Pr from §2 defines an injective algebra homomorphism $\mathrm{Pr} : Z(\mathfrak{g}) \hookrightarrow Z(\mathfrak{g}, e)$. As explained in the footnote to [P2, Question 5.1], this map is also surjective, so it is an algebra isomorphism

$$\mathrm{Pr} : Z(\mathfrak{g}) \xrightarrow{\sim} Z(\mathfrak{g}, e).$$

A $U(\mathfrak{g}, e)$-module V is of *central character* $\psi : Z(\mathfrak{g}) \to \mathbb{C}$ if $\mathrm{Pr}(z)v = \psi(z)v$ for all $z \in Z(\mathfrak{g})$ and $v \in V$. Analogously for $\mathfrak{g}_0$ we have the isomorphism

$$\mathrm{Pr}_0 : Z(\mathfrak{g}_0) \xrightarrow{\sim} Z(\mathfrak{g}_0, e),$$

and we say that a $U(\mathfrak{g}_0, e)$-module V is of central character $\psi_0 : Z(\mathfrak{g}_0) \to \mathbb{C}$ if $\mathrm{Pr}_0(z)v = \psi_0(v)$ for all $z \in Z(\mathfrak{g}_0)$ and $v \in V$. We want to relate the central character of an irreducible highest weight module over $U(\mathfrak{g}, e)$ to the central character of its maximal weight space over $U(\mathfrak{g}_0, e)$. We remark that the surjectivity of the map Pr will not be used in any of the arguments below but the surjectivity of Pr_0 is essential; if e is regular in $\mathfrak{g}_0$ then the surjectivity of Pr_0 is already clear from [K, §2].

Let Φ (resp. Φ_0) denote the root system of $\mathfrak{g}$ (resp. $\mathfrak{g}_0$) with respect to $\mathfrak{t}$ and W (resp. W_0) be the corresponding Weyl group. Let $\Phi_\pm := \left\{\alpha \in \Phi \;\middle|\; \alpha|_{\mathfrak{t}^e} \in \Phi^e_\pm\right\}$ so that

$$\Phi = \Phi_- \sqcup \Phi_0 \sqcup \Phi_+,$$

corresponding to the decomposition $\mathfrak{g} = \mathfrak{g}_- \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_+$. We stress that Φ_+ is *not* a positive system of roots in Φ; we reserve the notation Φ^+ for such a thing below. For each $\alpha \in \Phi$, fix a non-zero vector x_α in the α-root space of $\mathfrak{g}$ and let $\alpha^\vee := 2\alpha/(\alpha|\alpha) \in \mathfrak{t}^*$.

Now we define Harish-Chandra isomorphisms

$$\Psi : Z(\mathfrak{g}) \xrightarrow{\sim} S(\mathfrak{t})^W, \qquad \Psi_0 : Z(\mathfrak{g}_0) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}$$

for $\mathfrak{g}$ and $\mathfrak{g}_0$ as follows. Let Φ^+ be any system of positive roots in Φ and set $\Phi_0^+ := \Phi^+ \cap \Phi_0$, which is a system of positive roots in Φ_0. Set

$$\rho := \tfrac{1}{2} \sum_{\alpha \in \Phi^+} \alpha, \qquad \rho_0 := \tfrac{1}{2} \sum_{\alpha \in \Phi_+} \alpha + \tfrac{1}{2} \sum_{\alpha \in \Phi_0^+} \alpha.$$

The first term on the right hand side of the definition of ρ_0 is orthogonal to all the roots in Φ_0; it should be viewed as a shift in origin for the definition of Ψ_0. Then, the Harish-Chandra isomorphisms Ψ and Ψ_0 are determined on $z \in Z(\mathfrak{g})$ and $z_0 \in Z(\mathfrak{g}_0)$ by the equations

$$\begin{aligned} z &\equiv S_\rho(\Psi(z)) &&\left(\mathrm{mod}\ \textstyle\sum_{\alpha \in \Phi^+} U(\mathfrak{g}) x_\alpha\right), \\ z_0 &\equiv S_{\rho_0}(\Psi_0(z)) &&\left(\mathrm{mod}\ \textstyle\sum_{\alpha \in \Phi_0^+} U(\mathfrak{g}_0) x_\alpha\right). \end{aligned}$$

Although the definitions of Ψ and Ψ_0 involve the choice of Φ^+, it is known by [D, 7.4.5] that the Harish-Chandra isomorphisms Ψ and Ψ_0 are actually independent of this choice (though Ψ_0 does depend on the choice of Φ^e_+ because of the shift of origin in defining ρ_0).

Theorem 4.7. *There is a unique embedding $c : Z(\mathfrak{g}) \hookrightarrow Z(\mathfrak{g}_0)$ such that the following diagram commutes:*

$$\begin{array}{ccccc}
Z(\mathfrak{g}, e) & \xleftarrow{\mathrm{Pr}} & Z(\mathfrak{g}) & \xrightarrow{\Psi} & S(\mathfrak{t})^W \\
{\scriptstyle \pi_{-\gamma}}\downarrow & & \downarrow{\scriptstyle c} & & \downarrow{\scriptstyle \iota} \\
Z(\mathfrak{g}_0, e) & \xleftarrow{\mathrm{Pr}_0} & Z(\mathfrak{g}_0) & \xrightarrow{\Psi_0} & S(\mathfrak{t})^{W_0}
\end{array}$$

Here, $\iota : S(\mathfrak{t})^W \hookrightarrow S(\mathfrak{t})^{W_0}$ denotes the natural inclusion.

Proof. Given $\sigma, \tau \in \{+, 0, -\}$ let $\Phi_\sigma(\tau)$ denote the set of all $\alpha \in \Phi_\sigma$ such that the degree of x_α in the good grading is positive, zero or negative according to whether $\tau = +, 0$ or $-$, respectively. Note that $\Phi_\sigma(\tau) = -\Phi_{-\sigma}(-\tau)$, in particular, $\Phi_0(0)$ is a closed subsystem of Φ (it is the root system of the reductive Lie algebra $\mathfrak{h}_0$). Pick a system $\Phi_0^+(0)$ of positive roots in $\Phi_0(0)$. Then set

$$\Phi^+ := \Phi_0^+(0) \sqcup \Phi_+(0) \sqcup \Phi_+(-) \sqcup \Phi_0(-) \sqcup \Phi_-(-),$$

which is a system of positive roots in Φ. Using this choice we define the weights ρ, ρ_0 and the Harish-Chandra isomorphisms Ψ, Ψ_0, as explained above. Adopting the shorthand that $\sum X$ denotes $\sum_{x \in X} x$ for any finite subset X of $\mathfrak{t}^*$, we have that

$$\rho = \tfrac{1}{2} \sum \left(\Phi_0^+(0) \sqcup \Phi_+(0) \sqcup \Phi_+(-) \sqcup \Phi_0(-) \sqcup \Phi_-(-) \right),$$
$$\rho_0 = \tfrac{1}{2} \sum \left(\Phi_+(+) \sqcup \Phi_+(0) \sqcup \Phi_+(-) \sqcup \Phi_0^+(0) \sqcup \Phi_0(-) \right).$$

Recalling the weight γ from §4.1, we deduce that

$$\rho - \rho_0 = \tfrac{1}{2} \sum \Phi_-(-) - \tfrac{1}{2} \sum \Phi_+(+) = \sum \Phi_-(-) = \gamma.$$

Now we need to fix some PBW bases. Let $t_1, \dots, t_n$ be a basis for $\mathfrak{t}$. Also enumerate the elements of $\Phi_0^+(0)$ as $\alpha_1, \dots, \alpha_s$, the elements of $\Phi_0(-)$ as $\beta_1, \dots, \beta_q$, the elements of $\Phi_+(-) \sqcup \Phi_-(-)$ as $\beta_{q+1}, \dots, \beta_r$, and the elements of $\Phi_+(0)$ as $\nu_1, \dots, \nu_p$. Order the basis $\{t_i, x_\alpha \mid 1 \le i \le n, \alpha \in \Phi\}$ of $\mathfrak{g}$ so the $x_{-\beta_i}$ come first, then the $x_{-\nu_i}$, then the $x_{-\alpha_i}$, then the t_i, then the x_{α_i}, then the x_{ν_i}, then the x_{β_i}. This ordering determines a corresponding PBW basis for $U(\mathfrak{g})$. Using the same ordering on the basis $\{t_i, x_\alpha \mid 1 \le i \le n, \alpha \in \Phi_0\}$ of $\mathfrak{g}_0$ we also get a PBW basis for $U(\mathfrak{g}_0)$.

We next calculate $\mathrm{Pr}_0(z_0)$ for any $z_0 \in Z(\mathfrak{g}_0)$. We can write

$$z_0 = S_{\rho_0}(\Psi_0(z_0)) + u_0 + v_0$$

where $S_{\rho_0}(\Psi_0(z_0)) \in S(\mathfrak{t})$, u_0 is a linear combination of ordered PBW monomials ending in x_{α_i} $(1 \le i \le s)$ and v_0 is a linear combination of ordered PBW monomials ending in x_{β_i} $(1 \le i \le q)$. As z_0 is central it is of degree 0 in the good grading. Hence $S_{\rho_0}(\Psi_0(z_0)), u_0$ and v_0 are all of degree 0 too. Note u_0 only involves products of the basis vectors $x_{\alpha_1}, \dots, x_{\alpha_s}$ on the positive side, all of which are of degree 0. Hence it can only involve products of the basis vectors $x_{-\alpha_1}, \dots, x_{-\alpha_s}$ on the negative side because each $x_{-\beta_1}, \dots, x_{-\beta_q}$ is of strictly positive degree. Hence $u_0 \in U(\mathfrak{h}_0)$ and $\mathrm{Pr}_0(S_{\rho_0}(\Psi_0(z_0)) + u_0) = S_{\rho_0}(\Psi_0(z_0)) + u_0$. To compute $\mathrm{Pr}_0(v_0)$, our PBW monomials are ordered so that the basis vectors $x_{\beta_1}, \dots, x_{\beta_q}$ for $\mathfrak{m}_0$ appear on the right hand side, so we simply replace each x_{β_i} by the scalar $\chi(x_{\beta_i})$. Since all x_{β_i} are of strictly negative degree, it follows that $\mathrm{Pr}_0(v_0)$ is of strictly positive degree. We have shown that

$$\mathrm{Pr}_0(z_0) = S_{\rho_0}(\Psi_0(z_0)) + u_0 + \mathrm{Pr}_0(v_0) \in Z(\mathfrak{g}_0, e) \subseteq U(\mathfrak{p}_0)$$

with $S_{\rho_0}(\Psi_0(z_0)) \in S(\mathfrak{t})$, $u_0 \in \sum_{i=1}^s U(\mathfrak{h}_0)x_{\alpha_i}$ and $\mathrm{Pr}_0(v_0) \in \sum_{j>0} U(\mathfrak{p}_0)(j)$. In particular we see from this that z_0 can be recovered uniquely from $\mathrm{Pr}_0(z_0)$: it is the unique element of $Z(\mathfrak{g}_0)$ such that

$$S_{\rho_0}(\Psi_0(z_0)) \equiv \mathrm{Pr}_0(z_0) \pmod{\textstyle\sum_{i=1}^s U(\mathfrak{h}_0)x_{\alpha_i} + \sum_{j>0} U(\mathfrak{p}_0)(j)}.$$

Instead take an element $z \in Z(\mathfrak{g})$. We expand it as

$$z = S_\rho(\Psi(z)) + t + u + v$$

where $S_\rho(\Psi(z)) \in S(\mathfrak{t})$, t is a linear combination of ordered PBW monomials ending in x_{α_i} $(1 \le i \le s)$, u is a linear combination of ordered PBW monomials ending in x_{ν_i} $(1 \le i \le p)$, and v is a linear combination of ordered PBW monomials ending in x_{β_i} $(1 \le i \le r)$. We apply the map Pr and argue just like in the previous paragraph to get that

$$\mathrm{Pr}(z) = S_\rho(\Psi(z)) + t + u + \mathrm{Pr}(v) \in Z(\mathfrak{g}, e) \subseteq U(\widetilde{\mathfrak{p}})_0$$

where $S_\rho(\Psi(z)) \in S(\mathfrak{t})$, $t \in \sum_{i=1}^s U(\mathfrak{h})x_{\alpha_i}$, $u \in \sum_{i=1}^p U(\mathfrak{h})x_{\nu_i}$ and finally $\mathrm{Pr}(v) \in U(\widetilde{\mathfrak{p}})\mathfrak{k}^{\mathrm{ne}} + \sum_{j>0} U(\mathfrak{p})(j)$. Next we apply the graded homomorphism $S_{-\gamma} \circ \pi : U(\widetilde{\mathfrak{p}})_0 \to U(\mathfrak{p}_0)$ to this. It annihilates $U(\mathfrak{h})x_{\nu_i}$ and $U(\widetilde{\mathfrak{p}})_0 \cap U(\widetilde{\mathfrak{p}})\mathfrak{k}^{\mathrm{ne}}$. So we deduce recalling Theorem 4.3 that

$$\pi_{-\gamma}(\mathrm{Pr}(z)) = S_{-\gamma}(S_\rho((\Psi(z))) + w \in Z(\mathfrak{g}_0, e)$$

for $w \in \sum_{i=1}^s U(\mathfrak{h}_0)x_{\alpha_i} + \sum_{j>0} U(\mathfrak{p}_0)(j)$. Using the last sentence of the previous paragraph and the fact from [P2] that $\mathrm{Pr}_0 : Z(\mathfrak{g}_0) \hookrightarrow Z(\mathfrak{g}_0, e)$ is surjective, we deduce that $\pi_{-\gamma}(\mathrm{Pr}(z)) = \mathrm{Pr}_0(z_0)$ where z_0 is the unique element of $Z(\mathfrak{g}_0)$ such that $S_{\rho_0}(\Psi_0(z_0)) = S_{-\gamma}(S_\rho(\Psi(z)))$. Equivalently, by the first paragraph, $\Psi_0(z_0) = \Psi(z)$.

Now we can prove the theorem. Since Ψ and Ψ_0 are isomorphisms, there is obviously a unique map c so the right hand square commutes. For the left hand square, we have shown for $z \in Z(\mathfrak{g})$ that $\pi_{-\gamma}(\mathrm{Pr}(z)) = \mathrm{Pr}_0(z_0)$ where $\Psi_0(z_0) = \Psi(z)$. This means that $z_0 = c(z)$ hence $\pi_{-\gamma}(\mathrm{Pr}(z)) = \mathrm{Pr}_0(c(z))$. □

For $\Lambda \in \mathcal{L}$, Schur's lemma implies that $Z(\mathfrak{g}_0, e)$ acts diagonally on Λ. Let $\psi_0^\Lambda : Z(\mathfrak{g}_0) \to \mathbb{C}$ be the resulting central character, i.e. $\mathrm{Pr}_0(z)v = \psi_0^\Lambda(z)v$ for all $z \in Z(\mathfrak{g}_0)$ and $v \in \Lambda$. Let $\psi^\Lambda : Z(\mathfrak{g}) \to \mathbb{C}$ denote $\psi_0^\Lambda \circ c$.

Corollary 4.8. *Let V be a highest weight $U(\mathfrak{g}, e)$-module of type $\Lambda \in \mathcal{L}$. Then V is of central character $\psi^\Lambda : Z(\mathfrak{g}) \to \mathbb{C}$.*

Proof. Say V is generated by its maximal weight space V_λ and $f : V_\Lambda \to V_\lambda$ is an isomorphism of $U(\mathfrak{g}_0, e)$-modules. For $z \in Z(\mathfrak{g})$ and $v \in V_\Lambda$, the theorem implies that

$$\mathrm{Pr}(z)f(v) = f(\pi_{-\gamma}(\mathrm{Pr}(z))v) = f(\mathrm{Pr}_0(c(z))v) = \psi_0^\Lambda(c(z))f(v).$$

Hence $\mathrm{Pr}(z)$ acts on V_λ as the scalar $\psi^\Lambda(z)$. Since V_λ generates V as a $U(\mathfrak{g},e)$-module it follows that $\mathrm{Pr}(z)$ acts on all of V as $\psi^\Lambda(z)$. Hence V has central character ψ^Λ. $\square$

Remark 4.9. Given in addition a weight $o \in \mathfrak{t}^*$ orthogonal to all the roots in Φ, one can modify the above definitions of ρ and ρ_0 by adding $o \in \mathfrak{t}^*$ to them both ("change of origin"). Providing one also adds o to the weight ϵ defined in §5.1 below, all our subsequent results remain true as stated. The point is that although adding o changes the Harish-Chandra isomorphisms Ψ and Ψ_0, hence also the parametrization of central characters, it does not affect the maps c or ι in Theorem 4.7.

4.4. **Category $\mathcal{O}(e)$.** We want to prove that Verma modules have finite length. This is not a hard result if e is regular in $\mathfrak{g}_0$, but to prove it in general we need to appeal to some recent results of Losev. We first recall a little more of the background for this. Let $\mathfrak{l}$ be a Lagrangian subspace of $\mathfrak{k}$ (for example a natural choice is $\mathfrak{l} := \mathfrak{k}_+ = \bigoplus_{\alpha\in\Phi_+^e}\mathfrak{k}_\alpha$). Define the left $U(\mathfrak{g})$-module $Q_\mathfrak{l} = U(\mathfrak{g})/I_\mathfrak{l}$ and the algebra $Q_\mathfrak{l}^{\mathfrak{m}\oplus\mathfrak{l}}$ as in Remark 2.1. It is obvious from the definition of $Q_\mathfrak{l}^{\mathfrak{m}\oplus\mathfrak{l}}$ that there is a well-defined multiplication map

$$Q_\mathfrak{l}\otimes Q_\mathfrak{l}^{\mathfrak{m}\oplus\mathfrak{l}} \to Q_\mathfrak{l}, \quad (u+I_\mathfrak{l})\otimes(v+I_\mathfrak{l}) \mapsto uv+I_\mathfrak{l}$$

making $Q_\mathfrak{l}$ into a $(U(\mathfrak{g}), Q_\mathfrak{l}^{\mathfrak{m}\oplus\mathfrak{l}})$-bimodule. Identifying $Q_\mathfrak{l}^{\mathfrak{m}\oplus\mathfrak{l}}$ with $U(\mathfrak{g},e)$ using the isomorphism ν from Remark 2.1 and the isomorphism from Theorem 2.4, this makes $Q_\mathfrak{l}$ into a $(U(\mathfrak{g}),U(\mathfrak{g},e))$-bimodule too.

Let $\mathcal{C}(e)$ denote the category of all left $U(\mathfrak{g},e)$-modules. Let $\mathcal{W}(e)$ denote the category of all $\mathfrak{g}$-modules on which $x-\chi(x)$ acts locally nilpotently for all $x \in \mathfrak{m}\oplus\mathfrak{l}$. Note $Q_\mathfrak{l}$ belongs to $\mathcal{W}(e)$, hence tensoring with this bimodule defines a functor

$$Q_\mathfrak{l}\otimes_{U(\mathfrak{g},e)}? : \mathcal{C}(e) \to \mathcal{W}(e).$$

The important *Skryabin's theorem* asserts that this functor is an equivalence of categories; see [S] or [GG, Theorem 6.1].

Hence if L is any irreducible $U(\mathfrak{g},e)$-module then $Q_\mathfrak{l}\otimes_{U(\mathfrak{g},e)} L$ is an irreducible $U(\mathfrak{g})$-module, and its annihilator $\mathrm{Ann}_{U(\mathfrak{g})}(Q_\mathfrak{l}\otimes_{U(\mathfrak{g},e)} L)$ is a primitive ideal of $U(\mathfrak{g})$. For any primitive ideal P of $U(\mathfrak{g})$, we let $\mathcal{VA}(P) \subseteq \mathfrak{g}$ denote its associated variety; see e.g. [J, §9]. It is known that $\mathcal{VA}(P)$ is the closure of a single nilpotent orbit in $\mathfrak{g}$; see [J2, 3.10]. By [P2, Theorem 3.1] and [L, Theorem 1.2.2(ii),(ix)], it is known for any irreducible $U(\mathfrak{g},e)$-module L that

$$\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})}(Q_\mathfrak{l}\otimes_{U(\mathfrak{g},e)} L)) \supseteq \overline{G\cdot e}$$

with equality if and only if L is finite dimensional.

Theorem 4.10. *The number of isomorphism classes of irreducible highest weight modules for $U(\mathfrak{g},e)$ with prescribed central character $\psi : Z(\mathfrak{g}) \to \mathbb{C}$ is finite, i.e. the set $\{\Lambda \in \mathcal{L} \mid \psi^\Lambda = \psi\}$ is finite.*

Proof. By Corollary 4.8, $\psi^\Lambda = \psi$ implies $\psi_0^\Lambda = \psi_0$ for some central character $\psi_0 : Z(\mathfrak{g}_0) \to \mathbb{C}$ such that $\psi_0 \circ c = \psi$. Each W-orbit in $\mathfrak{t}^*$ is a union of finitely many W_0-orbits, hence there are finitely many such ψ_0. Therefore it suffices to prove for each $\psi_0 : Z(\mathfrak{g}_0) \to \mathbb{C}$ that the set $\{\Lambda \in \mathcal{L} \mid \psi_0^\Lambda = \psi_0\}$ is finite. In other words, replacing $\mathfrak{g}$ by $\mathfrak{g}_0$, we may assume that e is a distinguished nilpotent element in $\mathfrak{g}$ and need to prove that the number of isomorphism classes of *finite dimensional* irreducible $U(\mathfrak{g}, e)$-modules with fixed central character ψ is finite. This statement is immediate if e is regular in $\mathfrak{g}$ by [K, §2]. In general we use [L, Theorem 1.2.2], as follows. The map sending L to $\mathrm{Ann}_{U(\mathfrak{g},e)}(L)$ induces a bijection between isomorphism classes of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules of central character ψ and primitive ideals of $U(\mathfrak{g}, e)$ of finite codimension that contain $\mathrm{Pr}(\ker \psi)$. So we just need to show there are finitely many such primitive ideals. By [L, Theorem 1.2.2(ii),(iii)], if $P = \mathrm{Ann}_{U(\mathfrak{g},e)}(L)$ is a primitive ideal of $U(\mathfrak{g}, e)$ containing $\mathrm{Pr}(\ker \psi)$ then $\mathrm{Ann}_{U(\mathfrak{g})}(Q_{\mathfrak{l}} \otimes_{U(\mathfrak{g},e)} L)$ is a primitive ideal of $U(\mathfrak{g})$ containing $\ker \psi$. A well known consequence of Duflo's theorem [Du] is that there are only finitely many such primitive ideals in $U(\mathfrak{g})$. Hence there are only finitely many possibilities for P thanks to [L, Theorem 1.2.2(vi),(vii)]. □

Corollary 4.11. *For each $\Lambda \in \mathcal{L}$, the Verma module $M(\Lambda, e)$ has a composition series.*

Proof. Imitate the standard argument in the classical case from [D, 7.6.1], using Corollary 4.8, Theorem 4.5(1)–(2) and Theorem 4.10. □

Now we introduce an analogue of the Bernstein-Gelfand-Gelfand category $\mathcal{O}$: let $\mathcal{O}(e) = \mathcal{O}(e; \mathfrak{t}, \mathfrak{q})$ denote the category of all finitely generated $U(\mathfrak{g}, e)$-modules V that are semisimple over $\mathfrak{t}^e$ with finite dimensional $\mathfrak{t}^e$-weight spaces, such that the set $\{\lambda \in (\mathfrak{t}^e)^* | V_\lambda \neq \{0\}\}$ is contained in a finite union of sets of the form $\{\nu \in (\mathfrak{t}^e)^* \mid \nu \leq \mu\}$ for $\mu \in (\mathfrak{t}^e)^*$. As $U(\mathfrak{g}, e)$ is Noetherian, $\mathcal{O}(e)$ is closed under the operations of taking submodules, quotients and finite direct sums. The following statement follows routinely from Corollary 4.11 and Theorem 4.5.

Corollary 4.12. *Every object in $\mathcal{O}(e)$ has a composition series. Moreover the category $\mathcal{O}(e)$ decomposes as $\mathcal{O}(e) = \bigoplus_\psi \mathcal{O}_\psi(e)$, where the direct sum is over all central characters $\psi : Z(\mathfrak{g}) \to \mathbb{C}$, and $\mathcal{O}_\psi(e)$ denotes the Serre subcategory of $\mathcal{O}(e)$ generated by the irreducible modules $\{L(\Lambda, e) \mid \Lambda \in \mathcal{L}$ such that $\psi^\Lambda = \psi\}$.*

In particular, this shows that the irreducible objects in $\mathcal{O}(e)$ are all of the form $L(\Lambda, e)$ for $\Lambda \in \mathcal{L}$. In the case $e = 0$, $\mathcal{O}(e)$ is the usual BGG category $\mathcal{O}$ for the semisimple Lie algebra $\mathfrak{g}$ with respect to the maximal toral subalgebra $\mathfrak{t}$ and the Borel $\mathfrak{q}$. At the other extreme, if e is a distinguished nilpotent element of $\mathfrak{g}$, then $\mathcal{O}(e)$ is the category of all finite dimensional $U(\mathfrak{g}, e)$-modules that are semisimple over the Lie algebra center of $\mathfrak{g}$.

Remark 4.13. If e is a distinguished but not a regular nilpotent element of $\mathfrak{g}$ then $U(\mathfrak{g}, e)$ has primitive ideals of infinite codimension by [L, Theorem 1.2.2(viii)] and [P2, Theorem 3.1]. So there is no chance in this case that every primitive ideal of $U(\mathfrak{g}, e)$ is the annihilator of an irreducible highest weight module in our narrow sense (finite dimensional weight spaces).

5. Special cases

In this section we specialize further. First we discuss the case that e is of *standard Levi type* in the sense of [FP], i.e. it is a regular nilpotent element of $\mathfrak{g}_0$. In particular, we will formulate a precise conjecture for the classification of finite dimensional irreducible $U(\mathfrak{g}, e)$-modules in standard Levi type. Then we prove this conjecture for the standard choice of positive roots in type A, by translating some results from [BK2] into the present framework. We continue with the notation from the previous section; in particular, recall we fixed a parabolic subalgebra $\mathfrak{q}$ with Levi factor $\mathfrak{g}_0$ in §4.1.

5.1. **Standard Levi type.** Assume from now on that e is a regular nilpotent element of $\mathfrak{g}_0$. In that case, by [K, §2], the map Pr_0 is an isomorphism

$$\mathrm{Pr}_0 : Z(\mathfrak{g}_0) \xrightarrow{\sim} U(\mathfrak{g}_0, e).$$

As we have already observed in §§4.3–4.4, many things in the theory are simpler under this assumption. To start with, $\mathfrak{p}_0$ is actually a Borel subalgebra of $\mathfrak{g}_0$ with opposite nilradical $\mathfrak{n}_0$. Let Ψ and Ψ_0 be the Harish-Chandra isomorphisms for $\mathfrak{g}$ and $\mathfrak{g}_0$, defined as in §4.3. Recalling that $\Phi_+ = \left\{\alpha \in \Phi \;\middle|\; \alpha|_{\mathfrak{t}^e} \in \Phi_+^e\right\}$ is the set of roots corresponding to the nilradical $\mathfrak{g}_+$ of $\mathfrak{q}$, let

$$\epsilon := \tfrac{1}{2} \sum_{\alpha \in \Phi_+} \alpha + \tfrac{1}{2} \sum_{\substack{1 \leq i \leq r \\ \beta_i|_{\mathfrak{t}^e} = 0}} \beta_i.$$

This is just the weight ρ_0 from §4.3 for the system of positive roots in Φ_0 corresponding to the Borel subalgebra $\mathfrak{t} \oplus \mathfrak{n}_0$ of $\mathfrak{g}_0$ (though we don't necessarily want to fix this choice). With this in mind, the following lemma is essentially [K, Proposition 2.3]:

Lemma 5.1. *Let $\xi : U(\mathfrak{p}_0) \twoheadrightarrow S(\mathfrak{t})$ be the homomorphism induced by the natural projection $\mathfrak{p}_0 \twoheadrightarrow \mathfrak{t}$. Then the restriction of $S_{-\epsilon} \circ \xi$ defines an algebra isomorphism*

$$\xi_{-\epsilon} : U(\mathfrak{g}_0, e) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}$$

such that $\Psi_0 = \xi_{-\epsilon} \circ \mathrm{Pr}_0$.

We can now define an explicit set $\mathcal{L}$ parametrizing the Verma modules $M(\Lambda, e)$ and the irreducible highest weight modules $L(\Lambda, e)$ for $U(\mathfrak{g}, e)$: define

$$\mathcal{L} := \mathfrak{t}^*/W_0 = \mathrm{Spec}(S(\mathfrak{t})^{W_0}).$$

Thus each $\Lambda \in \mathcal{L}$ is a W_0-orbit of weights in $\mathfrak{t}^*$. For each $\Lambda \in \mathcal{L}$, let V_Λ denote the one dimensional irreducible $U(\mathfrak{g}_0, e)$-module obtained by lifting the irreducible $S(\mathfrak{t})^{W_0}$-module corresponding to Λ through the isomorphism $\xi_{-\epsilon} : U(\mathfrak{g}_0, e) \xrightarrow{\sim} S(\mathfrak{t})^{W_0}$ from Lemma 5.1. Also fix finally a Borel subalgebra $\mathfrak{b}_0$ of $\mathfrak{g}_0$ containing $\mathfrak{t}$ and let

$$\mathfrak{b} := \mathfrak{b}_0 \oplus \mathfrak{g}_+,$$

which is a Borel subalgebra of $\mathfrak{g}$ contained in the parabolic $\mathfrak{q}$. Let Φ_0^+ and $\Phi^+ = \Phi_0^+ \sqcup \Phi_+$ be the systems of positive roots in Φ_0 and Φ corresponding to

$\mathfrak{b}_0$ and $\mathfrak{b}$. Let $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$. For $\lambda \in \mathfrak{t}^*$, let $\mathbb{C}_{\lambda-\rho}$ be the one dimensional $\mathfrak{t}$-module of weight $\lambda-\rho$. Let

$$M(\lambda) := U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_{\lambda-\rho}$$

denote the usual Verma module for $\mathfrak{g}$ of highest weight $(\lambda-\rho)$, with unique irreducible quotient $L(\lambda)$. Note by Corollary 4.8 and Lemma 5.1 that the central character ψ^Λ of $L(\Lambda, e)$ is equal to the central character of $L(\lambda)$ for any $\lambda \in \Lambda$. All other notation used below is as explained in §4.4.

Conjecture 5.2. *For $\Lambda \in \mathcal{L}$, pick $\lambda \in \Lambda$ such that $(\lambda|\alpha^\vee) \notin \mathbb{Z}_{>0}$ for each $\alpha \in \Phi_0^+$. Then $L(\Lambda, e)$ is finite dimensional if and only if*

$$\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda))) = \overline{G\cdot e}.$$

We will verify this conjecture in type A (for the standard choice of positive roots) in Corollary 5.6 below. To formulate a stronger conjecture which was inspired by ideas of Premet, let

$$\mathfrak{u} := \mathfrak{n}_0 \oplus \mathfrak{g}_+,$$

which is a maximal nilpotent subalgebra of $\mathfrak{g}$ contained in $\mathfrak{q}$. Note that χ restricts to a character of $\mathfrak{u}$. Let $\mathcal{O}(\chi) = \mathcal{O}(\chi; \mathfrak{t}, \mathfrak{q})$ denote the category of all finitely generated $\mathfrak{g}$-modules M that are locally finite over $Z(\mathfrak{g})$ and semisimple over $\mathfrak{t}^e$, such that $x - \chi(x)$ acts locally nilpotently on M for all $x \in \mathfrak{u}$. This is the category $\mathcal{N}(\chi)$ from [MS] (with $\mathfrak{n}$ there equal to our $\mathfrak{u}$) except we have added the mild extra condition that the center of the Levi factor of $\mathfrak{q}$ containing $\mathfrak{t}$ acts semisimply. In the case $\chi = 0$ we note that $\mathcal{O}(\chi)$ is the usual BGG category $\mathcal{O}$ again. To define the basic objects in the category $\mathcal{O}(\chi)$, let R denote the quotient of $U(\mathfrak{g})$ by the left ideal generated by all $\{x - \chi(x) \mid x \in \mathfrak{u}\}$. This left ideal is invariant under right multiplication by elements of $U(\mathfrak{g}_0, e)$, hence R is a $(U(\mathfrak{g}), U(\mathfrak{g}_0, e))$-bimodule. For $\Lambda \in \mathcal{L}$, set

$$M(\Lambda, \chi) := R \otimes_{U(\mathfrak{g}_0, e)} V_\Lambda,$$

naturally an object of $\mathcal{O}(\chi)$ of central character ψ^Λ. In [MS, §2], it is shown that $M(\Lambda, \chi)$ has a unique irreducible quotient $L(\Lambda, \chi)$, and that every object in $\mathcal{O}(\chi)$ has a composition series involving only the $L(\Lambda, \chi)$ as composition factors.

Conjecture 5.3. *There is an equivalence of categories $\mathbb{W} : \mathcal{O}(\chi) \to \mathcal{O}(e)$ such that $\mathbb{W}M(\Lambda, \chi) \cong M(\Lambda, e)$ and $\mathbb{W}L(\Lambda, \chi) \cong L(\Lambda, e)$ for each $\Lambda \in \mathcal{L}$. Moreover, $\mathbb{W}$ should respect annihilators in the sense that*

$$\mathrm{Ann}_{U(\mathfrak{g})}(M) = \mathrm{Ann}_{U(\mathfrak{g})}(Q_{\mathfrak{l}} \otimes_{U(\mathfrak{g},e)} \mathbb{W}M)$$

for each $M \in \mathcal{O}(\chi)$.

We point out that Conjecture 5.2 follows from Conjecture 5.3. Indeed, using [MS, Theorem 5.1], one can check for $\lambda \in \Lambda$ as in Conjecture 5.2 that

$$\mathrm{Ann}_{U(\mathfrak{g})}(L(\Lambda, \chi)) = \mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda)).$$

By Conjecture 5.3 we get that $\mathrm{Ann}_{U(\mathfrak{g})}(Q_{\mathfrak{l}} \otimes_{U(\mathfrak{g},e)} L(\Lambda, e)) = \mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda))$, and then Conjecture 5.2 follows using [P2, Theorem 3.1] and [L, Theorem

1.2.2(ii),(ix)] (see the discussion just before Theorem 4.10). Combined with [Ba, Theorem 6.2] and the Kazhdan–Lusztig conjecture for $\mathfrak{g}$, Conjecture 5.3 would also mean that the composition multiplicities of all Verma modules $M(\Lambda, e)$ can be computed in terms of Kazhdan–Lusztig polynomials. In particular, the Kazhdan–Lusztig conjecture of [VD] (as we understand it) is a consequence, as is [BK2, Conjecture 7.17] in type A. Note finally that Conjecture 5.3 (hence also Conjecture 5.2) is true if $e \in \mathfrak{g}$ is a long root element. In this special case for the good grading arising from the $\operatorname{ad} h$-eigenspace decomposition of $\mathfrak{g}$, the equivalence of categories $\mathbb{W}$ is given simply by taking Whittaker vectors with respect to $\mathfrak{m} \oplus \mathfrak{k}_+$; see [P2, Theorem 7.1].

5.2. **Type A.** We now recast some of the results of [BK2] in the language of this paper. In particular we prove Conjecture 5.2 for the standard choice of positive roots in type A. So let $\mathfrak{g} := \mathfrak{gl}_N(\mathbb{C})$ equipped with the trace form $(.|.)$, $\mathfrak{t}$ be the set of diagonal matrices and $\mathfrak{b}$ be the set of upper triangular matrices. Let $\varepsilon_i \in \mathfrak{t}^*$ be the ith diagonal coordinate function. Then the root system is $\Phi = \Phi^+ \sqcup (-\Phi^+)$ where $\Phi^+ := \{\varepsilon_i - \varepsilon_j \mid 1 \leq i < j \leq N\}$ as usual.

Let $\mathbf{p}$ be a partition of N and draw its Young diagram like in the following example:

$$\begin{array}{c|c|c|c|c|}
\cline{2-3}
p_1 & 1 & 2 \\
\cline{2-4}
p_2 & 3 & 4 & 5 \\
\cline{2-5}
p_3 & 6 & 7 & 8 & 9 \\
\cline{2-5}
\multicolumn{1}{c}{} & \multicolumn{1}{c}{q_1} & \multicolumn{1}{c}{q_2} & \multicolumn{1}{c}{q_3} & \multicolumn{1}{c}{q_4}
\end{array}$$

We let n denote the number of rows and ℓ denote the number of columns in the Young diagram of $\mathbf{p}$. We index the rows of the diagram by $1, \dots, n$ from top to bottom, columns by $1, \dots, \ell$ from left to right, and boxes by $1, \dots, N$ along rows as in the example. Let p_i (resp. q_i) denote the number of boxes in the ith row (resp. ith column). Let $\operatorname{row}(i)$ and $\operatorname{col}(i)$ denote the row and column numbers of the ith box. Letting $e_{i,j}$ denote the ij-matrix unit, we let $e \in \mathfrak{g}_2$ be the nilpotent matrix

$$e = \sum_{\substack{1 \leq i,j \leq N \\ \operatorname{row}(i) = \operatorname{row}(j) \\ \operatorname{col}(i) = \operatorname{col}(j) - 1}} e_{i,j},$$

which clearly has Jordan type $\mathbf{p}$; e.g. $e = e_{1,2} + e_{3,4} + e_{4,5} + e_{6,7} + e_{7,8} + e_{8,9}$ in the above example. We define an even good grading for e by declaring that $e_{i,j}$ is of degree $2(\operatorname{col}(j) - \operatorname{col}(i))$. We call this the *standard good grading.* Now define the finite W-algebra $U(\mathfrak{g}, e)$ as in §2.2. As the good grading is even, $U(\mathfrak{g}, e)$ is simply a subalgebra of $U(\mathfrak{p})$. The Levi factor $\mathfrak{h}$ of $\mathfrak{p}$ satisfies

$$\mathfrak{h} \cong \mathfrak{gl}_{q_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{q_\ell}(\mathbb{C}).$$

We also fix the choice of the parabolic $\mathfrak{q}$ in §4.1 to be the span of the matrix units $\{e_{i,j} \mid \operatorname{row}(i) \leq \operatorname{row}(j)\}$. So the Levi factor $\mathfrak{g}_0$ of $\mathfrak{q}$ satisfies

$$\mathfrak{g}_0 \cong \mathfrak{gl}_{p_1}(\mathbb{C}) \oplus \cdots \oplus \mathfrak{gl}_{p_\ell}(\mathbb{C}).$$

The choice of $\mathfrak{q}$ determines a system of positive roots Φ^e_+ in the restricted root system Φ^e, which we call the *standard positive roots*.

We incorporate the following two shifts as indicated in Remarks 3.4 and 4.9:

$$\eta := \sum_{i=1}^N (n - q_{\mathrm{col}(i)} - q_{\mathrm{col}(i)+1} - \cdots - q_\ell)\varepsilon_i, \qquad o := -\tfrac{1}{2}(N-1)\sum_{i=1}^N \varepsilon_i.$$

Noting η does indeed extend to a character of $\mathfrak{p}$, the embedding θ from Theorem 3.3 shifted in this way is the restriction of $S_\eta : U(\mathfrak{p}) \to U(\mathfrak{p})$, which matches [BK1, (9.2)]. Also the choice of origin o means that the weight ρ from §4.3 is

$$\rho = -\varepsilon_2 - 2\varepsilon_3 - \cdots - (N-1)\varepsilon_N,$$

which agrees with the choice made in [BK2]. In [BK1] an *explicit* linear map $\Theta : \mathfrak{g}^e \hookrightarrow U(\mathfrak{g}, e)$ as in Theorem 3.6 was described. The images of a certain distinguished basis of $\mathfrak{g}^e$ under this explicit map Θ were denoted

$$\begin{aligned}
&\{D_i^{(r)} \in U(\mathfrak{g}, e) \mid 1 \le i \le n,\ 1 \le r \le p_i\},\\
&\{E_{i,j}^{(r)} \in U(\mathfrak{g}, e) \mid 1 \le i < j \le n,\ p_j - p_i < r \le p_j\},\\
&\{F_{i,j}^{(r)} \in U(\mathfrak{g}, e) \mid 1 \le i < j \le n,\ 0 < r \le p_i\}.
\end{aligned}$$

These elements belong to the zero, positive and negative restricted root spaces of $U(\mathfrak{g}, e)$, respectively. Recall the maps $\xi_{-\epsilon}$ from Lemma 5.1 and $\pi_{-\gamma}$ from Theorem 4.3.

Lemma 5.4. *$\xi_{-\epsilon}(\pi_{-\gamma}(D_i^{(r)}))$ is equal to the rth elementary symmetric function in $\{e_{j,j} + i - 1 \mid 1 \le j \le N, \mathrm{row}(j) = i\}$.*

Proof. We need to recall the explicit form of the element $D_i^{(r)}$ from [BK1, Corollary 9.4]:

$$D_i^{(r)} = \sum_{s=1}^r \sum_{\substack{i_1,\dots,i_s\\ j_1,\dots,j_s}} (-1)^{r-s+\#\{1<k\le s \mid \mathrm{row}(i_k)<i\}} S_\eta(e_{i_1,j_1}\cdots e_{i_s,j_s})$$

where the second sum is over all $1 \le i_1,\dots,i_s,j_1,\dots,j_s \le N$ such that

1. $\mathrm{col}(j_1) - \mathrm{col}(i_1) + \cdots + \mathrm{col}(j_s) - \mathrm{col}(i_s) = r - s$;
2. $\mathrm{col}(i_t) \le \mathrm{col}(j_t)$ for each $t = 1,\dots,s$;
3. if $\mathrm{row}(j_t) \ge i$ then $\mathrm{col}(j_t) < \mathrm{col}(i_{t+1})$ for each $t = 1,\dots,s-1$;
4. if $\mathrm{row}(j_t) < i$ then $\mathrm{col}(j_t) \ge \mathrm{col}(i_{t+1})$ for each $t = 1,\dots,s-1$;
5. $\mathrm{row}(i_1) = \mathrm{row}(j_s) = i$;
6. $\mathrm{row}(j_t) = \mathrm{row}(i_{t+1})$ for each $t = 1,\dots,s-1$.

We claim that the map $\pi : U(\mathfrak{p})_0 \to U(\mathfrak{p}_0)$ annihilates all $S_\eta(e_{i_1,j_1}\cdots e_{i_s,j_s})$ on the right hand side of this formula that have $\mathrm{row}(i_t) \ne i$ for some t. To see this, take such a monomial and the maximal such t. If $\mathrm{row}(i_t) < i$ then e_{i_t,j_t} can be commuted to the end of the monomial in view of (3), hence since it lies in a positive restricted root space it is mapped to zero by π. If $\mathrm{row}(i_t) > i$ then we let $1 \le u < t$ be maximal such that $\mathrm{row}(i_u) < \mathrm{row}(j_u)$. Again e_{i_u,j_u} can be commuted to the end of the monomial by (3) and π gives zero.

Using the claim and (6) we see that $\pi(D_i^{(r)})$ is given explicitly by the analogous expression summing over $1 \leq i_1, \dots, i_s, j_1, \dots, j_s \leq N$ satisfying the same conditions as before and also $\mathrm{row}(i_t) = \mathrm{row}(j_t) = i$ for all t. Applying $S_{-\gamma}$ then $S_{-\epsilon} \circ \xi$ and noting that $S_{-\epsilon-\gamma+\eta} = S_{-\epsilon-\delta}$ we see that $\xi_{-\epsilon}(\pi_{-\gamma}(D_i^{(r)}))$ is equal to the rth elementary symmetric function in

$$\{S_{-\epsilon-\delta}(e_{j,j}) \mid 1 \leq j \leq N, \mathrm{row}(j) = i\}.$$

It remains to show $S_{-\epsilon-\delta}(e_{j,j}) = e_{j,j}+i-1$. To see this, let $\mathrm{N}(j), \mathrm{NE}(j), \mathrm{E}(j), \dots$ denote the number of boxes to the north (strictly above and in the same column), north east (strictly above and strictly to the right), east (strictly to the right and in the same row), ... of the jth box. The weights δ from §4.1 and ϵ from §5.1 are then given explicitly by the formulae

$$\delta = \sum_{j=1}^{N} (\mathrm{NW}(j) + \mathrm{N}(j) + \mathrm{NE}(j) + \mathrm{E}(j) + \mathrm{S}(j) + 1 - n)\varepsilon_j,$$

$$\epsilon = -\sum_{j=1}^{N} (\mathrm{NW}(j) + \mathrm{N}(j) + \mathrm{NE}(j) + \mathrm{E}(j))\varepsilon_j,$$

recalling we have shifted by $-\eta$ and o as indicated in Remarks 3.4 and 4.9. Hence

$$\epsilon + \delta = \sum_{j=1}^{N} (\mathrm{S}(j) + 1 - n)\varepsilon_j = \sum_{j=1}^{N} (1 - \mathrm{row}(j))\varepsilon_j$$

as required to complete the proof. $\square$

A $\mathbf{p}$-*tableau* means a filling of boxes of the Young diagram of $\mathbf{p}$ with complex numbers. The map sending a tableau to the weight $\sum_{i=1}^{N} a_i \varepsilon_i$, where a_i is the entry in the ith box, defines a bijection from the set $\mathrm{Tab}(\mathbf{p})$ of all $\mathbf{p}$-tableaux to the set $\mathfrak{t}^*$. It induces a bijection from the set $\mathrm{Row}(\mathbf{p})$ of all row equivalence classes of $\mathbf{p}$-tableaux to the set $\mathcal{L} = \mathfrak{t}^*/W_0$ from §5.1. Let $\leq$ denote the partial order on $\mathbb{C}$ defined by $a \leq b$ if $b - a \in \mathbb{Z}_{\geq 0}$. We call a tableau *column strict* if its entries are strictly increasing up columns from bottom to top in this order.

Theorem 5.5. *Let $\Lambda \in \mathcal{L}$ and $A \in \mathrm{Row}(\mathbf{p})$ be the corresponding row equivalence class of $\mathbf{p}$-tableaux. Then $L(\Lambda, e)$ is finite dimensional if and only if A has a column strict representative.*

Proof. In [BK2, §6.1] a $U(\mathfrak{g}, e)$-module M is called a highest weight module of type A if it is generated by a vector v_+ that is annihilated by all the $E_{i,j}^{(r)}$ and such that $D_i^{(r)}$ acts on v_+ by multiplication by the rth elementary symmetric function in the elements $\{a_j + i - 1 \mid 1 \leq j \leq N, \mathrm{row}(j) = i\}$, where a_j is the entry in the jth box of some representative of A. In view of Lemma 5.4 and the explicit definition of V_Λ given just after Lemma 5.1, this is exactly the same as the notion of a highest weight module of type $\Lambda \in \mathcal{L}$ from §4.2. Hence the Verma modules $M(\Lambda, e)$ and their irreducible quotients $L(\Lambda, e)$ here are exactly the same as the modules $M(A)$ and $L(A)$ in [BK2, §6.1]. Given this, the present theorem is a restatement of [BK2, Theorem 7.9]. $\square$

Corollary 5.6. *Conjecture 5.2 holds in the present situation.*

Proof. To deduce this from Theorem 5.5, we need to recall some classical results describing the associated varieties of primitive ideals in $U(\mathfrak{g})$ in terms the Robinson-Schensted correspondence. Let $\lambda = \sum_{i=1}^{N} a_i \varepsilon_i \in \mathfrak{t}^*$. We define a tableau $A(\lambda)$ by starting from the empty tableau and then using the Robinson-Schensted row insertion algorithm to successively incorporate the complex numbers $a_1, \dots, a_N$. At the ith step we add a_i to the bottom row of the tableau unless there is an entry b already in the bottom row with $a_i < b$, in which case we pick the smallest such b, replace b by a_i then bump b into the next row up by the analogous procedure. See [F, §1.1] for a detailed account. By [J1, Corollary 3.3] (together with [J1, Lemma 2.4] to reduce to regular weights) it is known that $\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda)))$ is equal to the closure of the orbit consisting of all nilpotent matrices of Jordan type equal to the shape of the tableau $A(\lambda)$.

Now to prove the corollary we take $\Lambda \in \mathcal{L}$ and pick a representative $\lambda \in \Lambda$ such that $(\lambda|\alpha^\vee) \notin \mathbb{Z}_{>0}$ for all $\alpha \in \Phi_0^+$. Let A be the corresponding $\mathbf{p}$-tableau. Thus if $a < b$ are entries in the same row of A then a is located to the left of b. It is now an elementary combinatorial exercise to check that A is row equivalent to a column strict tableau if and only if $A(\lambda)$ is of shape $\mathbf{p}$. Combined with Theorem 5.5 and the preceding paragraph, we deduce that $L(\Lambda, e)$ is finite dimensional if and only if $\mathcal{VA}(\mathrm{Ann}_{U(\mathfrak{g})}(L(\lambda))) = \overline{G \cdot e}$. □

The result just proved also holds for an arbitrary good grading; the general case easily reduces to the standard good grading considered here using [BG, Theorem 2].

References

[A] T. Arakawa, Representation theory of W-algebras, *Invent. Math.* **169** (2007), 219–320.

[Ba] E. Backelin, Representation of the category $\mathcal{O}$ in Whittaker categories, *Int. Math. Res. Not.* **4** (1997), 153–172.

[BGG] J. Bernstein, I. M. Gelfand and S. I. Gelfand, A category of $\mathfrak{g}$-modules, *Func. Anal. Appl.* **10** (1976), 87–92.

[BT] J. de Boer and T. Tjin, Quantization and representation theory of finite W-algebras, *Comm. Math. Phys.* **158** (1993), 485–516.

[B1] J. Brundan, Centers of degenerate cyclotomic Hecke algebras and parabolic category $\mathcal{O}$, preprint, *arXiv:math/0607717*.

[B2] ———, Symmetric functions, parabolic category $\mathcal{O}$, and the Springer fiber, to appear in *Duke Math. J.*.

[BG] J. Brundan and S. M. Goodwin, Good grading polytopes, *Proc. London Math. Soc.* **94** (2007), 155–180.

[BK1] J. Brundan and A. Kleshchev, Shifted Yangians and finite W-algebras, *Adv. Math.* **200** (2006), 136–195.

[BK2] ———, Representations of shifted Yangians and finite W-algebras, to appear in *Mem. Amer. Math. Soc.*.

[BK3] ———, Schur-Weyl duality for higher levels, preprint, *arXiv:math/0605217*.

[C] R. Carter, *Finite Groups of Lie Type*, Wiley, N.Y., 1985.

[D] J. Dixmier, *Enveloping algebras*, Graduate Studies in Math. 11, Amer. Math. Soc., 1996.

[Du] M. Duflo, Sur la classification des idéaux primitifs dans l'algèbre enveloppante d'une algèbre de Lie semi-simple, *Ann. Math.* **105** (1977), 107–120.

[GG] W. L. Gan and V. Ginzburg, Quantization of Slodowy slices, *Internat. Math. Res. Notices* **5** (2002), 243–255.

[DK] A. De Sole and V. Kac, Finite vs affine W-algebras, *Jpn. J. Math.* **1** (2006), 137–261.

[D^{3}HK] A. D'Andrea, C. De Concini, A. De Sole, R. Heluani and V. Kac, Three equivalent definitions of finite W-algebras, appendix to [DK].

[EK] A. Elashvili and V. Kac, Classification of good gradings of simple Lie algebras, in *Lie groups and invariant theory* (E. B. Vinberg ed.), pp. 85–104, Amer. Math. Soc. Transl. **213**, AMS, 2005.

[FP] E. Friedlander and B. Parshall, Modular representation theory of Lie algebras, *Amer. J. Math.* **110** (1988), 1055–1094.

[F] W. Fulton, *Young tableaux*, LMS, 1997.

[J] J. C. Jantzen, *Nilpotent orbits in representation theory*, Progress in Math., vol. 228, Birkhäuser, 2004.

[J1] A. Joseph, Towards the Jantzen conjecture III, *Compositio Math.* **41** (1981), 23–30.

[J2] ———, On the associated variety of a primitive ideal, *J. Algebra* **93** (1985), 509–523.

[K] B. Kostant, On Whittaker modules and representation theory, *Invent. Math.* **48** (1978), 101–184.

[L] I. Losev, Quantized symplectic actions and W-algebras, preprint, *arXiv:0707.3108.*

[Ly] T. E. Lynch, *Generalized Whittaker vectors and representation theory*, PhD thesis, M.I.T., 1979.

[MS] D. Miličíc and W. Soergel, The composition series of modules induced from Whittaker modules, *Comment. Math. Helv.* **72** (1997), 503–520.

[P1] A. Premet, Special transverse slices and their enveloping algebras, *Adv. in Math.* **170** (2002), 1–55.

[P2] ———, Enveloping algebras of Slodowy slices and the Joseph ideal, *J. Eur. Math. Soc.* **9** (2007), 487–543.

[P3] ———, Primitive ideals, non-restricted representations and finite W-algebras, *Mosc. Math. J.* **7** (2007), 743–762.

[S] S. Skryabin, A category equivalence, appendix to [P1].

[VD] K. de Vos and P. van Driel, The Kazhdan–Lusztig conjecture for finite W-algebras, *Lett. Math. Phys.* **35** (1995), 333–344.

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